Corrigendum to “Simulation of multivariate diffusion bridges”

Mogens Bladt
Dept. of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
DK-2100 Copenhagen Ø
Denmark
bladt@math.ku.dk

Samuel Finch
Dept. of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
DK-2100 Copenhagen Ø
Denmark
stjfinch@gmail.com

Michael Sørensen
Dept. of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
DK-2100 Copenhagen Ø
Denmark
michael@math.ku.dk

June 3, 2020

Abstract

We correct an error in Theorem 1 in Bladt *et al.* (2016), where the initial distribution of an auxiliary diffusion process that is used to describe the distribution of the proposed approximate diffusion bridge is wrong. As a consequence we also correct the pseudo marginal Metropolis-Hastings algorithm and the alternative MCMC-algorithm that have an exact diffusion bridge as their target distribution. The same auxiliary diffusion plays a central role in the two algorithms.

Address for correspondance: Michael Sørensen, Dept. of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark.
E-mail: michael@math.ku.dk

We correct Theorem 1 in Bladt *et al.* (2016) that gives the distribution of the approximate diffusion bridges proposed in the paper. In Theorem 1 the initial distribution of the auxiliary diffusion is incorrect.

First we briefly describe the model and the processes that Theorem 1 is concerned with. Let $X = \{X_t\}_{t\geq 0}$ be a d-dimensional diffusion given by the stochastic differential equation

$$dX_t = \alpha(X_t)dt + \sigma(X_t)dW_t, \tag{1}$$

with $X_0 = A$. Here A is a random variable independent of the d-dimensional Wiener process W, and the coefficients α and σ are sufficiently regular to ensure that the equation has a unique strong solution that is a strong Markov process. We assume that X is ergodic with invariant probability density function ν and that $\sigma(x)$ is invertible for all x in the state space. If a and b are given points in the state space of X, a solution of (1) in the interval $[0,T]$ such that $X_0 = a$ and $X_T = b$ is called an (a,b) diffusion bridge.

Define another diffusion process X' as the solution to

$$dX'_t = \alpha(X'_t)dt + \sigma(X'_t)dW'_t, \quad X'_0 = a \tag{2}$$

with the Wiener process

$$dW'_t = \{I - (1-\gamma)\Pi(X_t, X'_t)\}\, dW_t + \sqrt{1-\gamma^2}u(X_t, X'_t)dU_t.$$

Here $\gamma \in [-1,1)$, U is a univariate standard Wiener process independent of W, I is the d-dimensional identity matrix and

$$\Pi(x,x') = u(x,x')u(x,x')^T,$$

where T denotes transposition, and $u(x,x')$ is the unit vector such that $\sigma(x')u(x,x')$ points in the direction $x - x'$. By Lemma 1 in Bladt *et al.* (2016), the sample path of X in $[0,t]$ is a function of the initial value A, the sample path of X' in $[0,t]$, and the sample path in $[0,t]$ of a standard univariate Wiener process, U', independent of X':

$$\{X_s\}_{0\le s\le t} = \mathcal{K}'_t\left(A, \{X'_s\}_{0\le s\le t}, \{U'_s\}_{0\le s\le t}\right). \tag{3}$$

The distribution of the approximate diffusion bridge is given by the following theorem, which replaces the incorrect Theorem 1 in Bladt *et al.* (2016) :

Theorem 1* *Suppose $A \sim \nu$, and define a process by*

$$Z_t = \begin{cases} X'_t & \text{if } \ 0 \le t \le \tau \\ X_t & \text{if } \ \tau < t \le T, \end{cases}$$

where $\tau = \inf\{t > 0 \,|\, X_t = X'_t\}$.

Then the distribution of $\{Z_t\}_{0\le t\le T}$ conditionally on the events $\tau \le T$ and $X_T = b$ equals the distribution of an (a,b) diffusion bridge, B, conditionally on the event that the bridge is hit by the process $\mathcal{K}'_T\left(A, \{B_t\}_{0\le t\le T}, \{U'_t\}_{0\le t\le T}\right)$. Here U' is a standard univariate Wiener process, and A, U' and B are independent.

What has changed relative to Bladt *et al.* (2016) is the initial distribution of the diffusion that hits the diffusion bridge. The approximate diffusion bridge is the process Z conditionally on $\tau \le T$ and $X_T = b$, which can easily be simulated by standard methods. The reason for the error in the original paper was a misinterpretation of a conditional probability. In order to avoid such misinterpretations, we give the proof of the theorem in more detail.

Proof of Theorem 1*. By the strong Markov property Z has the same distribution as X'. Moreover, the joint distribution of (Z,τ) is the same as the joint distribution of (X',τ), because τ depends only on the sample path $\{Z_t\}_{0\le t\le \tau} = \{X'_t\}_{0\le t\le \tau}$ (and of course on $\{X_t\}_{0\le t\le \tau}$). Hence

$$P(Z \in \cdot \,|\, X_T = b, \tau \le T) = P(Z \in \cdot \,|\, Z_T = b, \tau \le T) = P(X' \in \cdot \,|\, X'_T = b, \tau \le T).$$

The event $\{X'_T = b, \tau \leq T\}$ is the event that X' is an (a, b) diffusion bridge and that the diffusion bridge is hit by X, so the theorem follows because of (3). $\square$

We call the process $\mathcal{K}'_T\left(A, \{B_t\}_{0\leq t\leq T}, \{U'_t\}_{0\leq t\leq T}\right)$ the stationary diffusion associated with B. In Bladt *et al.* (2016) phrases like "$p^*_T(b)$-diffusion associated with" must in all cases be changed to "stationary diffusion associated with", and the words "with density function $p^*_T(b, \cdot)$" must be changed to "with the stationary distribution" . In particular, this redefines the probabilities $\pi_T(x)$ and π_T given by (2.20) in Corollary 1, where the random variable A must have the stationary distribution.

With these changes, the results in Sections 2.2 and 2.4 still hold and the alternative MCMC algorithm works. In the pseudo marginal Metropolis-Hastings algorithm in Section 2.3 the definition of the probability $\pi_T(x)$ has been changed as explained above, so the sequence of diffusions $\tilde{X}^{(1)}, \tilde{X}^{(2)}, \ldots$ associated to the sample path x, which defines the geometric random variable $T(x)$, must be a sequence of independent stationary diffusions associated with x. With this change the target distribution is the distribution of an exact diffusion bridge.

It is crucial that values of the stationary distribution can be simulated. We usually need to have an expression for the stationary density to be able to apply the methods of the paper, because this density function appears in the drift of the time-reversed diffusion given by (2.9). If it is difficult to simulate the stationary distribution, a simple approximate method is to simulate the diffusion (2.1) in a long time-interval, and use the value at the end point. As usual it is necessary to be aware of the potential effect of discretisation errors.

Much of the simulation study in Section 3 is concerned with the approximate method and is hence not affected by the error in Theorem 1. However, the results on the probability $\pi_T(x)$, on the Metropolis-Hastings algorithm and on the alternative MCMC algorithm are obviously affected. The results are still of interest, because when the length of the time interval T is sufficiently large, the distribution of a $p^*_T(b)$-diffusion associated with a sample path x is not far from that of a stationary diffusion associated with the same sample path. Note that the discussion of the reason why approximate bridges are almost exact for the Ornstein-Uhlenbeck process when γ is close to one still holds. This is because the arguments do not depend on the distribution of the random variable A.

Section 4 og 5 are unaffected by the error, because in the statistical application only approximate bridges are used, and only approximate bridge simulation is discussed. Finally, the proof of corollary 1 holds if $p^*_T(b, a)$ is replaced by the invariant probability density function $\nu(a)$ in line 5 of the proof.

References

Bladt, M., Finch, S., and Sørensen, M. (2016). Simulation of multivariate diffusion bridges. *J.R. Statist. Soc. B*, **78**, 343–369.