On the moments of some first passage times for exponential families of processes

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Summary:

For curved exponential families of stochastic processes a natural and often studied sequential procedure is to stop observation when a linear combination of the coordinates of the canonical process crosses a prescribed level. Conditions are given which ensure that such first passage times or a function of them have finite moments. Also results about L_p-convergence as the prescribed level tends to infinity are given.

Keywords:

Diffusion-type process; Large deviations; Local characteristics; Markov processes; Semimartingale; Sequential maximum likelihood estimation; Stopping time.

1 Introduction

For curved exponential families of stochastic processes a natural sequential procedure is to stop observation when a linear combination of the coordinates of the canonical process crosses a prescribed level. This type of sequential procedure has been studied by several authors, see e.g. the list of references in Sørensen (1986). For a general definition of and basic results on exponential families of stochastic processes, see Küchler and Sørensen (1989, 1994).

Sequential procedures of the type described above have likelihood functions which, exactly or approximately, are of the non-curved exponential type. This implies, as is well-known, several nice statistical properties. An interpretation of these stopping rules is that observation is continued until a prescribed amount of observed information has been obtained about a certain one-dimensional parameter function. General results on this type of sequential procedure were given by Sørensen (1986, 1994) and Stefanov (1986a,b, 1993).

If exponential families obtained by sequential sampling are non-curved, the maximum likelihood estimator for the mean value parameter is efficient in the sense that there is equality in the Cramér-Rao inequality. This has in much of the literature been a main motivation for studying stopping rules of the type described above. Since efficient estimators do not exist for curved exponential families (see Theorem 15.4 in Chensov, 1982, p. 219), such stopping rules are the only possibilities to obtain a so-called efficient sequential plan.

Conditions ensuring that a linear combination of the coordinates of the canonical process crosses a given level at a finite time are well-known, see Stefanov (1986b). Höpfner (1987) showed that these stopping times have finite moments for ergodic Markov processes with countable state-space. He also gave conditions ensuring that the stopping times divided by the level converges in L_p to a finite limit as the level tends to infinity. The present paper uses large deviation results for semi-martingales to give conditions ensuring the existence of moments and L_p-convergence for a rather general class of ergodic exponential families of processes.

In Section 2 the type of model and the stopping times considered in this paper are defined, and some preliminary results are given. The main results are presented in Section 3, while exponential families of diffusion-type processes and Markov processes with finite state-space are discussed in Section 4.

2 Preliminaries

Consider a measurable space $(\Omega, \mathcal{F})$ with a class of probability measures $\mathcal{P} = \{\mathcal{P}_\theta \in \Theta\}$, $\Theta \subseteq \mathbb{R}^k$ satisfying $\mathrm{int}\Theta \neq \emptyset$, and with a stochastic process X. Let $\{\mathcal{F}\}$ denote the right-continuous filtration generated by observation of X in the time-interval $[0, t]$, and let P_θ^t denote the restriction of P_θ to $\mathcal{F}_t$. We assume that there exists a probability measure P on $(\Omega, \mathcal{F})$ such that for all $t > 0$ and all $\theta \in \Theta$ we have $P_\theta^t << P^t$ and

$$L_t(\theta) = \frac{dP_\theta^t}{dP^t} = \exp\Big(\theta^T A_t - \varphi(\theta)^T S_t\Big). \tag{2.1}$$

Here $L_t(\theta)$ is the likelihood function corresponding to observation of X in $[0, t]$. We denote transposition of matrices by a T. In (2.1) the stochastic process S and the non-random function φ are $(m-k)-$dimensional, while A is a $k-$dimensional vector process which is right-continuous with limits from the left. We assume, moreover, that every coordinate of S_t is a non-decreasing predictable process satisfying that $S_0 = 0$ and $S_t \to \infty$ as $t \to \infty$.

This type of exponential families of stochastic processes covers several important classes of stochastic process models including exponential families of diffusions, exponential families of counting processes, and Markov processes with finite state-space. Further examples can be found in Sørensen (1986) and Küchler and Sørensen (1994).

We shall study stopping times of the type

$$\tau_u(\alpha, \beta) = \inf\Big\{t > 0 : \alpha^T A_t + \beta^T S_t > u\Big\}, \tag{2.2}$$

where $\alpha \in \mathbb{R}^k$, $\beta \in \mathbb{R}^{m-k}$ and $u > 0$. Sometimes, when there is no ambiguity, we will just write τ_u.

We shall work under the following condition.

Condition A(θ).

(i) The score vector

$$U_t(\theta) = A_t - \dot{\varphi}(\theta)^T S_t, \tag{2.3}$$

where $\dot{\varphi}(\theta) = \{\partial\varphi_i(\theta)/\partial\theta_j\}$, is a square integrable $P_\theta-$martingale and the (i,j)'th element of its quadratic characteristic is

$$< U(\theta) > \begin{matrix}(i,j)\\ t\end{matrix} = -\frac{\partial^2}{\partial\theta_i\partial\theta_j}\varphi(\theta)^T S_t. \tag{2.4}$$

(ii) Every coordinate of $\dot{\varphi}(\theta)\alpha + \beta$ is positive.

(iii) There exists a strictly increasing function f and a constant vector $c(\theta) \neq 0$ such that $f(0) = 0$ and $S_t/f(t) \to c(\theta)$ in $P_\theta-$probability as $t \to \infty$.

□

A condition implying condition $A(\theta)$ (i) is that $0 \in \mathrm{int}M(\theta, t)$ for all $t > 0$ and $\theta \in \mathrm{int}\Theta$, where $M(\theta, t)$ denotes the domain of the Laplace transform of S_t under P_θ, see Corollary 6.3 in Küchler and Sørensen (1992).

We further assume that we can find an exponential representation, where $A_t = (A_t^c, A_t^d)^T$ with A_t^c k_1-dimensional, and where the corresponding decomposition of the score vector $U_t(\theta) = (U_t^c(\theta), U_t^d(\theta))$ has the property that $U_t^c(\theta)$ is a continuous martingale, whereas $U_t^d(\theta)$ is a purely discontinuous martingale. This assumption essentially means that the diffusion and the jump mechanism of the process are parameterized separately. It implies that the quadratic characteristic of the martingale

$$M_t = \alpha^T U_t(\theta) \tag{2.5}$$

has the form

$$< M >_t = \pi^c(\alpha)^T S_t + \pi^d(\alpha)^T S_t \tag{2.6}$$

where the vectors $\pi^c(d)$ and $\pi^d(\alpha)$ are given by

$$\pi^c(\alpha) = -\sum_{i,j=1}^{k_1} \alpha_i \alpha_j \frac{\partial^2}{\partial\theta_i \partial\theta_j} \varphi(\theta), \tag{2.7}$$

$$\pi^d(\alpha) = -\sum_{i,j=k_1+1}^{k} \alpha_i \alpha_j \frac{\partial^2}{\partial\theta_i \partial\theta_j} \varphi(\theta). \tag{2.8}$$

In (2.6), $\pi^c(\alpha)^T S_t$ is the quadratic characteristic of the continuous martingale part of M, while $\pi^d(\alpha)^T S_t$ is the quadratic characteristic of the purely discontinuous part of M.

Under Condition $A(\theta)$ the following results about $\tau_u(\alpha, \beta)$ hold.

Lemma 2.1

$$\left(\alpha^T A_t + \beta^T S_t\right) / f(t) \to (\dot{\varphi}(\theta)\alpha + \beta)^T c(\theta) \tag{2.9}$$

in P_θ–probability, and

$$P_\theta(\tau_u(\alpha, \beta) < \infty) = 1. \tag{2.10}$$

Proof:

Since

$$\alpha^T A_t + \beta^T S_t = M_t + (\dot{\varphi}(\theta)\alpha + \beta)^T S_t,$$

it follows by the law of large numbers for martingales (Liptser, 1980) that

$$\left(\alpha^T A_t + \beta^T S_t\right) / f(t) \to (\dot{\varphi}(\theta)\alpha + \beta)^T c(\theta) > 0$$

in $P_\theta-$probability. This proves the lemma. □

This result generalizes a result by Stefanov (1986b). In the next lemma a_+ denotes the positive part of a.

Lemma 2.2. Suppose

$$\Big[\alpha^T (A_t - A_{t-})\Big]_+ / f(t) \to 0 \tag{2.11}$$

in P_θ−probability as $t \to \infty$.

Then

$$u^{-1} f(\tau_u(\alpha,\beta)) \to \Big[(\dot{\varphi}(\theta)\alpha + \beta)^T c(\theta)\Big]^{-1} \tag{2.12}$$

in P_θ−probability as $u \to \infty$.

Proof: The result follows by noting that

$$\frac{f(\tau_u)}{u} = \frac{f(\tau_u)}{\alpha^T A_{\tau_u} + \beta^T S_{\tau_u} - D_{\tau_u}},$$

where $D_{\tau_u} = u - \alpha^T A_{\tau_u} - \beta^T S_{\tau_u}$, and that (2.11) implies $D_{\tau_u}/S_{\tau_u} \to 0$ in P_θ−probability. Now Lemma 2.1 can be applied since $\tau_u \uparrow \infty$ as $u \to \infty$. □

We shall impose the following condition on the martingale M.

Condition M(θ)**.**

The martingale $M = \alpha^T U(\theta)$ is quasi-left-continuous and its jump characteristic ν under P_θ has the form

$$\nu(\omega, dt, dx) = \sum_{i=1}^{m-k} K_i(dx) dS_t^{(i)}, \tag{2.13}$$

where K_i is a non-random measure satisfying that

$$\int x^2 K_i(dx) < \infty, \qquad i = 1, ..., m-k. \tag{2.14}$$

□

Under Condition M(θ) the quadratic characteristic of the purely discontinous martingale part of M is

$$\int_0^t \int_{\mathbb{R}\setminus\{0\}} x^2 \nu(ds, dx) = v(\alpha)^T S_t, \tag{2.15}$$

where

$$v_i(\alpha) = \int_{\mathbb{R}\setminus\{0\}} x^2 K_i(dx),$$

which is in accordance with (2.6). A condition implying (2.13) is $\nu(\omega, dt, dx) = K(dx) m(\omega, dt)$, where K satisfies (2.14). Note that (2.14) implies that

$$\int_0^t \int_{|x|>1} |x| \nu(ds, dx) < \infty, \qquad t > 0.$$

3 Results

In this section we give results about moments of $f(\tau_u)$ and L_p-convergence of $u^{-1}f(\tau_u)$.

Theorem 3.1. Suppose Condition A(θ) and Condition M(θ) are satisfied. Assume further the Cramér condition that there exists $\lambda_0 > 0$ such that

$$\int_{|x|>1} e^{\lambda x} K_i(dx) < \infty, \qquad i = 1, ..., m-k \text{ for } \lambda \in (0, \lambda_0].$$

Then for $g(t) = f^{-1}(\kappa t)$, where $\kappa > [(\dot{\varphi}(\theta)\alpha + \beta)^T c]^{-1}$,

$$P_\theta(\tau_u > g(u)) \le \inf_{\substack{\lambda \in \Lambda_a \\ q > 1}} \exp\left[q^{-1}(q-1)\lambda u\right]\left\{E_\theta\left(\exp\left[(q-1)a(\lambda)^T S_{g(u)}\right]\right)\right\}^{1/q}, \tag{3.1}$$

where

$$a(\lambda) = \frac{1}{2}\lambda^2\pi^c(\alpha) + h(\lambda) - \lambda(\dot{\varphi}(\theta)\alpha + \beta) \tag{3.2}$$

with

$$h_i(\lambda) = \int_{\mathbb{R}\setminus\{0\}} \left(e^{\lambda x} - 1 - \lambda x\right)K_i(dx)$$

and with $\pi^c(\alpha)$ given by (2.7). The set

$$\Lambda_a = \{\lambda \in (0, \lambda_0] : a(\lambda) < 0\} \tag{3.3}$$

is non-empty. By $a(\lambda) < 0$ we mean that every coordinate of $a(\lambda)$ is negative. □

Remark: It is obvious from the proof that (3.1) holds for any positive and strictly increasing function g satisfying that $f(t) - [(\dot{\varphi}(\theta)\alpha + \beta)^T c]^{-1} g^{-1}(t)$ is a positive increasing function. □

Proof:

$$P_\theta(\tau_u > g(u)) \le P_\theta\left(K_{g(u)} < u\right),$$

where

$$K_t = \alpha^T A_t + \beta^T S_t = M_t - \tilde{B}_t + f(t)(\dot{\varphi}(\theta)\alpha + \beta)^T c$$

with

$$\tilde{B}_t = (\dot{\varphi}(\theta)\alpha + \beta)^T (cf(t) - S_t).$$

We will therefore apply the large deviation result Liptser and Shiryaev (1989, Theorem 4.13.3) to the probability

$$P(K_t < \tilde{g}(t)) = P\left(X_t > (\dot{\varphi}(\theta)\alpha + \beta)^T cf(t) - \tilde{g}(t)\right)$$

where $X = \tilde{B} - M$. In order to do this, we need the local characteristics of the semimartingale X under P_θ. The jump characteristic and the continuous martingale characteristic of X equal those of M, i.e. ν and $\pi^c(\alpha)^T S_t$, respectively. The drift characteristic of X is $\tilde{B}$ minus the compensation of the jumps numerically larger than one, so our $\tilde{B}$ equals the process denoted in the same way in Liptser and Shiryaev's theorem. Now for $\lambda \in (0, \lambda_0]$ define

$$G_t(\lambda) = \lambda \tilde{B}_t + \frac{1}{2}\lambda^2 \pi^c(\alpha)^T S_t + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \Big(e^{\lambda x} - 1 - \lambda x\Big)\nu(ds, dx).$$

Under Condition M(θ)

$$\int_0^t \int_{\mathbb{R}\setminus\{0\}} \Big(e^{\lambda x} - 1 - \lambda x\Big)\nu(ds, dx) = h(\alpha)^T S_t.$$

Thus

$$G_t(\lambda) = \lambda(\dot{\varphi}(\theta)\alpha + \beta)^T c f(t) + a(\lambda)^T S_t.$$

Let us next show that the set Λ_a is non-empty. First note that

$$h_i'(\lambda) = \int_{\mathbb{R}\setminus\{0\}} x\Big(e^{\lambda x} - 1\Big)K_i(dx)$$

when $\lambda \in (0, \lambda_0)$. Hence $h_i'(0+) = 0$ and $a'(0+) = -(\dot{\varphi}(\theta)\alpha + \beta)$. Now, since $a(0) = 0$ and every coordinate of $a'(0+)$ is negative, we can find $\bar{\lambda} \in (0, \lambda_0)$ such that every coordinate of $a(\bar{\lambda})$ is negative.

We have now checked the conditions in Liptser and Shiryaev's theorem, and their formula (4.13.26) yields that for every positive and non-decreasing function ϕ_t

$$P(X_t > \delta\phi_t) \leq \Big\{E_\theta\Big(\exp\big[-(q-1)\phi_t\big(\lambda\delta - \phi_t^{-1}G_t(\lambda)\big)\big]\Big)\Big\}^{1/q}$$

for every $\lambda \in \Lambda_a$, $q > 1$ and $\delta > 0$. By choosing $\delta = (\dot{\varphi}(\theta)\alpha + \beta)^T c$ and $\phi_t = f(t) - \delta^{-1}\tilde{g}(t)$ we obtain

$$P_\theta(X_t > \delta\phi_t) \leq \exp\big[q^{-1}(q-1)\lambda\tilde{g}(t)\big]\Big\{E_\theta\Big(\exp\Big[(q-1)a(\lambda)^T S_t\Big]\Big)\Big\}^{1/q},$$

provided $\tilde{g}$ is a function such that $f - \delta^{-1}\tilde{g}$ is positive and non-decreasing.

For $t = g(u)$ and $\tilde{g}(t) = g^{-1}(t) = \kappa^{-1}f(t)$, we get that $\phi_t = f(t)(1 - (\delta\kappa)^{-1}) > 0$ and

$$P_\theta\big(K_{g(u)} < u\big) \leq \exp\big[q^{-1}(q-1)\lambda u\big]\Big\{E_\theta\Big(\exp\Big[(q-1)a(\lambda)^T S_{g(u)}\Big]\Big)\Big\}^{1/q},$$

which proves the theorem. □

With the notation

$$\psi_{\lambda,q}(t) = \Big\{E_\theta\Big(\exp\Big[(q-1)a(\lambda)^T S_t\Big]\Big)\Big\}^{1/q}$$

we have the following result.

Theorem 3.2. Suppose Condition A(θ) and Condition M(θ) are satisfied and that there exist $q > 1$ and $\lambda \in \Lambda_a$ such that

$$\int_0^\infty \psi_{\lambda,q}\big(f^{-1}(x)\big)x^{p-1}dx < \infty$$

for some $p > 0$. Then the p'th moment of $f(\tau_u)$ under P_θ is finite. □

Proof: The result follows because

$$E_\theta\Big(\big(u^{-1}f(\tau_u)\big)^p\Big) = \int_0^\infty P_\theta\Big(\tau_u > f^{-1}\Big(ux^{1/p}\Big)\Big)dx$$

and because the integrand for x sufficiently large is dominated by

$$\exp\big[q^{-1}(q-1)\lambda u\big]\psi_{\lambda,q}\Big(f^{-1}\Big(ux^{1/p}\Big)\Big),$$

which has been assumed integrable. □

Theorem 3.3. Suppose Condition A(θ) and Condition M(θ) are satisfied and that there exist $q > 1$ and $\lambda \in \Lambda_a$ such that

$$\psi_{\lambda,q}\big(f^{-1}(x)\big) = O\Big(e^{-bx}x^\rho\Big) \text{ as } x \to \infty,$$

where $b > 0$ and $\rho \geq 0$. Then the family of random variables $\{(u^{-1}f(\tau_u))^p : u \geq \epsilon > 0\}$ is uniformly integrable under P_θ for every $p > 0$ and $\epsilon > 0$. Under the conditions of Lemma 2.2

$$u^{-1}f(\tau_u) \to \Big[(\dot{\varphi}(\theta_0)\alpha + \beta)^T c(\theta)\Big]^{-1}$$

in L_p $(p > 0)$ as $u \to \infty$ under P_θ. □

Proof: With $\mu = q^{-1}(q-1)\lambda$ and with the constant k_1 suitably chosen we have for c sufficiently large that

$$
\begin{aligned}
&E_\theta\Big[\big(u^{-1}f(\tau_u)\big)^p 1_{\{(u^{-1}f(\tau_u))^p>c\}}\Big] \\
&= cP_\theta\Big[\tau_u > f^{-1}\Big(uc^{1/p}\Big)\Big] + \int_c^\infty P_\theta\Big[\tau_u > f^{-1}\Big(uy^{1/p}\Big)\Big]dy \\
&\le ce^{\mu u}\psi_{\lambda,q}\Big(f^{-1}\Big(uc^{1/p}\Big)\Big) + e^{\mu u}\int_c^\infty \psi_{\lambda,q}\Big(f^{-1}\Big(uy^{1/p}\Big)\Big)dy \\
&\le ce^{\mu u}\psi_{\lambda,q}\Big(f^{-1}\Big(uc^{1/p}\Big)\Big) + e^{\mu u}u^{-p}p\int_{uc^{1/p}}^\infty \psi\big(f^{-1}(z)\big)z^{p-1}dz \\
&\le k_1u^\rho c^{\rho/p+1}e^{-u\left(bc^{1/p}-\mu\right)} + k_1e^{\mu u}u^{-p}p\int_{uc^{1/p}}^\infty e^{-bz}z^{\rho+p-1}dz \\
&\le k_1u^\rho c^{\rho/p+1}e^{-u\left(bc^{1/p}-\mu\right)}\left\{1+p\int_0^\infty e^{-buc^{1/p}y}(y+1)^{p+\rho-1}dy\right\} \\
&\le k_1\epsilon^\rho c^{\rho/p+1}e^{-\epsilon\left(bc^{1/p}-\mu\right)}\left\{1+p\int_0^\infty e^{-b\epsilon c^{1/p}y}(y+1)^{p+\rho-1}dy\right\},
\end{aligned}
$$

for $c > ((\rho\epsilon^{-1}+\mu)/b)^p$. This expression goes to zero as $c \to \infty$, which proves the first claim. The result on L_p-convergence follows in view of Lemma 2.2 . □

4 Examples

4.1 Diffusion processes

Consider the class of stochastic differential equations

$$
dX_t = [a_t(X) + \theta b_t(X)]dt + c_t(X)dW_t, \quad t > 0, \tag{4.1}
$$

$\theta \in \Theta \subseteq \mathbb{R}$, with initial condition $X_0 = x_0$. All quantities are one-dimensional, $c > 0$, and the functionals a_t, b_t and c_t depend on X only through $\{X_s : s \le t\}$. As usual W is a standard Wiener process. We assume that a, b and c satisfy conditions ensuring that (4.1) has a unique weak solution for all $\theta \in \Theta$. Examples of models of this kind are the Brownian motion with drift, the Ornstein-Uhlenbeck processes, the Bessel processes and the non-Markov processes studied by Küchler and Mensch (1992).

If X is observed in the time-interval $[0,t]$, the likelihood function is, under suitable regularity conditions (see e.g. Liptser and Shiryaev, 1977), given by

$$
L_t(\theta) = \exp\left(\theta A_t - \frac{1}{2}\theta^2 S_t\right),
$$

where

$$S_t = \int_0^t b_s(X)^2 c_s(X)^{-2} ds$$

and

$$A_t = \int_0^t b_s(X) c_s(X)^{-2} d\tilde{X}_s$$

with

$$\tilde{X}_t = X_t - x_0 - \int_0^t a_s(X) ds.$$

This model is of the type considered in this paper.

Since a solution of (4.1) does not jump, condition $M(\theta)$ and the Cramér condition are automatically satisfied. Thus we need only assume Condition $A(\theta)$ to obtain that

$$P_\theta(\tau_u > g(u)) \leq \inf_{\substack{\lambda \in \Lambda_a \\ q > 1}} \exp\left[q^{-1}(q-1)\lambda u\right] \psi_{\lambda,q}(g(u)). \tag{4.2}$$

For diffusion-type processes $a(\lambda)$ has the simple form

$$a(\lambda) = \frac{1}{2}\lambda^2\alpha^2 - \lambda(\alpha\theta + \beta),$$

and

$$\Lambda_a = \big(0, 2\alpha^{-2}(\alpha\theta + \beta)\big).$$

For the particular case of the Ornstein-Uhlenbeck process

$$dX_t = \theta X_t dt + dW_t, \quad X_0 = x_0,$$

condition $A(\theta)$ is satisfied with $f(t) = t$ and $c(\theta) = -(2\theta)^{-1}$ when $\theta < 0$. From results in Küchler and Sørensen (1992) it follows that

$$\psi_{\lambda,q}(t) = O\Bigg(\exp\left[-\frac{t}{2q}\left\{\sqrt{\theta^2 - 2(q-1)a(\lambda)} + \theta\right\}\right]\Bigg)$$

for all $q < 1$ and $\lambda \in \Lambda_a$, so the conditions of Theorem 3.3 are satisfied. Hence $\{(u^{-1}\tau_u)^p : u \geq \epsilon > 0\}$ is uniformly integrable for every $p > 0$ and $\epsilon > 0$, and

$$u^{-1}\tau_u \to -2\theta(\alpha\theta + \beta)^{-1}$$

in $L_p(P_\theta)$ as $u \to \infty$ for every $p > 0$.

4.2 Finite state-space Markov processes

A continuous time Markov process with finite state-space $\{1, ..., m\}$ and intensity matrix $\{\lambda_{ij}\}$, where $\lambda_{ij} > 0$ for $i \neq j$, has, when observed in $[0, t]$, the likelihood function

$$L_t(\theta) = \exp\left\{\sum_{i=1}^{m}\left[\sum_{j\neq i}\theta_{ij}K_t^{(i,j)} - \left(\sum_{j\neq i}e^{\theta_{ij}} + 1 - m\right)S_t^{(i)}\right]\right\}, \qquad (4.3)$$

provided the initial state $X_0 = x_0$ is fixed. Here $\theta_{ij} = \log\lambda_{ij}$, the process $K_t^{(i,j)}$ is the number of transitions from state i to state j in the time interval $[0, t]$, and $S_t^{(i)}$ is the time the process has spent in state i before t. The assumption that $\lambda_{ij} > 0$ for all $i \neq j$ is only made to simplify the exposition.

We consider stopping times of the form

$$\tau_u = \inf\left\{t > 0 : \sum_{i=1}^{m}\left(\beta_i S_t^{(i)} + \sum_{j\neq i}\alpha_{ij}K_t^{(i,j)}\right) > u\right\}. \qquad (4.4)$$

The score vector is a square integrable $P_\theta-$martingale with quadratic characteristic (2.4) and $A(\theta)$, (iii) is satisfied with $f(t) = t$ and $c_i(\theta) = \lambda_{i.}(\theta)^{-1}\left[\sum_j \lambda_{j.}(\theta)^{-1}\right]^{-1}$, where $\lambda_{i.}(\theta) = \sum_{j\neq i}\exp(\theta_{ij})$. The jump characteristic of M has the form (2.12) with

$$K_i(dx) = \sum_{j\neq i}\alpha_{ij}e^{\theta_{ij}}\delta_{\{j\}}(dx),$$

where $\delta_{\{j\}}$ is the Dirac-measure at j. The Cramér condition is obviously satisfied. Hence by Theorem 3.1

$$P_\theta(\tau_u > u) \leq \inf_{\substack{\lambda \in \Lambda_u \\ q > 1}} \exp\left[q^{-1}(q-1)\lambda u\right]\psi_{\lambda,q}(u).$$

For finite state-space Markov processes

$$a_i(\lambda) = \sum_{j\neq i}\alpha_{ij}e^{\theta_{ij}}\left(e^{j\lambda} - 1 - (j+1)\lambda\right) - \lambda\beta_i.$$

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