LARGE DEVIATIONS FOR SOLUTIONS TO STOCHASTIC RECURRENCE EQUATIONS UNDER KESTEN'S CONDITION

D. BURACZEWSKI, E. DAMEK, T. MIKOSCH AND J. ZIENKIEWICZ

ABSTRACT. In this paper we prove large deviations results for partial sums constructed from the solution to a stochastic recurrence equation. We assume Kesten's condition [17] under which the solution of the stochastic recurrence equation has a marginal distribution with power law tails, while the noise sequence of the equations can have light tails. The results of the paper are analogs of those obtained by A.V. and S.V. Nagaev [21, 22] in the case of partial sums of iid random variables. In the latter case, the large deviation probabilities of the partial sums are essentially determined by the largest step size of the partial sum. For the solution to a stochastic recurrence equation, the magnitude of the large deviation probabilities is again given by the tail of the maximum summand, but the exact asymptotic tail behavior is also influenced by clusters of extreme values, due to dependencies in the sequence. We apply the large deviation results to study the asymptotic behavior of the ruin probabilities in the model.

1. INTRODUCTION

Through the last 40 years, the stochastic recurrence equation

$$Y_n = A_n\, Y_{n-1} + B_n\,, \qquad n \in \mathbb{Z}\,, \tag{1.1}$$

and its stationary solution have attracted much attention. Here (A_i, B_i), $i \in \mathbb{Z}$, is an iid sequence, $A_i > 0$ a.s. and B_i assumes real values. (In what follows, we write $A, B, Y, \ldots$, for generic elements of the strictly stationary sequences (A_i), (B_i), $(Y_i), \ldots$, and we also write c for any positive constant whose value is not of interest.)

It is well known that if $\mathbb{E} \log A < 0$ and $\mathbb{E} \log^+ |B| < \infty$, there exists a unique strictly stationary ergodic solution (Y_i) to the stochastic recurrence equation (1.1) with representation

$$Y_n = \sum_{i=-\infty}^{n} A_{i+1} \cdots A_n\, B_i\,, \quad n \in \mathbb{Z}\,,$$

where, as usual, we interpret the summand for $i = n$ as B_n.

One of the most interesting results for the stationary solution (Y_i) to the stochastic recurrence equation (1.1) was discovered by Kesten [17]. He proved under general conditions that the marginal distributions of (Y_i) have power law tails. For later use, we formulate a version of this result due to Goldie [10].

Theorem 1.1. (Kesten [17], Goldie [10]) *Assume that the following conditions hold:*

2000 *Mathematics Subject Classification.* Primary 60F10; secondary 91B30, 60G70.

Key words and phrases. Stochastic recurrence equation, large deviations, ruin probability.

D. Buraczewski and E. Damek were partially supported by the NCN grant N N201 392337. Thomas Mikosch's research is partly supported by the Danish Natural Science Research Council (FNU) Grants 09-072331 "Point process modelling and statistical inference" and 10-084172 "Heavy tail phenomena: Modeling and estimation". J. Zienkiewicz was supported by the MNiSW grant N N201 397137. D. Buraczewski was also supported by the European Commission via IEF Project (contract number PIEF-GA-2009-252318 - SCHREC).

- *There exists $\alpha > 0$ such that*

$$\mathbb{E}A^\alpha = 1\,. \tag{1.2}$$

- *$\rho = \mathbb{E}(A^\alpha \log A)$ and $\mathbb{E}|B|^\alpha$ are both finite.*
- *The law of $\log A$ is non-arithmetic.*
- *For every x, $\mathbb{P}\{Ax + B = x\} < 1$.*

Then Y is regularly varying with index $\alpha > 0$. In particular, there exist constants $c_\infty^+, c_\infty^- \ge 0$ such that $c_\infty^+ + c_\infty^- > 0$ and

$$\mathbb{P}\{Y > x\} \sim c_\infty^+\, x^{-\alpha}\,, \quad \text{and} \quad \mathbb{P}\{Y \le -x\} \sim c_\infty^-\, x^{-\alpha} \quad \text{as } x \to \infty\,. \tag{1.3}$$

Moreover, if $B \equiv 1$ a.s. then the constant c_∞^+ takes on the form

$$c_\infty := \mathbb{E}[(1+Y)^\alpha - Y^\alpha]/(\alpha\rho)\,,$$

Goldie [10] also showed that similar results remain valid for the stationary solution to stochastic recurrence equations of the type $Y_n = f(Y_{n-1}, A_n, B_n)$ for suitable functions f satisfying some contraction condition.

The power law tails (1.3) stimulated research on the extremes of the sequence (Y_i) . Indeed, if (Y_i) were iid with tail (1.3) and $c_\infty^+ > 0$, then the maximum sequence $M_n = \max(Y_1, \dots, Y_n)$ would satisfy the limit relation

$$\lim_{n\to\infty} \mathbb{P}\{(c_\infty^+\, n)^{-1/\alpha} M_n \le x\} = e^{-x^{-\alpha}} = \Phi_\alpha(x)\,, \quad x > 0\,, \tag{1.4}$$

where Φ_α denotes the Fréchet distribution, i.e. one of the classical extreme value distributions; see Gnedenko [11]; cf. Embrechts et al. [6], Chapter 3. However, the stationary solution (Y_i) to (1.1) is not iid and therefore one needs to modify (1.4) as follows: the limit has to be replaced by Φ_α^θ for some constant $\theta \in (0,1)$, the so-called *extremal index* of the sequence (Y_i); see de Haan et al. [12]; cf. [6], Section 8.4.

The main objective of this paper is to derive another result which is a consequence of the power law tails of the marginal distribution of the sequence (Y_i): we will prove large deviation results for the partial sum sequence

$$\mathcal{S}_n = Y_1 + \cdots + Y_n\,, \quad n \ge 1\,, \quad \mathcal{S}_0 = 0\,.$$

This means we will derive exact asymptotic results for the left and right tails of the partial sums $\mathcal{S}_n$. Since we want to compare these results with those for an iid sequence we recall the corresponding classical results due to A.V. and S.V. Nagaev [21, 22] and Cline and Hsing [3].

Theorem 1.2. *Assume that (Y_i) is an iid sequence with a regularly varying distribution, i.e. there exists an $\alpha > 0$, constants $p, q \ge 0$ with $p + q = 1$ and a slowly varying function L such that*

$$\mathbb{P}\{Y > x\} \sim p\,\frac{L(x)}{x^\alpha} \quad \text{and} \quad \mathbb{P}\{Y \le -x\} \sim q\,\frac{L(x)}{x^\alpha} \quad \text{as } x \to \infty. \tag{1.5}$$

Then the following relations hold for $\alpha > 1$ and suitable sequences $b_n \uparrow \infty$:

$$\lim_{n\to\infty} \sup_{x \ge b_n} \left| \frac{\mathbb{P}\{\mathcal{S}_n - \mathbb{E}\mathcal{S}_n > x\}}{n\,\mathbb{P}\{|Y| > x\}} - p \right| = 0 \tag{1.6}$$

and

$$\lim_{n\to\infty} \sup_{x \ge b_n} \left| \frac{\mathbb{P}\{\mathcal{S}_n - \mathbb{E}\mathcal{S}_n \le -x\}}{n\,\mathbb{P}\{|Y| > x\}} - q \right| = 0\,. \tag{1.7}$$

If $\alpha > 2$ one can choose $b_n = \sqrt{an\log n}$, where $a > \alpha - 2$, and for $\alpha \in (1,2]$, $b_n = n^{\delta + 1/\alpha}$ for any $\delta > 0$.

For $\alpha \in (0,1]$, *(1.6) and (1.7) remain valid if the centering* $\mathbb{E}\mathcal{S}_n$ *is replaced by* 0 *and* $b_n = n^{\delta+1/\alpha}$ *for any* $\delta > 0$.

For $\alpha \in (0,2]$ one can choose a smaller bound b_n if one knows the slowly varying function L appearing in (1.5). A functional version of Theorem 1.2 with multivariate regularly varying summands was proved in Hult et al. [13] and the results were used to prove asymptotic results about multivariate ruin probabilities. Large deviation results for iid heavy-tailed summands are also known when the distribution of the summands is subexponential, including the case of regularly varying tails; see the recent paper by Denisov et al. [5] and the references therein. In this case, the regions where the large deviations hold very much depend on the decay rate of the tails of the summands. For semi-exponential tails (such as for the log-normal and the heavy-tailed Weibull distributions) the large deviation regions (b_n,∞) are much smaller than those for summands with regularly varying tails. In particular, $x = n$ is not necessarily contained in (b_n,∞).

The aim of this paper is to study large deviation probabilities for a particular dependent sequence (Y_n) as described in Kesten's Theorem 1.1. For dependent sequences (Y_n) much less is known about the large deviation probabilities for the partial sum process $(\mathcal{S}_n)$. Gantert [8] proved large deviation results of logarithmic type for mixing subexponential random variables. Davis and Hsing [4] and Jakubowski [14, 15] proved large deviation results of the following type: there exist sequences $s_n \to \infty$ such that

$$\frac{\mathbb{P}\{\mathcal{S}_n > a_n s_n\}}{n\,\mathbb{P}\{Y > a_n\, s_n\}} \to c_\alpha$$

for suitable positive constants c_α under the assumptions that Y is regularly varying with index $\alpha \in (0,2)$, $n\,P(|Y| > a_n) \to 1$ and (Y_n) satisfies some mixing conditions. Both Davis and Hsing [4] and Jakubowski [14, 15] could not specify the rate at which the sequence (s_n) grows to infinity, and an extension to $\alpha > 2$ was not possible. These facts limit the applicability of these results, for example for deriving the asymptotics of ruin probabilities for the random walk $(\mathcal{S}_n)$. Large deviations results for particular stationary sequences (Y_n) with regularly varying finite-dimensional distributions were proved in Mikosch and Samorodnitsky [19] in the case of linear processes with iid regularly varying noise and in Konstantinides and Mikosch [18] for solutions (Y_n) to the stochastic recurrence equation (1.1), where B is regularly varying with index $\alpha > 1$ and $\mathbb{E}A^\alpha < 1$. This means that Kesten's condition (1.2) is not satisfied in this case and the regular variation of (Y_n) is due to the regular variation of B. For these processes, large deviation results and ruin bounds are easier to derive by applying the "heavy-tail large deviation heuristics": a large value of $\mathcal{S}_n$ happens in the most likely way, namely it is due to one very large value in the underlying regularly varying noise sequence, and the particular dependence structure of the sequence (Y_n) determines the clustering behavior of the large values of $\mathcal{S}_n$. This intuition fails when one deals with the partial sums $\mathcal{S}_n$ under the conditions of Kesten's Theorem 1.1: here a large value of $\mathcal{S}_n$ is not due to a single large value of the B_n's or A_n's but to large values of the products $A_1 \cdots A_n$.

The paper is organized as follows. In Section 2 we prove an analog to Theorem 1.2 for the partial sum sequence $(\mathcal{S}_n)$ constructed from the solution to the stochastic recurrence equation (1.1) under the conditions of Kesten's Theorem 1.1. The proof of this result is rather technical: it is given in Section 3 where we split the proof into a series of auxiliary results. There we treat the different cases $\alpha \le 1$, $\alpha \in (1,2]$ and $\alpha > 2$ by different tools and methods. In particular, we will use exponential tail inequalities which are suited for the three distinct situations. In contrast to the iid situation described in Theorem 1.2, we will show that the x-region where the large deviations hold cannot be chosen as an infinite interval (b_n,∞) for a suitable lower bound $b_n \to \infty$, but one also needs upper bounds $c_n \ge b_n$. In Section 4 we apply the large deviation results to get precise asymptotic bounds for the ruin probability related to the random walk $(\mathcal{S}_n)$. This ruin bound is an analog of the celebrated result by Embrechts and Veraverbeke [7] in the case of a random walk with iid step sizes.

2. Main result

The following is the main result of this paper. It is an analog of the well known large deviation result of Theorem 1.2.

Theorem 2.1. *Assume that the conditions of Theorem 1.1 are satisfied and additionally there exists $\varepsilon > 0$ such that $\mathbb{E}A^{\alpha+\varepsilon}$ and $\mathbb{E}|B|^{\alpha+\varepsilon}$ are finite. Then the following relations hold:*

(1) *For $\alpha \in (0,2]$, $M > 2$,*

$$\sup_n \sup_{n^{1/\alpha}(\log n)^M \le x} \frac{\mathbb{P}\{\mathcal{S}_n - d_n > x\}}{n\,\mathbb{P}\{|Y| > x\}} < \infty\,, \tag{2.1}$$

If additionally $e^{s_n} \ge n^{1/\alpha}(\log n)^M$ and $\lim_{n\to\infty} s_n/n = 0$ then

$$\lim_{n\to\infty} \sup_{n^{1/\alpha}(\log n)^M \le x \le e^{s_n}} \left| \frac{\mathbb{P}\{\mathcal{S}_n - d_n > x\}}{n\,\mathbb{P}\{|Y| > x\}} - \frac{c_\infty^+\, c_\infty}{c_\infty^+ + c_\infty^-} \right| = 0\,, \tag{2.2}$$

where $d_n = 0$ or $d_n = \mathbb{E}\mathcal{S}_n$ according as $\alpha \in (0,1]$ or $\alpha \in (1,2]$.

(2) *For $\alpha > 2$ and any $c_n \to \infty$,*

$$\sup_n \sup_{c_n\, n^{0.5}\log n \le x} \frac{\mathbb{P}\{\mathcal{S}_n - \mathbb{E}\mathcal{S}_n > x\}}{n\,\mathbb{P}\{|Y| > x\}} < \infty\,. \tag{2.3}$$

If additionally $c_n n^{0.5}\log n \le e^{s_n}$ and $\lim_{n\to\infty} s_n/n = 0$ then

$$\lim_{n\to\infty} \sup_{c_n\, n^{0.5}\log n \le x \le e^{s_n}} \left| \frac{\mathbb{P}\{\mathcal{S}_n - \mathbb{E}\mathcal{S}_n > x\}}{n\,\mathbb{P}\{|Y| > x\}} - \frac{c_\infty^+\, c_\infty}{c_\infty^+ + c_\infty^-} \right| = 0\,. \tag{2.4}$$

Clearly, if we exchange the variables B_n by $-B_n$ in the above results we obtain the corresponding asymptotics for the left tail of $\mathcal{S}_n$. For example, for $\alpha > 1$ the following relation holds uniformly for the x-regions indicated above:

$$\lim_{n\to\infty} \frac{\mathbb{P}\{\mathcal{S}_n - n\mathbb{E}Y \le -x\}}{n\,\mathbb{P}\{|Y| > x\}} = \frac{c_\infty^- c_\infty}{c_\infty^+ + c_\infty^-}\,.$$

Remark 2.2. The deviations of Theorem 2.1 from the iid case (see Theorem 1.2) are two-fold. First, the extremal clustering in the sequence (Y_n) manifests in the presence of the additional constants c_∞ and $c_\infty^\pm$. Second, the precise large deviation bounds (2.2) and (2.4) are proved for x-regions bounded from above by a sequence e^{s_n} for some $s_n \to \infty$ with $s_n/n \to 0$. Mikosch and Wintenberger [20] extended Theorem 2.1 to more general classes of stationary sequences (Y_t). In particular, they proved similar results for stationary Markov chains with regularly varying finite-dimensional distributions, satisfying a drift condition. The solution (Y_t) to (1.1) is a special case of this setting if the distributions of A, B satisfy some additional conditions. Mikosch and Wintenberger [20] use a regeneration argument to explain that the large deviation results do not hold uniformly in the unbounded x-regions (b_n, ∞) for suitable sequences (b_n), $b_n \to \infty$.

3. Proof of the main result

3.1. Basic decompositions.

In what follows, it will be convenient to use the following notation

$$\Pi_{ij} = \begin{cases} A_i \cdots A_j & i \le j \\ 1 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \Pi_j = \Pi_{1j}\,,$$

and

$$\widetilde{Y}_i = \Pi_{2i}B_1 + \Pi_{3i}B_2 + \cdots + \Pi_{ii}B_{i-1} + B_i\,, \quad i \geq 1\,.$$

Since $Y_i = \Pi_i Y_0 + \widetilde{Y}_i$ the following decomposition is straightforward:

$$\mathcal{S}_n = Y_0 \sum_{i=1}^n \Pi_i + \sum_{i=1}^n \widetilde{Y}_i =: Y_0\,\eta_n + \widetilde{\mathcal{S}}_n\,, \tag{3.1}$$

where

$$\widetilde{\mathcal{S}}_n = \widetilde{Y}_1 + \cdots + \widetilde{Y}_n \quad \text{and} \quad \eta_n = \Pi_1 + \cdots + \Pi_n\,, \quad n \geq 1\,. \tag{3.2}$$

In view of (3.1) and Lemma 3.1 below it suffices to bound the ratios

$$\frac{\mathbb{P}\{\widetilde{\mathcal{S}}_n - \widetilde{d}_n > x\}}{n\,\mathbb{P}\{|Y| > x\}}$$

uniformly for the considered x-regions, where $\widetilde{d}_n = \mathbb{E}\widetilde{\mathcal{S}}_n$ for $\alpha > 1$ and $\widetilde{d}_n = 0$ for $\alpha \leq 1$.

The proof of the following bound is given at the end of this subsection.

Lemma 3.1. *Let (s_n) be a sequence such that $s_n/n \to 0$. Then for any sequence (b_n) with $b_n \to \infty$ the following relations hold:*

$$\lim_{n\to\infty} \sup_{b_n \leq x \leq e^{s_n}} \frac{\mathbb{P}\{|Y_0|\,\eta_n > x\}}{n\,\mathbb{P}\{|Y| > x\}} = 0 \quad \textit{and} \quad \limsup_{n\to\infty} \sup_{b_n \leq x} \frac{\mathbb{P}\{|Y_0|\,\eta_n > x\}}{n\,\mathbb{P}\{|Y| > x\}} < \infty,$$

Before we further decompose $\widetilde{\mathcal{S}}_n$ we introduce some notation to be used throughout the proof: for any x in the considered large deviation regions,

- $m = [(\log x)^{0.5+\sigma}]$ for some positive number $\sigma < 1/4$, where $[\cdot]$ denotes the integer part.
- $n_0 = [\rho^{-1}\log x]$, where $\rho = \mathbb{E}(A^\alpha \log A)$.
- $n_1 = n_0 - m$ and $n_2 = n_0 + m$
- For $\alpha > 1$, let D be the smallest integer such that $-D\log\mathbb{E}A > \alpha - 1$. Notice that the latter inequality makes sense since $\mathbb{E}A < 1$ due to (1.2) and the convexity of the function $\psi(h) = \mathbb{E}A^h$, $h > 0$.
- For $\alpha \leq 1$, fix some $\beta < \alpha$ and let D be the smallest integer such that $-D\log\mathbb{E}A^\beta > \alpha - \beta$ where, by the same remark as above, $\mathbb{E}A^\beta < 1$.
- Let n_3 be the smallest integer satisfying

$$D\log x \leq n_3\,, \quad x > 1\,. \tag{3.3}$$

Notice that since the function $\Psi(h) = \log\psi(h)$ is convex, putting $\beta = 1$ if $\alpha > 1$, by the choice of D we have $\frac{1}{D} < \frac{\Psi(\alpha)-\Psi(\beta)}{\alpha-\beta} < \Psi'(\alpha) = \rho$, therefore $n_2 < n_3$ if x is sufficiently large.

For fixed n, we change the indices $i \to j = n - i + 1$ and, abusing notation and suppressing the dependence on n, we reuse the notation

$$\widetilde{Y}_j = B_j + \Pi_{jj}B_{j+1} + \cdots + \Pi_{j,n-1}B_n.$$

Writing $n_4 = \min(j + n_3, n)$, we further decompose $\widetilde{Y}_j$:

$$\widetilde{Y}_j = \widetilde{U}_j + \widetilde{W}_j = B_j + \Pi_{jj}B_{j+1} + \cdots + \Pi_{j,n_4-1}B_{n_4} + \widetilde{W}_j. \tag{3.4}$$

Clearly, $\widetilde{W}_j$ vanishes if $j \geq n - n_3$ and therefore the following lemma is nontrivial only for $n > n_3$. The proof is given at the end of this subsection.

Lemma 3.2. *For any small $\delta > 0$, there exists a constant $c > 0$ such that*

$$\mathbb{P}\Big\{\Big|\sum_{j=1}^{n}(\widetilde{W}_j - c_j)\Big| > x\Big\} \le c\, n\, x^{-\alpha-\delta}, \quad x > 1\,, \tag{3.5}$$

where $c_j = 0$ or $c_j = \mathbb{E}\widetilde{W}_j$ according as $\alpha \le 1$ or $\alpha > 1$.

By virtue of (3.5) and (3.4) it suffices to study the probabilities $\mathbb{P}\Big\{\sum_{j=1}^{n}(\widetilde{U}_j - a_j) > x\Big\}$, where $a_j = 0$ for $\alpha \le 1$ and $a_j = \mathbb{E}\widetilde{U}_j$ for $\alpha > 1$.

We further decompose $\widetilde{U}_i$ into

$$\widetilde{U}_i = \widetilde{X}_i + \widetilde{S}_i + \widetilde{Z}_i, \tag{3.6}$$

where for $i \le n - n_3$,

$$\begin{aligned} \widetilde{X}_i &= B_i + \Pi_{ii}B_{i+1} + \cdots + \Pi_{i,i+n_1-2}B_{i+n_1-1}\,,\\ \widetilde{S}_i &= \Pi_{i,i+n_1-1}B_{i+n_1} + \cdots + \Pi_{i,i+n_2-1}B_{i+n_2}\,,\\ \widetilde{Z}_i &= \Pi_{i,i+n_2}B_{i+n_2+1} + \cdots + \Pi_{i,i+n_3-1}B_{i+n_3}\,. \end{aligned} \tag{3.7}$$

For $i > n - n_3$, define $\widetilde{X}_i, \widetilde{S}_i, \widetilde{Z}_i$ as follows: For $n_2 < n - i < n_3$ choose $\widetilde{X}_i, \widetilde{S}_i$ as above and

$$\widetilde{Z}_i = \Pi_{i,i+n_2}B_{i+n_2+1} + \cdots + \Pi_{i,n-1}B_n.$$

For $n_1 \le n - i \le n_2$, choose $\widetilde{Z}_i = 0$, $\widetilde{X}_i$ as before and

$$\widetilde{S}_i = \Pi_{i,i+n_1-1}B_{i+n_1} + \cdots + \Pi_{i,n-1}B_n.$$

Finally, for $n - i < n_1$, define $\widetilde{S}_i = 0, \widetilde{Z}_i = 0$ and

$$\widetilde{X}_i = B_i + \Pi_{ii}B_{i+1} + \cdots + \Pi_{i,n-1}B_n.$$

Let p_1, p, p_3 be the largest integers such that $p_1 n_1 \le n - n_1 + 1$, $p n_1 \le n - n_2$ and $p_3 n_1 \le n - n_3$, respectively. We study the asymptotic tail behavior of the corresponding block sums given by

$$X_j = \sum_{i=(j-1)n_1+1}^{jn_1} \widetilde{X}_i\,, \qquad S_j = \sum_{i=(j-1)n_1+1}^{jn_1} \widetilde{S}_i\,, \qquad Z_j = \sum_{i=(j-1)n_1+1}^{jn_1} \widetilde{Z}_i, \tag{3.8}$$

where j is less or equal p_1, p, p_3 respectively.

The remaining steps of the proof are organized as follows.

- Section 3.2. We show that the X_j's and Z_j's do not contribute to the considered large deviation probabilities. This is the content of Lemmas 3.4 and 3.5.
- Section 3.3. We provide bounds for the tail probabilities of S_j; see Proposition 3.6 and Lemma 3.8. These bounds are the main ingredients in the proof of the large deviation result.
- Section 3.4. In Proposition 3.9 we combine the bounds provided in the previous subsections.
- Section 3.5: We apply Proposition 3.9 to prove the main result.

Proof of Lemma 3.1. The infinite series $\eta = \sum_{i=0}^{\infty}\Pi_i$ has the distribution of the stationary solution to the stochastic recurrence equation (1.1) with $B \equiv 1$ a.s. and therefore, by Theorem 1.1, $P(\eta > x) \sim c_\infty x^{-\alpha}\,, \quad x \to \infty$. It follows from a slight modification of Jessen and Mikosch [16], Lemma 4.1(4), and the independence of Y_0 and η that

$$\mathbb{P}\{|Y_0|\,\eta > x\} \sim c\, x^{-\alpha}\log x\,, \quad x \to \infty\,. \tag{3.9}$$

Since $s_n/n \to 0$ as $n \to \infty$ we have

$$\sup_{b_n \le x \le e^{s_n}} \frac{\mathbb{P}\{|Y_0|\,\eta_n > x\}}{n\,\mathbb{P}\{|Y| > x\}} \le \sup_{b_n \le x \le e^{s_n}} \frac{\mathbb{P}\{|Y_0|\,\eta > x\}}{n\,\mathbb{P}\{|Y| > x\}} \to 0\,.$$

There exist $c_0, x_0 > 0$ such that $P\{|Y_0| > y\} \le c_0 y^{-\alpha}$ for $y > x_0$. Therefore

$$\mathbb{P}\{|Y_0|\,\eta_n > x\} \le \mathbb{P}\{x/\eta_n \le x_0\} + c_0 x^{-\alpha}\mathbb{E}\eta_n^\alpha \mathbf{1}_{\{x/\eta_n > x_0\}} \le c x^{-\alpha}\mathbb{E}\eta_n^\alpha\,.$$

By Bartkiewicz et al. [1], $\mathbb{E}\eta_n^\alpha \le cn$. Hence

$$I_n = \sup_{b_n \le x} \frac{\mathbb{P}\{|Y_0|\,\eta_n > x\}}{n\,\mathbb{P}\{|Y| > x\}} \le \sup_{b_n \le x} \frac{c x^{-\alpha}\mathbb{E}\eta_n^\alpha}{n\,\mathbb{P}\{|Y| > x\}} < \infty.$$

This concludes the proof. □

Proof of Lemma 3.2. Assume first that $\alpha > 1$. Since $\mathbb{E}\widetilde{W}_j$ is finite, $-D\log \mathbb{E}A > \alpha - 1$ and $D \log x \le n_3$, we have for some positive δ

$$\mathbb{E}|\widetilde{W}_j| \le \frac{(\mathbb{E}A)^{n_3}}{1-\mathbb{E}A}\mathbb{E}|B| \le c\, e^{D\,\log x\,\log \mathbb{E}A} \le c\, x^{-(\alpha-1)-\delta}\,, \tag{3.10}$$

and hence by Markov's inequality

$$\mathbb{P}\Big\{\Big|\sum_{j=1}^n (\widetilde{W}_j - \mathbb{E}\widetilde{W}_j)\Big| > x\Big\} \le 2\,x^{-1}\sum_{j=1}^n \mathbb{E}|\widetilde{W}_j| \le c\,n\,x^{-\alpha-\delta}.$$

If $\beta < \alpha \le 1$ an application of Markov's inequality yields for some positive δ,

$$\begin{aligned}\mathbb{P}\Big\{\sum_{j=1}^n \widetilde{W}_j > x\Big\} &\le x^{-\beta}\sum_{j=1}^n \mathbb{E}|\widetilde{W}_j|^\beta \le x^{-\beta}\frac{n\mathbb{E}|B|^\beta(\mathbb{E}A^\beta)^{n_3}}{(1-\mathbb{E}A^\beta)} \\ &\le c x^{-\beta} n e^{D\,\log x\,\log \mathbb{E}A^\beta} \le c\,n\,x^{-\alpha-\delta}\,.\end{aligned}$$

In the last step we used the fact that $-D\log \mathbb{E}A^\beta > \alpha - \beta$. This concludes the proof of the lemma. □

3.2. **Bounds for $\mathbb{P}\{X_j > x\}$ and $\mathbb{P}\{Z_j > x\}$.** We will now study the tail behavior of the single block sums X_1, Z_1 defined in (3.8). We start with a useful auxiliary result.

Lemma 3.3. *Assume $\psi(\alpha+\epsilon) = \mathbb{E}A^{\alpha+\epsilon} < \infty$ for some $\epsilon > 0$. Then there is a constant $C = C(\epsilon) > 0$ such that $\psi(\alpha+\gamma) \le C\, e^{\rho\gamma}$ for $|\gamma| \le \epsilon/2$, where $\rho = \mathbb{E}(A^\alpha \log A)$.*

Proof. By a Taylor expansion and since $\psi(\alpha) = 1$, $\psi'(\alpha) = \rho$, we have for some $\theta \in (0,1)$,

$$\psi(\alpha+\gamma) = 1 + \rho\gamma + 0.5\psi''(\alpha+\theta\gamma)\gamma^2\,. \tag{3.11}$$

If $|\theta\gamma| < \epsilon/2$ then, by assumption, $\psi''(\alpha+\theta\gamma) = \mathbb{E}A^{\alpha+\theta\gamma}(\log A)^2$ is bounded by a constant $c > 0$. Therefore,

$$\psi(\alpha+\gamma) \le 1 + \rho\gamma + c\gamma^2 = e^{\log(1+\rho\gamma+c\,\gamma^2)} \le C\, e^{\rho\gamma}.$$

□

The following lemma ensures that the X_i's do not contribute to the considered large deviation probabilities.

Lemma 3.4. *There exist positive constants C_1, C_2, C_3 such that*

$$\mathbb{P}\{X_1 > x\} \le \mathbb{P}\{\underline{X}_1 > x\} \le C_1\, x^{-\alpha} e^{-C_2\, (\log x)^{C_3}}\,, \quad x > 1,$$

where

$$\underline{X}_1 = \sum_{i=1}^{n_1} (|B_i| + \Pi_{ii}|B_{i+1}| + \cdots + \Pi_{i,i+n_1-2}|B_{i+n_1-1}|).$$

Proof. We have $\underline{X}_1 = \sum_{k=m+1}^{n_0} R_k$, where for $m < k \le n_0$,

$$R_k = \Pi_{1,n_0-k}|B_{n_0-k+1}| + \cdots + \Pi_{i,i+n_0-k-1}|B_{i+n_0-k}| + \cdots + \Pi_{n_1,n_1+n_0-k-1}|B_{n_1+n_0-k}|\,.$$

Notice that for x sufficiently large,

$$\Big\{ \sum_{k=m+1}^{n_0} R_k > x \Big\} \subset \bigcup_{k=m+1}^{n_0} \{R_k > x/k^3\}.$$

Indeed, on the set $\{R_k \le x/k^3\,, m < k \le n_0\}$ we have for some $c > 0$ and sufficiently large x, by the definition of $m = [(\log x)^{0.5+\sigma}]$,

$$\sum_{k=m+1}^{n_0} R_k \le \frac{x}{m+1} \sum_{k=1}^{\infty} \frac{1}{k^2} \le c\, \frac{x}{(\log x)^{0.5+\sigma}} < x\,.$$

We conclude that, with $I_k = \mathbb{P}\{R_k > x/k^3\}$,

$$\mathbb{P}\Big\{ \sum_{k=m+1}^{n_0} R_k > x \Big\} \le \sum_{k=m+1}^{n_0} I_k\,.$$

Next we study the probabilities I_k. Let $\delta = (\log x)^{-0.5}$. By Markov's inequality,

$$I_k \le (x/k^3)^{-(\alpha+\delta)}\, \mathbb{E}R_k^{\alpha+\delta} \le (x/k^3)^{-(\alpha+\delta)} n_0^{\alpha+\delta} (\mathbb{E}A^{\alpha+\delta})^{n_0-k} \mathbb{E}|B|^{\alpha+\delta}\,.$$

By Lemma 3.3 and the definition of $n_0 = [\rho^{-1} \log x]$,

$$I_k \le c\,(x/k^3)^{-(\alpha+\delta)}\, n_0^{\alpha+\delta}\, e^{(n_0-k)\rho\delta} \le c\, x^{-\alpha}\, k^{3(\alpha+\delta)}\, n_0^{\alpha+\delta} e^{-k\rho\delta}\,.$$

Since $k \ge (\log x)^{0.5+\sigma} \ge m$ there are positive constants ζ_1, ζ_2 such that $k\delta \ge k^{\zeta_1} (\log x)^{\zeta_2}$ and therefore for sufficiently large x and appropriate positive constants C_1, C_2, C_3,

$$\sum_{k=m+1}^{n_0} I_k \le c\, x^{-\alpha}\, n_0^{\alpha+\delta} \sum_{k=m+1}^{n_1} e^{-\rho\, k^{\zeta_1}\, (\log x)^{\zeta_2}} k^{3(\alpha+\delta)} \le C_1\, x^{-\alpha}\, e^{-C_2\, (\log x)^{C_3}}\,.$$

This finishes the proof. □

The following lemma ensures that the Z_i's do not contribute to the considered large deviation probabilities.

Lemma 3.5. *There exist positive constants C_4, C_5, C_6 such that*

$$\mathbb{P}\{Z_1 > x\} \le \mathbb{P}\{\underline{Z}_1 > x\} \le C_4\, x^{-\alpha} e^{-C_5\, (\log x)^{C_6}}\,, \quad x > 1,$$

where

$$\underline{Z}_1 = \sum_{i=1}^{n_1} (\Pi_{i,i+n_2}|B_{i+n_2+1}| + \cdots + \Pi_{i,i+n_3-1}|B_{i+n_3}|)\,.$$

Proof. We have $\underline{Z}_1 = \sum_{k=1}^{n_3-n_2} \widetilde{R}_k$, where

$$\widetilde{R}_k = \Pi_{1,n_2+k}|B_{n_2+k+1}| + \cdots + \Pi_{i,i+n_2+k-1}|B_{i+n_2+k}| + \cdots + \Pi_{n_1,n_1+n_2+k-1}|B_{n_1+n_2+k}|.$$

As in the proof of Lemma 3.4 we notice that, with $J_k = \mathbb{P}\{\widetilde{R}_k > x/(n_2+k)^3\}$, for x sufficiently large

$$\mathbb{P}\{\sum_{k=1}^{n_3-n_2} \widetilde{R}_k > x\} \le \sum_{k=1}^{n_3-n_2} J_k.$$

Next we study the probabilities J_k. Choose $\delta = (n_2+k)^{-0.5} < \epsilon/2$ with ϵ as in Lemma 3.3. By Markov's inequality,

$$J_k \le ((n_2+k)^3/x)^{\alpha-\delta}\mathbb{E}\widetilde{R}_k^{\alpha-\delta} \le ((n_2+k)^3/x)^{\alpha-\delta}\, n_1^{\alpha-\delta}(\mathbb{E}A^{\alpha-\delta})^{n_2+k}\mathbb{E}|B|^{\alpha-\delta}\,.$$

By Lemma 3.3 and since $n_2+k = n_0+m+k$,

$$(\mathbb{E}A^{\alpha-\delta})^{n_2+k} \le c\,e^{-\delta\rho(n_2+k)} \le c\,x^{-\delta}e^{-\delta\rho(m+k)}\,.$$

There is $\zeta_3 > 0$ such that $\delta(m+k) \ge (\log x + k)^{\zeta_3}$. Hence, for appropriate constants $C_4, C_5, C_6 > 0$,

$$\sum_{k=1}^{n_3-n_2} J_k \le c\,x^{-\alpha}n_1^{\alpha-\delta}\sum_{k=1}^{n_3-n_2}(n_2+k)^{3(\alpha-\delta)}e^{-\rho(\log x+k)^{\zeta_3}} \le C_4\,x^{-\alpha}e^{-C_5\,(\log x)^{C_6}}\,.$$

This finishes the proof. □

3.3. **Bounds for $\mathbb{P}\{S_j > x\}$.** The next proposition is a first major step towards the proof of the main result. For the formulation of the result and its proof, recall the definitions of $\widetilde{S}_i$ and S_i from (3.7) and (3.8), respectively.

Proposition 3.6. *Assume that $c_\infty^+ > 0$ and let (b_n) be any sequence such that $b_n \to \infty$. Then the following relation holds:*

$$\lim_{n\to\infty}\sup_{x\ge b_n}\Big|\frac{\mathbb{P}\{S_1 > x\}}{n_1\mathbb{P}\{Y > x\}} - c_\infty\Big| = 0\,. \tag{3.12}$$

If $c_\infty^+ = 0$ then

$$\lim_{n\to\infty}\sup_{x\ge b_n}\frac{\mathbb{P}\{S_1 > x\}}{n_1\mathbb{P}\{|Y| > x\}} = 0\,. \tag{3.13}$$

The proof depends on the following auxiliary result whose proof is given in Appendix B.

Lemma 3.7. *Assume that Y and η_k (defined in (3.2)) are independent and $\psi(\alpha+\epsilon) = \mathbb{E}A^{\alpha+\epsilon} < \infty$ for some $\epsilon > 0$. Then for $n_1 = n_0 - m = [\rho^{-1}\log x] - [(\log x)^{0.5+\sigma}]$ for some $\sigma < 1/4$ and any sequences $b_n \to \infty$ and $r_n \to \infty$ the following relation holds:*

$$\lim_{n\to\infty}\sup_{r_n\le k\le n_1, b_n\le x}\Big|\frac{\mathbb{P}\{\eta_k\,Y > x\}}{k\,\mathbb{P}\{Y > x\}} - c_\infty\Big| = 0\,,$$

provided $c_\infty^+ > 0$. If $c_\infty^+ = 0$ then

$$\lim_{n\to\infty}\sup_{r_n\le k\le n_1, b_n\le x}\frac{\mathbb{P}\{\eta_k\,Y > x\}}{k\,\mathbb{P}\{|Y| > x\}} = 0.$$

Proof of Proposition 3.6. For $i \le n_1$, consider

$$\begin{aligned}&\widetilde{S}_i + S_i' \\ &= \Pi_{i,n_1} B_{n_1+1} + \cdots + \Pi_{i,i+n_1-2} B_{i+n_1-1} + \widetilde{S}_i + \Pi_{i,i+n_2} B_{i+n_2+1} + \cdots + \Pi_{i,n_2+n_1-1} B_{n_2+n_1} \\ &= \Pi_{i,n_1} (B_{n_1+1} + A_{n_1+1} B_{n_1+2} + \cdots + \Pi_{n_1+1,n_2+n_1-1} B_{n_2+n_1}) \,.\end{aligned}$$

Notice that

$$\mathbb{P}\{|S_1' + \cdots + S_{n_1}'| > x\} \le n_1 \, \mathbb{P}\{|S_1'| > x/n_1\}.$$

Therefore and by virtue of Lemmas 3.4 and 3.5 there exist positive constants C_7, C_8, C_9 such that

$$\mathbb{P}\{|S_1' + \cdots + S_{n_1}'| > x\} \le C_7 \, x^{-\alpha} \, e^{-C_8 (\log x)^{C_9}} \,, \quad x \ge 1 \,.$$

Therefore and since $S_1 = \sum_{i=1}^{n_1} \widetilde{S}_i$ it suffices for (3.12) to show that

$$\lim_{n\to\infty} \sup_{x \ge b_n} \Big| \frac{\mathbb{P}\{S_1 + \sum_{i=1}^{n_1} S_i' > x\}}{n_1 \mathbb{P}\{Y > x\}} - c_\infty \Big| = 0 \,.$$

We observe that

$$S_1 + \sum_{i=1}^{n_1} S_i' =: U T_1 \quad \text{and} \quad T_1 + T_2 \stackrel{d}{=} Y \,,$$

where

$$\begin{aligned} U &= \Pi_{1,n_1} + \Pi_{2,n_1} + \cdots + \Pi_{n_1,n_1} \,, \\ T_1 &= B_{n_1+1} + \Pi_{n_1+1,n_1+1} B_{n_1+2} + \cdots + \Pi_{n_1+1,n_2+n_1-1} B_{n_2+n_1} \,, \\ T_2 &= \Pi_{n_1+1,n_2+n_1} B_{n_2+n_1+1} + \Pi_{n_1+1,n_2+n_1+1} B_{n_2+n_1+2} + \cdots \,. \end{aligned}$$

Since $U =_d \eta_{n_1}$ and $Y =_d T_1 + T_2$, in view of Lemma 3.7 we obtain

$$\lim_{n\to\infty} \sup_{x \ge b_n} \Big| \frac{\mathbb{P}\{U(T_1 + T_2) > x\}}{n_1 \, \mathbb{P}\{Y > x\}} - c_\infty \Big| = 0 \,,$$

provided $c_\infty^+ > 0$ or

$$\lim_{n\to\infty} \sup_{x \ge b_n} \frac{\mathbb{P}\{U(T_1 + T_2) > x\}}{n_1 \, \mathbb{P}\{|Y| > x\}} = 0 \,,$$

if $c_\infty^+ = 0$. Thus to prove the proposition it suffices to justify the existence of some positive constants C_{10}, C_{11}, C_{12} such that

$$\mathbb{P}\{|UT_2| > x\} \le C_{10} \, x^{-\alpha} e^{-C_{11} \, (\log x)^{C_{12}}} \,, \quad x > 1 \,. \tag{3.14}$$

For this purpose we use the same argument as in the proof of Lemma 3.4. First we write

$$\mathbb{P}\{|UT_2| > x\} \le \sum_{k=0}^{\infty} \mathbb{P}\{U \, \Pi_{n_1+1,n_1+n_2+k} \, |B_{n_1+n_2+k+1}| > x/(\log x + k)^3\}.$$

Write $\delta = (\log x + k)^{-0.5}$. Then by Lemma 3.3, Markov's inequality and since $n_2 = n_0 + m$,

$$\begin{aligned}&\mathbb{P}\{U \, \Pi_{n_1+1,n_1+n_2+k} \, |B_{n_1+n_2+k+1}| > x/(\log x + k)^3\} \\ &\le (\log x + k)^{3(\alpha-\delta)} x^{-(\alpha-\delta)} \, \mathbb{E} U^{\alpha-\delta} (\mathbb{E} A^{\alpha-\delta})^{n_2+k} \mathbb{E}|B|^{\alpha-\delta} \\ &\le c \, (\log x + k)^{3(\alpha-\delta)} x^{-(\alpha-\delta)} \, e^{-(n_2+k)\rho\delta} \\ &\le c \, e^{-(m+k)\rho\delta} (\log x + k)^{3(\alpha-\delta)} x^{-\alpha}.\end{aligned}$$

There is $\zeta > 0$ such that $(m+k)\delta \geq (\log x + k)^\zeta$ and therefore,

$$\begin{aligned}\mathbb{P}\{|UT_2| > x\} &\leq c\, x^{-\alpha} \sum_{k=0}^{\infty} e^{-(\log x + k)^\zeta \rho} (\log x + k)^{3(\alpha-\delta)} \\ &\leq c\, x^{-\alpha} e^{-(\log x)^\zeta \rho/2}\,.\end{aligned}$$

This proves (3.14) and the lemma. □

Observe that if $|i-j| > 2$ then S_i and S_j are independent. For $|i-j| \leq 2$ we have the following bound:

Lemma 3.8. *The following relation holds for some constant $c > 0$:*

$$\sup_{i \geq 1, |i-j| \leq 2} \mathbb{P}\{|S_i| > x, |S_j| > x\} \leq c\, n_1^{0.5}\, x^{-\alpha}, \quad x > 1\,.$$

Proof. Assume without loss of generality that $i = 1$ and $j = 2, 3$. Then we have

$$\begin{aligned}|S_1| &\leq \big(\Pi_{1,n_1} + \cdots + \Pi_{n_1,n_1}\big) \\ &\quad \times \big(|B_{n_1+1}| + \Pi_{n_1+1,n_1+1}|B_{n_1+2}| + \cdots + \Pi_{n_1+1,n_1+n_2-1}|B_{n_2+n_1}|\big) =: U_1 T_1', \\ |S_2| &\leq \big(\Pi_{n_1+1,2n_1} + \cdots + \Pi_{2n_1,2n_1}\big) \\ &\quad \times \big(|B_{2n_1+1}| + \Pi_{2n_1+1,2n_1+1}|B_{2n_1+2}| + \cdots + \Pi_{2n_1+1,2n_1+n_2-1}|B_{2n_1+n_2}|\big) =: U_2 T_2', \\ |S_3| &\leq \big(\Pi_{2n_1+1,3n_1} + \cdots + \Pi_{3n_1,3n_1}\big) \\ &\quad \times \big(|B_{3n_1+1}| + \Pi_{3n_1+1,3n_1+1}|B_{3n_1+2}| + \cdots + \Pi_{3n_1+1,3n_1+n_2-1}|B_{3n_1+n_2}|\big) =: U_3 T_3'\,.\end{aligned}$$

We observe that $U_1 \stackrel{d}{=} \eta_{n_1}$, U_i, $i = 1,2,3$, are independent, U_i is independent of T_i' for each i, and the T_i''s have power law tails with index $\alpha > 0$. We conclude from (3.12) that

$$\begin{aligned}\mathbb{P}\{|S_1| > x, |S_2| > x\} &\leq \mathbb{P}\{T_1' > x\, n_1^{-1/(2\alpha)}\} + \mathbb{P}\{T_1' \leq x\, n_1^{-1/(2\alpha)},\ U_1 T_1' > x\,, U_2 T_2' > x\} \\ &\leq c\, n_1^{0.5}\, x^{-\alpha} + \mathbb{P}\{n_1^{-1/(2\alpha)} U_1 > 1,\ U_2 T_2' > x\} \\ &\leq c\, n_1^{0.5}\, x^{-\alpha} + \mathbb{P}\{U_1 > n_1^{1/(2\alpha)}\}\, \mathbb{P}\{U_2 T_2' > x\} \\ &\leq c\, n_1^{0.5}\, x^{-\alpha}\,.\end{aligned}$$

In the same way we can bound $\mathbb{P}\{|S_1| > t, |S_3| > t\}$. We omit details. □

3.4. **Semi-final steps in the proof of the main theorem.** In the following proposition, we combine the various tail bounds derived in the previous sections. For this reason, recall the definitions of X_i, S_i and Z_i from (3.8) and that p_1, p, p_3 are the largest integers such that $p_1 n_1 \leq n - n_1 + 1$, $p n_1 \leq n - n_2$ and $p_3 n_1 \leq n - n_3$, respectively.

Proposition 3.9. *Assume the conditions of Theorem 2.1. In particular, consider the following x-regions:*

$$\Lambda_n = \begin{cases} (n^{1/\alpha} (\log n)^M, \infty) & \text{for } \alpha \in (0,2],\ M > 2, \\ (c_n n^{0.5} \log n, \infty) & \text{for } \alpha > 2,\ c_n \to \infty, \end{cases}$$

and introduce a sequence $s_n \to \infty$ such that $e^{s_n} \in \Lambda_n$ and $s_n = o(n)$. Then the following relations hold:

$$\frac{c_\infty^+ c_\infty}{c_\infty^+ + c_\infty^-} \geq \limsup_{n\to\infty} \sup_{x\in\Lambda_n} \frac{\mathbb{P}\{\sum_{j=1}^p (S_j - c_j) > x\}}{n\,\mathbb{P}\{|Y| > x\}}, \tag{3.15}$$

$$0 = \lim_{n\to\infty} \sup_{x\in\Lambda_n, \log x\leq s_n} \Big| \frac{\mathbb{P}\{\sum_{j=1}^p (S_j - c_j) > x\}}{n\,\mathbb{P}\{|Y| > x\}} - \frac{c_\infty^+ c_\infty}{c_\infty^+ + c_\infty^-} \Big|, \tag{3.16}$$

$$0 = \lim_{n\to\infty} \sup_{x\in\Lambda_n} \frac{\mathbb{P}\Big\{|\sum_{j=1}^{p_1} (X_j - e_j)| > x\Big\}}{n\,\mathbb{P}\{|Y| > x\}}, \tag{3.17}$$

$$0 = \lim_{n\to\infty} \sup_{x\in\Lambda_n} \frac{\mathbb{P}\Big\{|\sum_{j=1}^{p_3} (Z_j - z_j)| > x\Big\}}{n\,\mathbb{P}\{|Y| > x\}}, \tag{3.18}$$

where $c_j = e_j = z_j = 0$ for $\alpha \leq 1$ and $c_j = \mathbb{E}S_j$, $e_j = \mathbb{E}X_j$, $z_j = \mathbb{E}Z_j$ for $\alpha > 1$.

Proof. We split the proof into the different cases corresponding to $\alpha \leq 1$, $\alpha \in (1, 2]$ and $\alpha > 2$.

The case $1 < \alpha \leq 2$.

Step 1: Proof of (3.15) and (3.16). Since $M > 2$, we can choose ξ so small that

$$2 + 4\xi < M \quad \text{and} \quad \xi < 1/(4\alpha), \tag{3.19}$$

and we write $y = x/(\log n)^{2\xi}$. Consider the following disjoint partition of Ω:

$$\Omega_1 = \bigcap_{j=1}^p \{|S_j| \leq y\},$$
$$\Omega_2 = \bigcup_{1\leq i<k\leq p} \{|S_i| > y, |S_k| > y\},$$
$$\Omega_3 = \bigcup_{k=1}^p \{|S_k| > y, |S_i| \leq y \text{ for all } i \neq k\}.$$

Then for $A = \{\sum_{j=1}^p (S_j - c_j) > x\}$,

$$\mathbb{P}\{A\} = \mathbb{P}\{A\cap\Omega_1\} + \mathbb{P}\{A\cap\Omega_2\} + \mathbb{P}\{A\cap\Omega_3\} =: I_1 + I_2 + I_3. \tag{3.20}$$

Next we treat the terms I_i, $i = 1, 2, 3$, separately.

Step 1a: Bounds for I_2. We prove

$$\lim_{n\to\infty} \sup_{x\in\Lambda_n} (x^\alpha/n)\, I_2 = 0. \tag{3.21}$$

We have

$$I_2 \leq \sum_{1\leq i<k\leq p} \mathbb{P}\{|S_i| > y, |S_k| > y\}.$$

For $k \geq i + 3$, S_k and S_i are independent and then, by (3.12),

$$\mathbb{P}\{|S_i| > y, |S_k| > y\} = (\mathbb{P}\{|S_1| > y\})^2 \leq c\,(n_1(y))^2\, y^{-2\alpha},$$

where $n_1(y)$ is defined in the same way as $n_1 = n_1(x)$ with x replaced by y. Also notice that $n_1(y) \leq n_1(x)$. For $k = i + 1$ or $i + 2$, we have by Lemma 3.8

$$\mathbb{P}\{|S_i| > y, |S_k| > y\} \leq c\,(n_1(y))^{0.5}\, y^{-\alpha}.$$

Summarizing the above estimates and observing that (3.19) holds, we obtain for $x \in \Lambda_n$,

$$\begin{aligned} I_2 &\le c\,[p^2 n_1^2 y^{-2\alpha} + p n_1^{0.5} y^{-\alpha}] \\ &\le c\,n\,x^{-\alpha}\left[x^{-\alpha} n\,(\log n)^{4\xi\alpha} + (\log n)^{2\xi\alpha} n_1^{-0.5}\right] \\ &\le c\,n\,x^{-\alpha}\left[(\log n)^{(4\xi-M)\alpha} + (\log n)^{2\xi\alpha-0.5}\right]. \end{aligned}$$

This proves (3.21).

Step 1b: Bounds for I_1. We will prove

$$\lim_{n\to\infty} \sup_{x\in\Lambda_n} (x^\alpha/n)\, I_1 = 0\,. \tag{3.22}$$

For this purpose, we write $S_j^y = S_j \mathbf{1}_{\{|S_j|\le y\}}$ and notice that $\mathbb{E}S_j = \mathbb{E}S_j^y + \mathbb{E}S_j\mathbf{1}_{\{|S_j|>y\}}$. Elementary computations show that

$$|S_1|^\alpha \le n_1^{\max(\alpha,1)} (2m+1)^{\max(\alpha,1)} \mathbb{E}|B|^\alpha\,. \tag{3.23}$$

Therefore by the Hölder and Minkowski inequalities, and by (3.12)

$$\begin{aligned} |\mathbb{E}S_j\mathbf{1}_{\{|S_j|>y\}}| &\le (\mathbb{E}|S_j|^\alpha)^{1/\alpha}\,(\mathbb{P}\{|S_j|>y\})^{1-1/\alpha} \\ &\le c\,(\log x)^{1.5+\sigma} y^{-\alpha+1}\,(n_1(y))^{1-1/\alpha} \\ &\le c\,(\log x)^{1.5+\sigma+2\xi(\alpha-1)} x^{-\alpha+1} n_1\,. \end{aligned}$$

Let now $\gamma > 1/\alpha$ and $n^{1/\alpha}(\log n)^M \le x \le n^\gamma$. Since $p\,n_1 \le n$ and (3.19) holds,

$$p\,|\mathbb{E}S_j\mathbf{1}_{\{|S_j|>y\}}| \le c\,(\log x)^{1.5+\sigma+2\xi(\alpha-1)} x^{-\alpha+1}\,n = o(x)\,. \tag{3.24}$$

If $x > n^\gamma$ then

$$x > (\log x)^M n^{1/\alpha} \quad\text{and}\quad x^{-\alpha} < (\log x)^{-M\alpha} n^{-1}.$$

Hence

$$p\,|\mathbb{E}S_j\mathbf{1}_{\{|S_j|>y\}}| \le c\,x(\log x)^{1.5+\sigma+2\xi(\alpha-1)}(\log x)^{-M\alpha} = o(x)\,. \tag{3.25}$$

Using the bounds (3.24) and (3.25), we see that for x sufficiently large,

$$\begin{aligned} I_1 &\le \mathbb{P}\Big\{\Big|\sum_{j=1}^p (S_j^y - ES_j^y)\Big| > 0.5\,x\Big\} \\ &= \mathbb{P}\Big\{\Big|\Big(\sum_{1\le j\le p, j\in\{1,4,7,\dots\}} + \sum_{1\le j\le p, j\in\{2,5,8,\dots\}} + \sum_{1\le j\le p, j\in\{3,6,9,\dots\}}\Big)(S_j^y - ES_j^y)\Big| > 0.5\,x\Big\} \\ &\le 3\,\mathbb{P}\Big\{\Big|\sum_{1\le j\le p, j\in\{1,4,7,\dots\}} (S_j^y - ES_j^y)\Big| > x/6\Big\}\,. \end{aligned} \tag{3.26}$$

In the last step, for the ease of presentation, we slightly abused notation since the number of summands in the 3 partial sums differs by a bounded number of terms which, however, do not contribute to the asymptotic tail behavior of I_1. Since the summands $S_1^y, S_4^y, \dots$ are iid and bounded, we may apply Prokhorov's inequality (A.1) to the random variables $R_k = S_k^y - \mathbb{E}S_1^y$ in (3.26) with $y = x/(\log n)^{2\xi}$ and $B_p = p\,\mathrm{var}(S_1^y)$. Then $a_n = x/(2y) = 0.5\,(\log n)^{2\xi}$ and since, in view of (3.23), $\mathrm{var}(S_1^y) \le y^{2-\alpha}\mathbb{E}|S_1|^\alpha$,

$$I_1 \le c\Big(\frac{p\,\mathrm{var}(S_1^y)}{xy}\Big)^{a_n} \le c\Big((\log n)^{(1.5+\sigma)\,\alpha+2\xi(\alpha-1)-1}\Big)^{a_n}\Big(\frac{n}{x^\alpha}\Big)^{a_n}.$$

Therefore, for $x \in \Lambda_n$,

$$(x^\alpha/n)\,I_1 \le c\,(\log n)^{((1.5+\sigma)\alpha+2\xi(\alpha-1))a_n - M\alpha(a_n-1)},$$

which tends to zero if $M > 2$, ξ satisfies (3.19) and σ is sufficiently small.

Step 1c: Bounds for I_3. Here we assume $c_\infty^+ > 0$. In this case, we can bound I_3 by using the following inequalities: for every $\epsilon > 0$, there is n_0 such that for $n \ge n_0$, uniformly for $x \in \Lambda_n$ and every fixed $k \ge 1$,

$$(1-\epsilon)c_\infty \le \frac{\mathbb{P}\{A \cap \{|S_k| > y, |S_i| \le y, i \ne k\}\}}{n_1 \mathbb{P}\{Y > x\}} \le (1+\epsilon)c_\infty. \tag{3.27}$$

Write $z = x/(\log n)^\xi$ and introduce the probabilities, for $k \ge 1$,

$$\begin{aligned} J_k &= \mathbb{P}\Big\{A \cap \{|\sum_{j\ne k}(S_j - c_j)| > z,\ |S_k| > y, |S_i| \le y, i \ne k\}\Big\}, \\ V_k &= \mathbb{P}\Big\{A \cap \{|\sum_{j\ne k}(S_j - c_j)| \le z,\ |S_k| > y, |S_i| \le y, i \ne k\}\Big\}. \end{aligned} \tag{3.28}$$

Write $S = \sum (S_j - c_j)$, where summation is taken over the set $\{j : 1 \le j \le p\,, j \ne k, k \pm 1, k \pm 2\}$. For n sufficiently large J_k is dominated by

$$\mathbb{P}\{|S| > z - 8y,\ |S_k| > y, |S_i| \le y, i \ne k\} \le \mathbb{P}\{|S_k| > y\}\, \mathbb{P}\{|S| > 0.5\, z,\ |S_i| \le y, i \ne k\}\,.$$

Applying the Prokhorov inequality (A.1) in the same way as in step 1b, we see that

$$\mathbb{P}\{|S| > 0.5\, z,\ |S_i| \le y, i \ne k\} \le c\, n\, z^{-\alpha} \le c\, (\log n)^{-(M-\xi)\alpha}\,,$$

and by Markov's inequality,

$$\mathbb{P}\{|S_1| > y\} \le c \frac{n_1(y)}{y^\alpha} \le c\, \frac{n_1}{y^\alpha}.$$

Therefore

$$\sup_{x \in \Lambda_n} (x^\alpha / n_1) J_k \quad \le \quad c\, (\log n)^{3\alpha\xi - M\alpha} \to 0.$$

Thus it remains to bound the probabilities V_k. We start with sandwich bounds for V_k:

$$\mathbb{P}\{S_k - c_k > x + z,\ |S| \le z - 8y,\ |S_i| \le y, i \ne k\} \le V_k \tag{3.29}$$

$$\le \quad \mathbb{P}\{S_k - c_k > x - z,\ |S| \le z + 8y,\ |S_i| \le y, i \ne k\}\,. \tag{3.30}$$

By (3.12), for every small $\epsilon > 0$, n sufficiently large and uniformly for $x \in \Lambda_n$, we have

$$(1-\epsilon)c_\infty \le \frac{\mathbb{P}\{S_k - c_k > x + z\}}{n_1\, \mathbb{P}\{Y > x\}} \le \frac{\mathbb{P}\{S_k - c_k > x - z\}}{n_1\, \mathbb{P}\{Y > x\}} \le (1+\epsilon)c_\infty\,, \tag{3.31}$$

where we also used that $\lim_{n\to\infty}(x+z)/x = 1$. Then the following upper bound is immediate from (3.30):

$$\frac{V_k}{n_1\, \mathbb{P}\{Y > y\}} \le \frac{\mathbb{P}\{S_k - c_k > x - z\}}{n_1\, \mathbb{P}\{Y > x\}} \le (1+\epsilon)c_\infty\,.$$

From (3.29) we have

$$\frac{V_k}{n_1\, \mathbb{P}\{Y > y\}} \ge \frac{\mathbb{P}\{S_k - c_k > x + z\}}{n_1\, \mathbb{P}\{Y > x\}} - L_k\,.$$

In view of the lower bound in (3.31), the first term on the right-hand side yields the desired lower bound if we can show that L_k is negligible. Indeed, we have

$$\begin{aligned} n_1\mathbb{P}\{Y>x\}L_k &= \mathbb{P}\{\{S_k-c_k>x+z\}\cap\big[\{|S|>z-8y\}\cup\bigcup_{i\neq k}\{|S_i|>y\}\big]\} \\ &\le \mathbb{P}\{S_k-c_k>x+z,|S|>z-8y\}+\sum_{i\neq k}\mathbb{P}\{S_k-c_k>x+z,|S_i|>y\} \\ &\le \mathbb{P}\{S_k-c_k>x+z\}\Big[\mathbb{P}\{|S|>z-8y\}+p\,\mathbb{P}\{|S_1|>y\}\Big] \\ &\quad +c\sum_{|i-k|\le 2,i\neq k}\mathbb{P}\{S_k-c_k>x+z,|S_i|>y\} \end{aligned}$$

Similar bounds as in the proofs above yield that the right-hand side is of the order $o(n_1/x^\alpha)$, hence $L_k=o(1)$. We omit details. Thus we obtain (3.27).

Step 1d: Final bounds. Now we are ready for the final steps in the proof of (3.16) and (3.15). Suppose first $c_\infty^+>0$ and $\log x\le s_n$. In view of the decomposition (3.20) and steps 1a and 1b we have as $n\to\infty$ and uniformly for $x\in\Lambda_n$,

$$\begin{aligned} \frac{\mathbb{P}\{\sum_{j=1}^p(S_j-c_j)>x\}}{n\,\mathbb{P}\{Y>x\}} &\sim \frac{I_3}{n\,\mathbb{P}\{Y>x\}} \\ &\sim \frac{n_1}{n}\frac{\sum_{k=1}^p\mathbb{P}\{\sum_{j=1}^p(S_j-\mathbb{E}S_j)>x,\ |S_k|>y,|S_j|\le y,j\neq k\}}{n_1\mathbb{P}\{Y>x\}}. \end{aligned}$$

In view of step 1c, in particular (3.27), the last expression is dominated from above by $(p\,n_1/n)(1+\epsilon)c_\infty\le(1+\epsilon)c_\infty$ and from below by

$$\frac{n_1p}{n}(1-\epsilon)c_\infty\ge\frac{n-n_2-n_1}{n}(1-\epsilon)c_\infty\ge(1-\epsilon)c_\infty\Big(1-\frac{3s_n}{n\rho}\Big).$$

Letting first $n\to\infty$ and then $\epsilon\to0$, (3.15) follows and (3.16) is also satisfied provided the additional condition $\lim_{n\to\infty}s_n/n=0$ holds.

If $c_\infty^+=0$ then $I_3=o(n\,\mathbb{P}\{|Y|>x\})$. Let now $x\in\Lambda_n$ and recall the definition of V_k from (3.28). Then for every small δ and sufficiently large x

$$\begin{aligned} \frac{\mathbb{P}\{\sum_{j=1}^p(S_j-c_j)>x\}}{n\,\mathbb{P}\{|Y|>x\}} &\sim \frac{I_3}{n\,\mathbb{P}\{|Y|>x\}} \\ &\le \frac{n_1}{n}\frac{\sum_{k=1}^pV_k}{n_1\mathbb{P}\{|Y|>x\}} \\ &\le \sup_{x\in\Lambda_n}\frac{\mathbb{P}\{S_1>x(1-\delta)-|c_1|\}}{n_1\mathbb{P}\{|Y|>x\}} \end{aligned}$$

and (3.15) follows from the second part of Lemma 3.7.

Step 2: Proof of (3.17) **and** (3.18)**.** We restrict ourselves to (3.17) since the proof of (3.18) is analogous. Write $B=\{|\sum_{j=1}^{p_1}(X_j-e_j)|>x\}$. Then

$$\mathbb{P}\{B\}\le\mathbb{P}\Big\{B\cap\bigcup_{k=1}^{p_1}\{|X_k|>y\}\Big\}+\mathbb{P}\Big\{B\cap\{|X_j|\le y\text{ for all }j\le p_1\}\Big\}=P_1+P_2.$$

By Lemma 3.4,

$$P_1\le p_1\,\mathbb{P}\{|X_1|>y\}\le C_1p_1y^{-\alpha}e^{-C_2(\log y)^{C_3}}=o(nx^{-\alpha}).$$

For the estimation of P_2 consider the random variables $X_j^y=X_j\mathbf{1}_{\{|X_j|\le y\}}$ and proceed as in step 1b.

The case $\alpha > 2$.

The proof is analogous to $\alpha \in (1,2]$. We indicate differences in the main steps.

Step 1b. First we bound the large deviation probabilities of the truncated sums (it is an analog of step 1b of Proposition 3.9). We assume without loss of generality that $c_n \le \log n$. Our aim now is to prove that for $y = xc_n^{-0.5}$:

$$\lim_{n\to\infty} \sup_{x\in\Lambda_n} \frac{x^\alpha}{n}\mathbb{P}\Big\{\Big|\sum_{j=1}^p (S_j - \mathbb{E}S_j)\Big| > x,\ |S_j| \le y \text{ for all } j \le p\Big\} = 0. \tag{3.32}$$

We proceed as in the proof of (3.22) with the same notation $S_j^y = S_j \mathbf{1}_{\{|S_j|\le y\}}$. As in the proof mentioned, $p\,|\mathbb{E}S_j\mathbf{1}_{\{|S_j|>y\}}| = o(x)$ and so we estimate the probability of interest by

$$I := 3\,\mathbb{P}\Big\{\Big|\sum_{1\le j\le p\,,j\in\{1,4,7,\dots\}} (S_j^y - \mathbb{E}S_1^y)\Big| > x/6\Big\} \tag{3.33}$$

The summands in the latter sum are independent and therefore one can apply the two-sided version of the Fuk-Nagaev inequality (A.3) to the random variables in (3.33): with $a_n = \beta x/y = c_n^{0.5}\beta$ and $p\,\mathrm{var}(S_1^y) \le cpn_1^2 \le cnn_1$,

$$I \le \Big(c\frac{p\,n_1^{(1.5+\sigma)\alpha}}{xy^{\alpha-1}}\Big)^{a_n} + \exp\Big\{-\frac{(1-\beta)^2x^2}{2e^\alpha\, cnn_1}\Big\}. \tag{3.34}$$

We suppose first that $x \le n^{0.75}$. Then the first quantity in (3.34) multiplied by x^α/n is dominated by

$$\Big(c(\log n)^{(1.5+\sigma)\alpha}c_n^{0.5(\alpha-1)}\Big)^{a_n}(n/x^\alpha)^{a_n-1} \le c_n^{-0.5a_n(1+\alpha)+\alpha}\frac{(c(\log n)^{(1.5+\sigma)\alpha})^{a_n}}{(n^{0.5\alpha-1}(\log n)^\alpha)^{a_n-1}} \to 0\,.$$

The second quantity in (3.34) multiplied by x^α/n is dominated by

$$\frac{x^\alpha}{n}\exp\Big\{-\frac{(1-\beta)^2c_n^2(\log n)^2}{2e^\alpha cn_1}\Big\} \le n^{\alpha\gamma-1}n^{-c\,c_n^2} \to 0.$$

If $x > n^{0.75}$ then $xn^{-0.5} > x^\delta$ for an appropriately small δ. Then the first quantity in (3.34) is dominated by

$$\Big(c(\log x)^{(1.5+\sigma)\alpha}\Big)^{a_n}c_n^{0.5a_n(\alpha-1)}(n/x^\alpha)^{a_n-1} \le c_n^{-0.5a_n(1+\alpha)+\alpha}\frac{(c(\log x)^{(1.5+\sigma)\alpha})^{a_n}}{(n^{0.5\alpha-1}x^{\alpha\delta})^{a_n-1}} \to 0.$$

The second quantity is dominated by

$$\frac{x^\alpha}{n}\exp\Big\{-\frac{(1-\beta)^2c_n^2x^{2\delta}(\log n)^2}{2e^\alpha cn_1}\Big\} \le x^\alpha e^{-cx^\delta} \to 0,$$

which finishes the proof of (3.32).

Step 1c. We prove that, for any $\varepsilon \in (0,1)$, sufficiently large n and fixed $k \ge 1$, the following relation holds uniformly for $x \in \Lambda_n$,

$$(1-\varepsilon)c_\infty \le \frac{\mathbb{P}\{\sum_{j=1}^p(S_j - \mathbb{E}S_j) > x,\ |S_k| > y, |S_i| \le y, i \ne k\}}{n_1\,\mathbb{P}\{Y > x\}} \le (1+\varepsilon)c_\infty. \tag{3.35}$$

Let $z \in \Lambda_n$ be such that $x/z \to \infty$. As for $\alpha \in (1,2]$, one proves

$$\frac{x^\alpha}{n_1}\mathbb{P}\Big\{\sum_{j=1}^p(S_j - \mathbb{E}S_j) > x, |\sum_{j\ne k}(S_j - \mathbb{E}S_j)| > z,\ |S_k| > y, |S_j| \le y, j \ne k\Big\} \to 0.$$

Apply the Fuk-Nagaev inequality (A.3) to bound

$$\mathbb{P}\Big\{|\sum_{j\neq k}(S_j-\mathbb{E}S_j)|>\frac{z}{2},|S_j|\le y,j\neq k\Big\}.$$

In the remainder of the proof one can follow the arguments of the proof in step 1c for $\alpha\in(1,2]$.

The case $\alpha\le 1$.

The proof is analogous to the case $1<\alpha\le 2$; instead of Prokhorov's inequality (A.1) we apply S.V. Nagaev's inequality (A.2). We omit further details.

□

3.5. **Final steps in the proof of Theorem 2.1.** We have for small $\varepsilon>0$,

$$\begin{aligned}
&\mathbb{P}\{\sum_{i=1}^n(\widetilde{S}_i-\mathbb{E}\widetilde{S}_i)>x(1+2\varepsilon)\}-\mathbb{P}\{|\sum_{i=1}^n(\widetilde{X}_i-\mathbb{E}\widetilde{X}_i)|>x\varepsilon\}-\mathbb{P}\{|\sum_{i=1}^n(\widetilde{Z}_i-\mathbb{E}\widetilde{Z}_i)|>x\varepsilon\}\\
&\le\ \mathbb{P}\{\widetilde{\mathcal{S}}_n-\widetilde{d}_n>x\}\\
&\le\ \mathbb{P}\{\sum_{i=1}^n(\widetilde{S}_i-\mathbb{E}\widetilde{S}_i)>x(1-2\varepsilon)\}+\mathbb{P}\{\sum_{i=1}^n(\widetilde{X}_i-\mathbb{E}\widetilde{X}_i)>x\varepsilon\}+\mathbb{P}\{\sum_{i=1}^n(\widetilde{Z}_i-\mathbb{E}\widetilde{Z}_i)>x\varepsilon\}.
\end{aligned}\tag{3.36}$$

Divide the last two probabilities in the first and last lines by $n\mathbb{P}\{|Y|>x\}$. Then these ratios converge to zero for $x\in\Lambda_n$, in view of (3.17), (3.18) and Lemmas 3.4 and 3.5. Now

$$\begin{aligned}
\mathbb{P}\{\sum_{i=pn_1+1}^{n}(\widetilde{S}_i-\mathbb{E}\widetilde{S}_i)>x(1-2\varepsilon)\} &=\mathbb{P}\{\sum_{i=pn_1+1}^{n-n_1}(\widetilde{S}_i-\mathbb{E}\widetilde{S}_i)>x(1-2\varepsilon)\}\\
&\le\mathbb{P}\{\sum_{i=pn_1+1}^{n-n_1}(\underline{\widetilde{S}_i}>x(1-2\varepsilon)-\sum_{i=pn_1+1}^{n-n_1}|\mathbb{E}\widetilde{S}_i|\},
\end{aligned}$$

where $\underline{\widetilde{S}_i}=\Pi_{i,i+n_1-1}|B_{i+n_1}|+\cdots+\Pi_{i,i+n_2-1}|B_{i+n_2}|$.

Notice that if $i\le n-n_2$ then (for $\alpha>1$)

$$\mathbb{E}\widetilde{S}_i=\mathbb{E}\widetilde{S}_1$$

and for $n-n_2<i\le n-n_1$

$$\mathbb{E}\widetilde{S}_i=(\mathbb{E}A)^{n_1}\Big(1+\mathbb{E}A+...+(\mathbb{E}A)^{n-i-n_1}\Big)\mathbb{E}B.$$

Hence there is C such that

$$\sum_{i=pn_1+1}^{n-n_1}|\mathbb{E}\widetilde{S}_i|\le 2n_1C$$

and so by Proposition 3.6

$$\begin{aligned}\mathbb{P}\{\sum_{i=pn_1+1}^{n-n_1}(\underline{\widetilde{S}_i} > x(1-2\varepsilon) - \sum_{i=pn_1+1}^{n-n_1}|\mathbb{E}\widetilde{S}_i|\} &\le \mathbb{P}\{\sum_{i=1}^{n_1}(\underline{\widetilde{S}_i} > \frac{x(1-2\varepsilon)-2n_1C}{2}\}\\ &+ \mathbb{P}\{\sum_{i=n_1}^{2n_1}(\underline{\widetilde{S}_i} > \frac{x(1-2\varepsilon)-2n_1C}{2}\},\\ &\le C_1 n_1 x^{-\alpha} = o\Big(n\mathbb{P}\{|Y|>x\}\Big),\end{aligned}$$

provided $\lim_{n\to\infty} s_n/n = 0$. Taking into account (3.16) and the sandwich (3.36), we conclude that (2.2) holds.

If the x-region is not bounded from above and $n > n_1(x)$ then the above calculations together with Lemma 3.1 show (2.1). If $n \le n_1(x)$, then

$$\mathbb{P}\{\mathcal{S}_n - \widetilde{d}_n > x\} \le C_1 x^{-\alpha} e^{-C_2(\log x)^{C_3}}.$$

and again (2.1) holds.

4. Ruin probabilities

In this section we study the *ruin probability* related to the centered partial sum process $T_n = \mathcal{S}_n - \mathbb{E}\mathcal{S}_n$, $n \ge 0$, i.e. for given $u > 0$ and $\mu > 0$ we consider the probability

$$\psi(u) = \mathbb{P}\{\sup_{n\ge 1}\big[T_n - \mu\, n\big] > u\}\,.$$

We will work under the assumptions of Kesten's Theorem 1.1. Therefore the random variables Y_i are regularly varying with index $\alpha > 0$. Only for $\alpha > 1$ the variable Y has finite expectation and therefore we will assume this condition throughout. Notice that the random walk $(T_n - n\,\mu)$ has dependent steps and negative drift. Since (Y_n) is ergodic we have $n^{-1}(T_n - n\,\mu) \stackrel{\text{a.s.}}{\to} -\mu$ and in particular $\sup_{n\ge 1}(T_n - n\,\mu) < \infty$ a.s.

It is in general difficult to calculate $\psi(u)$ for a given value u, and therefore most results on ruin study the asymptotic behavior of $\psi(u)$ when $u \to \infty$. If the sequence (Y_i) is iid it is well known (see Embrechts and Veraverbeke [7] for a special case of subexponential step distribution and Mikosch and Samorodnitsky [19] for a general regularly varying step distribution) that

$$\psi_{ind}(u) \sim \frac{u\,\mathbb{P}\{Y > u\}}{\mu\,(\alpha-1)}\,, \quad u \to \infty\,. \tag{4.1}$$

(We write ψ_{ind} to indicate that we are dealing with iid steps.) If the step distribution has exponential moments the ruin probability $\psi_{ind}(u)$ decays to zero at an exponential rate; see for example the celebrated Cramér-Lundberg bound in Embrechts et al. [6], Chapter 2.

It is the main aim of this section to prove the following analog of the classical ruin bound (4.1):

Theorem 4.1. *Assume that the conditions of Theorem 1.1 are satisfied and additionally $B \ge 0$ a.s. and there exists $\varepsilon > 0$ such that $\mathbb{E}A^{\alpha+\varepsilon}$ and $\mathbb{E}B^{\alpha+\varepsilon}$ are finite, $\alpha > 1$ and $c_\infty^+ > 0$. The following asymptotic result for the ruin probability holds for fixed $\mu > 0$, as $u \to \infty$:*

$$\begin{aligned}\mathbb{P}\big\{\sup_{n\ge 1}(\mathcal{S}_n - \mathbb{E}\mathcal{S}_n - n\mu) > u\big\} \quad &\sim \quad \frac{c_\infty}{\mu(\alpha-1)}u^{-\alpha+1}\\ &\sim \quad \frac{c_\infty}{c_\infty^+}\frac{u\,\mathbb{P}\{Y > u\}}{\mu(\alpha-1)}\,.\end{aligned} \tag{4.2}$$

Remark 4.2. We notice that the dependence in the sequence (Y_t) manifests in the constant c_∞/c_∞^+ in relation (4.2) which appears in contrast to the iid case; see (4.1).

To prove our result we proceed similarly as in the proof of Theorem 2.1. First notice that in view of (3.9),

$$\mathbb{P}\{\sup_{n\geq 1}(Y_0\eta_n - \mathbb{E}(Y_0\eta_n)) > u\} \leq \mathbb{P}\{Y_0\eta > u\} = o(u^{1-\alpha}).$$

Thus, it is sufficient to prove

$$u^{\alpha-1}\mathbb{P}\big\{\sup_{n\geq 1}(\widetilde{\mathcal{S}}_n - \mathbb{E}\widetilde{\mathcal{S}}_n - n\mu) > u\big\} \sim \frac{c_\infty}{\mu(\alpha-1)}\,,$$

for $\widetilde{\mathcal{S}}_n$ defined in (3.2). Next we change indices as indicated after (3.3). However, this time we cannot fix n and therefore we will proceed carefully; the details will be explained below. Then we further decompose $\widetilde{\mathcal{S}}_n$ into smaller pieces and study their asymptotic behavior.

Proof of Theorem 4.1. The following lemma shows that the centered sums $(\widetilde{\mathcal{S}}_n - \mathbb{E}\widetilde{\mathcal{S}}_n)_{n\geq uM}$ for large M do not contribute to the asymptotic behavior of the ruin probability as $u\to\infty$.

Lemma 4.3. *The following relation holds:*

$$\lim_{M\to\infty}\limsup_{u\to\infty} u^{\alpha-1}\mathbb{P}\big\{\sup_{n>uM}(\widetilde{\mathcal{S}}_n - \mathbb{E}\widetilde{\mathcal{S}}_n - n\mu) > u\big\} = 0\,.$$

Proof. Fix a large number M and define the sequence $N_l = uM2^l$, $l\geq 0$. Assume for the ease of presentation that (N_l) constitutes a sequence of even integers; otherwise we can take $N_l = [uM]2^l$. Observe that

$$\mathbb{P}\big\{\sup_{n>uM}(\widetilde{\mathcal{S}}_n - \mathbb{E}\widetilde{\mathcal{S}}_n - n\mu) > u\big\} \leq \sum_{l=0}^{\infty} p_l\,,$$

where $p_l = \mathbb{P}\big\{\max_{n\in[N_l,N_{l+1})}(\widetilde{\mathcal{S}}_n - \mathbb{E}\widetilde{\mathcal{S}}_n - n\mu) > u\big\}$. For every fixed l, in the events above we make the change of indices $i\to j = N_{l+1}-i$ and write, again abusing notation,

$$\widetilde{Y}_j = B_j + \Pi_{jj}B_{j+1} + \cdots + \Pi_{j,N_{l+1}-2}B_{N_{l+1}-1}\,.$$

With this notation, we have

$$p_l = \mathbb{P}\big\{\max_{n\in(0,N_l]}\sum_{i=n}^{N_{l+1}-1}(\widetilde{Y}_i - \mathbb{E}\widetilde{Y}_i - \mu) > u\big\}\,.$$

Using the decomposition (3.4) with the adjustment $n_4 = \min(j+n_3, N_{l+1}-1)$, we write $\widetilde{Y}_j = \widetilde{U}_j + \widetilde{W}_j$. Then, by Lemma 3.2, for small $\delta>0$,

$$\begin{aligned}
p_{l1} &= \mathbb{P}\big\{\max_{n\in(0,N_l]}\sum_{i=n}^{N_{l+1}-1}(\widetilde{W}_i - \mathbb{E}\widetilde{W}_i - \mu/4) > u/4\big\}\\
&\leq \mathbb{P}\big\{\sum_{i=N_l}^{N_{l+1}-1}(\widetilde{W}_i - \mathbb{E}\widetilde{W}_i) + \sum_{i=1}^{N_l-1}\widetilde{W}_i > u/4 + N_l\mu/4\big\}\\
&\leq \mathbb{P}\big\{\sum_{i=1}^{N_{l+1}-1}(\widetilde{W}_i - \mathbb{E}\widetilde{W}_i) > u/4 + N_l(\mu/4 - E\widetilde{W}_1)\big\}\\
&\leq cN_{l+1}N_l^{-\alpha-\delta} \leq c(uM2^l)^{1-\alpha-\delta}\,.
\end{aligned}$$

We conclude that for every $M>0$,

$$\sum_{l=0}^{\infty} p_{l1} = o(u^{1-\alpha}) \qquad \text{as } u\to\infty.$$

As in (3.7) we further decompose $\widetilde{U}_i = \widetilde{X}_i + \widetilde{S}_i + \widetilde{Z}_i$, making the definitions precise in what follows. Let p be the smallest integer such that $pn_1 \ge N_{l+1} - 1$ for $n_1 = n_1(u)$. For $i = 1, \ldots, p-1$ define X_i as in (3.8), and $X_p = \sum_{i=(p-1)n_1+1}^{N_{l+1}-1} \widetilde{X}_i$. Now consider

$$\begin{aligned} p_{l2} &= \mathbb{P}\big\{ \max_{n\in(0,N_l]} \sum_{i=n}^{N_{l+1}-1} (\widetilde{X}_i - \mathbb{E}\widetilde{X}_i - \mu/4) > u/4 \big\} \\ &\le \mathbb{P}\big\{ \sum_{i=N_l}^{N_{l+1}-1} (\widetilde{X}_i - \mathbb{E}\widetilde{X}_i) + \max_{n\in(0,N_l]} \sum_{i=n}^{N_l-1} (\widetilde{X}_i - \mathbb{E}\widetilde{X}_i) > u/4 + N_l\mu/4 \big\} \\ &\le \mathbb{P}\big\{ \max_{k\le p/2} \sum_{i=k}^{p} (X_i - \mathbb{E}X_i) > u/8 + N_l\mu/8 \big\} \\ &\quad + \mathbb{P}\big\{ \max_{k\le p/2} \sum_{i=k}^{p} (X_i - \mathbb{E}X_i) \le u/8 + N_l\mu/8 \,,\, \max_{k\le p/2} \max_{1\le j<n_1} \sum_{i=kn_1-j}^{kn_1} (\widetilde{X}_i - \mathbb{E}\widetilde{X}_i) > u/8 + N_l\mu/8 \big\} \\ &= p_{l21} + p_{l22}\,. \end{aligned}$$

The second quantity is estimated by using Lemma 3.4 as follows for fixed $M > 0$

$$p_{l22} \le c\,p\,\mathbb{P}\big\{ \sum_{i=1}^{n_1} \widetilde{X}_i > u/8 + N_l\mu/8 \big\} \le C_1\,p\,N_l^{-\alpha} e^{-C_2(\log N_l)^{C_3}} = o(u^{1-\alpha}) 2^{-(\alpha-1)l}\,,$$

where C_i, $i = 1, 2, 3$, are some positive constants. Therefore for every fixed M,

$$\sum_{l=0}^{\infty} p_{l22} = o(u^{1-\alpha}), \qquad \text{as } u \to \infty.$$

Next we treat p_{l21}. We observe that X_i and X_j are independent for $|i-j| > 1$. Splitting summation in p_{l21} into the subsets of even and odd integers, we obtain an estimate of the following type

$$p_{l21} \quad \le \quad c_1 \mathbb{P}\big\{ \max_{k\le p/2} \sum_{k\le 2i\le p} (X_{2i} - \mathbb{E}X_{2i}) > c_2(u + N_l) \big\}\,,$$

where the summands are now independent. By the law of large numbers, for any $\epsilon \in (0,1)$, $k \le p/2$, large l,

$$\mathbb{P}\big\{ \sum_{2i\le k} (X_{2i} - \mathbb{E}X_{2i}) > -\epsilon c_2(u + N_l) \big\} \ge 1/2\,.$$

An application of the maximal inequality (A.5) in the Appendix and an adaptation of Proposition 3.9 yield

$$p_{l21} \le 2\mathbb{P}\big\{ \sum_{2i\le p} (X_{2i} - \mathbb{E}X_{2i}) > (1-\epsilon)c_2(u + N_l) \big\} \le c N_l^{1-\alpha}\,.$$

Using the latter bound and summarizing the above estimates, we finally proved that

$$\lim_{M\to\infty} \limsup_{u\to\infty} u^{\alpha-1} \sum_{l=0}^{\infty} p_{l2} = 0\,.$$

Similar arguments show that the sums involving the $\widetilde{S}_i$'s and $\widetilde{Z}_i$'s are negligible as well. This proves the lemma. □

In view of Lemma 4.3 it suffices to study the following probabilities for sufficiently large $M>0$:

$$\mathbb{P}\big\{\max_{n\le Mu}(\widetilde{\mathcal{S}}_n-\mathbb{E}\widetilde{\mathcal{S}}_n-n\mu)>u\big\}.$$

Write $N_0=\lfloor Mu\rfloor$, change again indices $i\to j=N_0-i+1$ and write, abusing notation,

$$\widetilde{Y}_j=B_j+\Pi_{jj}B_{j+1}+\cdots+\Pi_{j,N_0-1}B_{N_0}.$$

Then we decompose $\widetilde{Y}_j$ as in the proof of Lemma 4.3. Reasoning in the same way as above, one proves that the probabilities related to the quantities $\widetilde{W}_i$, $\widetilde{X}_i$ and $\widetilde{Z}_i$ are of lower order than $u^{1-\alpha}$ as $u\to\infty$ and, thus, it remains to study the probabilities

$$\mathbb{P}\Big\{\max_{n\le N_0}\sum_{i=n}^{N_0}(\widetilde{S}_i-\mathbb{E}\widetilde{S}_i-\mu)>u\Big\}, \tag{4.3}$$

where $\widetilde{S}_i$ were defined in (3.7).

Take $n_1=\lfloor\log N_0/\rho\rfloor$, $p=\lfloor N_0/n_1\rfloor$ and denote by S_i the sums of n_1 consecutive $\widetilde{S}_i$'s as defined in (3.8). Observe that for any n such that $n_1(k-1)+1\le n\le kn_1$, $k-1\le p$ we have

$$\begin{aligned}\sum_{i=n}^{N_0}(\widetilde{S}_i-\mathbb{E}\widetilde{S}_i-\mu)&\le 2n_1(\mathbb{E}\widetilde{S}_1+\mu)+\sum_{i=(k-1)n_1+1}^{(p+1)n_1}(\widetilde{S}_i-\mathbb{E}\widetilde{S}_i-\mu)\\&\le 2n_1(\mathbb{E}\widetilde{S}_1+\mu)+\sum_{i=k-1}^{p}(S_i-\mathbb{E}S_i-n_1\mu)\end{aligned}$$

and

$$\begin{aligned}\sum_{i=n}^{N_0}(\widetilde{S}_i-\mathbb{E}\widetilde{S}_i-\mu)&\ge -2n_1(\mathbb{E}\widetilde{S}_1+\mu)+\sum_{i=kn_1}^{pn_1}(\widetilde{S}_i-\mathbb{E}\widetilde{S}_i-\mu)\\&\ge -2n_1(\mathbb{E}\widetilde{S}_1+\mu)+\sum_{i=k}^{p-1}(S_i-\mathbb{E}S_i-n_1\mu)\,.\end{aligned}$$

Therefore and since n_1 is of order $\log u$, instead of the probabilities (4.3) it suffices to study

$$\psi_p(u)=\mathbb{P}\Big\{\max_{n\le p}\sum_{i=n}^{p}(S_i-\mathbb{E}S_i-n_1\mu)>u\Big\}.$$

Choose $q=[M/\varepsilon_1^{\alpha+1}]+1$ for some small ε_1 and large M. Then the random variables

$$R_k=\sum_{i=(k-1)q}^{kq-3}S_i,\quad k=1,\ldots,r=\lfloor p/q\rfloor\,,$$

are independent and we have

$$
\begin{aligned}
\psi_p(u) &\le \mathbb{P}\Big\{\max_{n\le qr}\sum_{\substack{n\le i\le qr\\ i\neq kq-2,kq-1}}(S_i-\mathbb{E}S_i-n_1\mu)>u(1-3\varepsilon_1)\Big\}\\
&\quad+\mathbb{P}\Big\{\max_{j\le r}\sum_{k=j}^{r}(S_{kq-2}-\mathbb{E}S_{kq-2}-n_1\mu)>\varepsilon_1 u\Big\}\\
&\quad+\mathbb{P}\Big\{\max_{j\le r}\sum_{k=j}^{r}(S_{kq-1}-\mathbb{E}S_{kq-1}-n_1\mu)>\varepsilon_1 u\Big\}\\
&\quad+\mathbb{P}\Big\{\max_{qr<n<p}\sum_{i=n}^{p}(S_i-\mathbb{E}S_i-n_1\mu)>\varepsilon_1 u\Big\}=:\sum_{i=1}^{4}\psi_p^{(i)}(u)\,.
\end{aligned}
$$

The quantities $\psi_p^{(i)}(u)$, $i=2,3$, can be estimated in the same way; we focus on $\psi_p^{(2)}(u)$. Applying Petrov's inequality (A.4) and Proposition 3.9, we obtain for some constant c not depending on ε_1,

$$
\begin{aligned}
\psi_p^{(2)}(u) &\le \mathbb{P}\Big\{\max_{j\le r}\sum_{k=j}^{r}(S_{kq-2}-\mathbb{E}S_{kq-2})>\varepsilon_1\,u\Big\}\\
&\le c\,\mathbb{P}\Big\{\sum_{k=1}^{r}(S_{kq-2}-\mathbb{E}S_{kq-2})>\varepsilon_1 u/2\Big\}\\
&\le c\,r\,n_1(\varepsilon_1\,u)^{-\alpha}\le c\,\varepsilon_1 u^{-\alpha+1}.
\end{aligned}
$$

Hence we obtain $\lim_{\varepsilon_1\downarrow 0}\limsup_{u\to\infty}u^{\alpha-1}\psi_p^{(2)}(u)=0$. By (3.15), for some constant c,

$$
\psi_p^{(4)}(u)\le c\frac{qn_1}{(\varepsilon_1 u)^{\alpha}}\le c\frac{Mu}{r(\varepsilon_1 u)^{\alpha}}.
$$

Since $r\ge q>M/\varepsilon_1^{\alpha+1}$ for large u we conclude for such u that $r^{-1}\le M^{-1}\varepsilon_1^{\alpha+1}$ and therefore $\psi_p^{(4)}(u)\le c\varepsilon_1 u^{1-\alpha}$ and $\lim_{\varepsilon_1\downarrow 0}\limsup_{u\to\infty}u^{\alpha-1}\psi_p^{(4)}(u)=0$.

Since A and B are non-negative we have for large u with $\mu_0=\mu\,(q-2)$,

$$
\begin{aligned}
\psi_p^{(1)}(u) &\le \mathbb{P}\Big\{\max_{j\le r}\sum_{i=1}^{j}(R_i-\mathbb{E}R_i-\mu_0 n_1)>u(1-3\varepsilon_1)-qn_1(\mathbb{E}S_1+\mu)\Big\}\\
&\le \mathbb{P}\Big\{\max_{j\le r}\sum_{i=1}^{j}(R_i-\mathbb{E}R_i-\mu_0 n_1)>u(1-4\varepsilon_1)\Big\}\,.
\end{aligned}
$$

Combining the bounds above we proved for large u, small ε_1 and some constant $c>0$ independent of ε_1 that

$$
\psi_p(u)\le \mathbb{P}\Big\{\max_{j\le r}\sum_{i=1}^{j}(R_i-\mathbb{E}R_i-\mu_0 n_1)>u(1-4\varepsilon_1)\Big\}+c\,\varepsilon_1\,u^{-\alpha+1}.
$$

Similar arguments as above show that

$$
\psi_p(u)\ge \mathbb{P}\Big\{\max_{j\le r}\sum_{i=1}^{j}(R_i-\mathbb{E}R_i-\mu_0 n_1)>u(1+4\varepsilon_1)\Big\}-c\,\varepsilon_1\,u^{-\alpha+1}.
$$

Thus we reduced the problem to study an expression consisting of independent random variables R_i and the proof of the theorem is finished if we can show the following result. Write

$$\Omega_r = \Big\{ \max_{j\le r} \sum_{i=1}^{j} (R_i - \mathbb{E}R_i - n_1\mu_0) > u \Big\} .$$

Lemma 4.4. *The following relation holds*

$$\lim_{M\to\infty} \limsup_{u\to\infty} \Big| u^{\alpha-1}\mathbb{P}\{\Omega_r\} - \frac{c_\infty\, c_\infty^+}{\mu(\alpha-1)} \Big| = 0 .$$

Proof. Fix some $\varepsilon_0>0$ and choose some large M. Define $C_0 = (q-2)c_\infty c_\infty^+$. Reasoning as in the proof of (3.16), we obtain for any integers $0\le j<k\le r$ and large u

$$1-\varepsilon_0 \le u^\alpha\, \frac{\mathbb{P}\{\sum_{i=j+1}^{k}\big(R_i-\mathbb{E}R_i\big)>u\}}{n_1(k-j)C_0} \le 1+\varepsilon_0 \tag{4.4}$$

Choose $\varepsilon,\delta>0$ small to be determined later in dependence on ε_0. Eventually, both $\varepsilon,\delta>0$ will become arbitrarily small when ε_0 converges to zero. Define the sequence $k_0=0$, $k_l=[\delta n_1^{-1}(1+\varepsilon)^{l-1}u]$, $l\ge 1$. Without loss of generality we will assume $k_{l_0} = Mun_1^{-1}$ for some integer number l_0. For $\eta>\varepsilon_0(2l_0)^{-1}$ consider the independent events

$$D_l = \Big\{ \max_{k_l<j\le k_{l+1}} \sum_{i=k_l+1}^{j} (R_i-\mathbb{E}R_i) > 2\eta u \Big\}, \qquad l=0,\dots,l_0-1.$$

Define the disjoint sets

$$W_l = \Omega_r \cap D_l \cap \bigcap_{m\ne l} D_m^c , \quad l=0,\dots,l_0-1 .$$

We will show that

$$\Big| \mathbb{P}\{\Omega_r\} - \sum_{l=0}^{l_0-1}\mathbb{P}\{W_l\}\Big| \le o(u^{1-\alpha}) , \quad u\to\infty . \tag{4.5}$$

First we observe that $\Omega_r \subset \bigcup_{l=0}^{l_0-1} D_l$. Indeed, on $\bigcap_{l=0}^{l_0-1} D_l^c$ we have

$$\max_{j\le r} \sum_{i=1}^{j} (R_i-\mathbb{E}R_i-n_1\mu_0) \le l_0 2\eta u \le \varepsilon_0 u ,$$

and therefore Ω_r cannot hold for small ε_0. Next we prove that

$$\mathbb{P}\{\bigcup_{m\ne l}(D_m\cap D_l)\} = o(u^{1-\alpha}) , \quad u\to\infty . \tag{4.6}$$

Then (4.5) will follow. First apply Petrov's inequality (A.4) to $\mathbb{P}\{D_l\}$ with q_0 arbitrarily close to one and power $p_0\in(1,\alpha)$. Notice that $\mathbb{E}|R_i|^{p_0}$ is of the order qn_1, hence m_{p_0} is of the order $\delta\varepsilon qu \le c\delta\varepsilon M\varepsilon_1^{-\alpha-1}u$. Next apply (4.4). Then one obtains for sufficiently large u, and small ε,δ, and some constant c' depending on $\varepsilon,\delta,\varepsilon_0,\varepsilon_1$,

$$\begin{aligned} \mathbb{P}\{D_l\} &\le q_0^{-1}\mathbb{P}\Big\{ \sum_{i=k_l+1}^{k_{l+1}} (R_i-\mathbb{E}R_i) > \eta u \Big\} \\ &\le q_0^{-1} n_1(k_{l+1}-k_l)(1+\varepsilon_0)C_0(\eta u)^{-\alpha} \le c' u^{1-\alpha} . \end{aligned}$$

Hence $\mathbb{P}\{\cup_{m\ne l}(D_l\cap D_m)\} = O(u^{2(1-\alpha)})$ as desired for (4.6) if all the parameters $\varepsilon,\delta,\varepsilon_0,\varepsilon_1$ are fixed.

Thus we showed (4.5) and it remains to find suitable bounds for the probabilities $\mathbb{P}\{W_l\}$. On the set W_l we have

$$\begin{aligned}\max_{j\le k_l}\sum_{i=1}^{j}(R_i-\mathbb{E}R_i-\mu_0 n_1) &\le \max_{j\le k_l}\sum_{i=1}^{j}(R_i-\mathbb{E}R_i)\le 2\eta l u\le \varepsilon_0 u\,,\\ \max_{k_{l+1}<j\le r}\sum_{i=k_l+1}^{j}(R_i-\mathbb{E}R_i) &\le 2\eta l_0 u\le\varepsilon_0 u\,.\end{aligned}$$

Therefore for small ε_0 and large u on the event W_l,

$$\begin{aligned}&\max_{j\le r}\sum_{i=1}^{j}(R_i-\mathbb{E}R_i-\mu_0 n_1)=\max_{k_l<j\le r}\sum_{i=1}^{j}(R_i-\mathbb{E}R_i-\mu_0 n_1)\\ &\le \sum_{i=1}^{k_l}(R_i-\mathbb{E}R_i-\mu_0 n_1)+\max_{k_l<j\le k_{l+1}}\sum_{i=k_l+1}^{j}(R_i-\mathbb{E}R_i)+\max_{k_{l+1}<j\le r}\sum_{i=k_{l+1}+1}^{j}(R_i-\mathbb{E}R_i)\\ &\le 2\varepsilon_0 u-k_l\mu_0 n_1+\max_{k_l<j\le k_{l+1}}\sum_{i=k_l+1}^{j}(R_i-\mathbb{E}R_i).\end{aligned}$$

Petrov's inequality (A.4) and (4.4) imply for $l\ge 1$ and large u that

$$\begin{aligned}\mathbb{P}\{W_l\} &\le \mathbb{P}\Big\{\max_{k_l<j\le k_{l+1}}\sum_{i=k_l+1}^{j}(R_i-\mathbb{E}R_i)\ge(1-2\varepsilon_0)u+\mu_0 n_1 k_l\Big\}\\ &\le q_0^{-1}\mathbb{P}\Big\{\sum_{i=k_l+1}^{k_{l+1}}(R_i-\mathbb{E}R_i)\ge(1-3\varepsilon_0)u+\mu_0 n_1 k_l\Big\}\\ &\le q_0^{-1}\frac{(k_{l+1}-k_l)n_1(1+\varepsilon_0)C_0}{((1-3\varepsilon_0)+\mu_0\delta(1+\varepsilon)^{l-1})^{\alpha}u^{\alpha}}\\ &= q_0^{-1}\frac{\delta\varepsilon(1+\varepsilon)^{l-1}(1+\varepsilon_0)C_0}{((1-3\varepsilon_0)+\mu_0\delta(1+\varepsilon)^{l-1})^{\alpha}}u^{1-\alpha}.\end{aligned}$$

For $l=0$ we use exactly the same arguments, but in this case $(k_1-k_0)n_1=\delta u$ and $k_0=0$. Thus we arrive at the upper bound

$$\begin{aligned}\sum_{l=0}^{l_0-1}\mathbb{P}\{W_l\} &\le q_0^{-1}(1+\varepsilon_0)C_0\Big(\frac{\delta}{(1-3\varepsilon_0)^{\alpha}}+\sum_{i=1}^{l_0-1}\frac{\delta\varepsilon(1+\varepsilon)^{l-1}}{((1-3\varepsilon_0)+\mu_0\delta(1+\varepsilon)^{l-1})^{\alpha}}\Big)u^{1-\alpha}\\ &= q_0^{-1}(1+\varepsilon_0)\,A(\varepsilon,\delta,\varepsilon_0,l_0)u^{1-\alpha}.\end{aligned}\tag{4.7}$$

To estimate $\mathbb{P}\{W_l\}$ from below first notice that on W_l, for large u,

$$\begin{aligned}\max_{j\le r}\sum_{i=1}^{j}(R_i-\mathbb{E}R_i-\mu_0 n_1) &\ge \sum_{i=1}^{k_{l+1}}(R_i-\mathbb{E}R_i-\mu_0 n_1)\\ &\ge \sum_{i=k_l+1}^{k_{l+1}}(R_i-\mathbb{E}R_i)-k_{l+1}\mu_0 n_1-k_l\mathbb{E}R_1\\ &\ge \sum_{i=k_l+1}^{k_{l+1}}(R_i-\mathbb{E}R_i)-k_{l+1}\mu_0 n_1-\varepsilon_0 u\,.\end{aligned}$$

By (4.6) and (4.4), we have for $l \geq 1$ and as $u \to \infty$,

$$\begin{aligned}
\mathbb{P}\{W_l\} &\geq \mathbb{P}\Big\{\Big\{\sum_{i=k_l+1}^{k_{l+1}} (R_i - \mathbb{E}R_i) > (1+\varepsilon_0)u + \mu_0 n_1 k_{l+1}\Big\} \cap D_l \cap \bigcap_{m\neq l} D_m^c\Big\} \\
&\geq \mathbb{P}\Big\{\sum_{i=k_l}^{k_{l+1}-1} (R_i - \mathbb{E}R_i) > (1+\varepsilon_0)u + \mu_0 n_1 k_{l+1}\Big\} - \mathbb{P}\Big\{D_l \cap \bigcup_{m\neq l} D_m\Big\} \\
&\geq \frac{(k_{l+1}-k_l)n_1(1-\varepsilon_0)C_0}{((1+\varepsilon_0)u + \mu_0 k_{l+1} n_1)^\alpha} - o(u^{1-\alpha}) \geq \frac{(1-2\varepsilon_0)C_0\delta(1+\varepsilon)^{l-1}\varepsilon}{((1+\varepsilon_0)+\mu_0\delta(1+\varepsilon)^l)^\alpha} u^{1-\alpha}\,.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{l=0}^{l_0-1} \mathbb{P}\{W_l\} &\geq (1-2\varepsilon_0)C_0\Big(\frac{\delta}{(1+\varepsilon_0+\mu_0\delta)^\alpha} + \sum_{l=1}^{l_0-1} \frac{\delta(1+\varepsilon)^{l-1}\varepsilon}{((1+\varepsilon_0)+\mu_0\delta(1+\varepsilon)^l)^\alpha}\Big) u^{1-\alpha} \\
&= (1-2\varepsilon_0)C_0 B(\varepsilon,\delta,\varepsilon_0,l_0) u^{1-\alpha}\,.
\end{aligned}$$

Thus we proved that

$$\begin{aligned}
(1-2\varepsilon_0)B(\varepsilon,\delta,\varepsilon_0,l_0) &\leq \liminf_{u\to\infty} C_0^{-1} u^{\alpha-1} \sum_{l=0}^{l_0-1} \mathbb{P}\{W_l\} \\
&\leq \limsup_{u\to\infty} C_0^{-1} u^{\alpha-1} \sum_{l=0}^{l_0-1} \mathbb{P}\{W_l\} \leq q_0^{-1}(1+\varepsilon_0)A(\varepsilon,\delta,\varepsilon_0,l_0)\,. \qquad (4.8)
\end{aligned}$$

Finally, we will justify that the upper and lower bounds are close for small $\varepsilon, \delta, \varepsilon_0$, large M and q_0 close to 1. For a real number s which is small in absolute value define the functions

$$f_s(x) = (1+s+\mu_0 x)^{-\alpha} \quad \text{and} \quad F_s(x) = (1+s+\mu_0 x)\, f_s(x) \quad \text{on } [\delta, M]\,.$$

Let $x_l = \delta(1+\varepsilon)^{l-1}$, $l = 1, \ldots, l_0$. Since $x_{l+1} - x_l = \delta\varepsilon(1+\varepsilon)^{l-1}$ are uniformly bounded by εM and f_s is Riemann integrable on $[0, \infty)$, choosing ε small, we have

$$\begin{aligned}
A(\varepsilon,\delta,\varepsilon_0,l_0) &= \sum_{l=1}^{l_0-1} f_{-3\varepsilon_0}(x_l)(x_{l+1}-x_l) \\
&\leq \int_\delta^M f_{-3\varepsilon_0}(x)\,dx = \frac{F_{-3\varepsilon_0}(\delta) - F_{-3\varepsilon_0}(M) + \varepsilon_0}{\mu_0(\alpha-1)}\,.
\end{aligned}$$

Thus we obtain the bound

$$\lim_{q_0\uparrow 1} \lim_{\varepsilon_0\downarrow 0} \lim_{M\to\infty} \lim_{\delta\downarrow 0} q_0^{-1}(1+\varepsilon_0)A(\varepsilon,\delta,\varepsilon_0,l_0) = (\mu_0(\alpha-1))^{-1}\,. \qquad (4.9)$$

Proceeding in a similar way,

$$B(\varepsilon,\delta,\varepsilon_0,l_0) \geq \int_\delta^M f_{\varepsilon_0}(x)dx = \frac{F_{\varepsilon_0}(\delta) - F_{\varepsilon_0}(M) - \varepsilon_0}{\mu_0(\alpha-1)}\,.$$

The right-hand side converges to $(\mu_0(\alpha-1))^{-1}$ by letting $\delta \downarrow 0$, $M \to \infty$ and $\varepsilon_0 \downarrow 0$. The latter limit relation in combination with (4.8) and (4.9) proves the lemma. □

Appendix A. Inequalities for sums of independent random variables

In this section, we consider a sequence (X_n) of independent random variables and their partial sums $R_n = X_1 + \cdots + X_n$. We always write $B_n = \mathrm{var}(R_n)$ and $m_p = \sum_{j=1}^n \mathbb{E}|X_j|^p$ for $p > 0$. First, we collect some of the classical tail estimates for R_n.

Lemma A.1. *The following inequalities hold.*
Prokhorov's inequality; *cf. Petrov* [23], *p. 77: Assume that the X_n's are centered, $|X_n| \le y$ for all $n \ge 1$ and some $y > 0$. Then*

$$\mathbb{P}\{R_n \ge x\} \le \exp\Big\{ -\frac{x}{2\,y}\mathrm{arsinh}\big(\frac{xy}{2\,B_n}\big)\Big\}, \quad x > 0\,. \tag{A.1}$$

S. V. Nagaev's inequality; *see* [22]: *Assume $m_p < \infty$ for some $p > 0$. Then*

$$\mathbb{P}\{R_n > x\} \le \sum_{j=1}^n \mathbb{P}\{X_j > y\} + \Big(\frac{e\,m_p}{xy^{p-1}}\Big)^{x/y}, \quad x, y > 0\,. \tag{A.2}$$

Fuk-Nagaev inequality; *cf. Petrov* [23], *p. 78: Assume that the X_n's are centered, $p > 2$, $\beta = p/(p+2)$ and $m_p < \infty$. Then*

$$\mathbb{P}\{R_n > x\} \le \sum_{j=1}^n \mathbb{P}\{X_j > y\} + \Big(\frac{m_p}{\beta x y^{p-1}}\Big)^{\beta x/y} + \exp\Big\{ -\frac{(1-\beta)^2 x^2}{2e^p B_n}\Big\}, \quad x, y > 0\,. \tag{A.3}$$

Petrov's inequality; *cf. Petrov* [23], *p. 81: Assume that the X_n's are centered and $m_p < \infty$ for some $p \in (1, 2]$. Then for every $q_0 < 1$, with $L = 1$ for $p = 2$ and $L = 2$ for $p \in (1, 2)$,*

$$\mathbb{P}\{\max_{i\le n} R_i > x\} \le q_0^{-1}\mathbb{P}\{R_n > x - [(L/(1-q_0))^{-1} m_p]^{1/p}\}, \quad x \in \mathbb{R}\,. \tag{A.4}$$

Lévy-Ottaviani-Skorokhod inequality; *cf. Petrov* [23], *Theorem 2.3 on p. 51. If $\mathbb{P}\{R_n - R_k \ge -c)\} \ge q$, $k = 1, \ldots, n-1$, for some constants $c \ge 0$ and $q > 0$, then*

$$\mathbb{P}\{\max_{i\le n} R_i > x\} \le q^{-1}\mathbb{P}\{R_n > x - c\}, \quad x \in \mathbb{R}\,. \tag{A.5}$$

Appendix B. Proof of Lemma 3.7

Assume first $c_\infty^+ > 0$. We have by independence of Y and η_k, for any $k \ge 1$, $x > 0$ and $r > 0$,

$$\frac{\mathbb{P}\{\eta_k Y > x\}}{k\,\mathbb{P}\{Y > x\}} = \Big(\int_{(0,x/r]} + \int_{[x/r,\infty)}\Big)\frac{\mathbb{P}\{Y > x/z\}}{k\,\mathbb{P}\{Y > x\}}\, d\mathbb{P}(\eta_k \le z) = I_1 + I_2\,.$$

For every $\varepsilon \in (0, 1)$ there is $r > 0$ such that for $x \ge r$ and $z \le x/r$,

$$\frac{\mathbb{P}\{Y > x/z\}}{\mathbb{P}\{Y > x\}} \in z^\alpha[1-\varepsilon, 1+\varepsilon] \quad \text{and} \quad \mathbb{P}\{Y > x\}x^\alpha \ge c_\infty^+ - \varepsilon\,.$$

Hence for sufficiently large x,

$$I_1 \in k^{-1}\mathbb{E}\eta_k^\alpha \mathbf{1}_{\{\eta_k \le x/r\}}[1-\varepsilon, 1+\varepsilon] \quad \text{and} \quad I_2 \le c\,k^{-1}x^\alpha \mathbb{P}\{\eta_k > x/r\} \le c\,k^{-1}\,\mathbb{E}\eta_k^\alpha \mathbf{1}_{\{\eta_k > x/r\}}\,.$$

We have

$$I_1 \in (k^{-1}\mathbb{E}\eta_k^\alpha - k^{-1}\mathbb{E}\eta_k^\alpha \mathbf{1}_{\{\eta_k > x/r\}})[1-\varepsilon, 1+\varepsilon]$$

and by virtue of Bartkiewicz et al. [1], $\lim_{k\to\infty} k^{-1}\mathbb{E}\eta_k^\alpha = c_\infty$. Therefore it is enough to prove that

$$\lim_{n\to\infty} \sup_{r_n \le k \le n_1, b_n \le x} k^{-1}\mathbb{E}\eta_k^\alpha \mathbf{1}_{\{\eta_k > x\}} = 0. \tag{B.1}$$

By the Hölder and Markov inequalities we have for $\epsilon > 0$,

$$\mathbb{E}\eta_k^\alpha \mathbf{1}_{\{\eta_k > x\}} \le (\mathbb{E}\eta_k^{\alpha+\epsilon})^{\alpha/(\alpha+\epsilon)}\big(\mathbb{P}\{\eta_k > x\}\big)^{\epsilon/(\alpha+\epsilon)} \le x^{-\epsilon}\,\mathbb{E}\eta_k^{\alpha+\epsilon}\,. \tag{B.2}$$

Next we study the order of magnitude of $\mathbb{E}\eta_k^{\alpha+\epsilon}$. By definition of η_k,

$$\begin{aligned}\mathbb{E}\eta_k^{\alpha+\epsilon} &= \mathbb{E}A^{\alpha+\epsilon}\mathbb{E}(1+\eta_{k-1})^{\alpha+\epsilon} \\ &= \mathbb{E}A^{\alpha+\epsilon}\big(\mathbb{E}(1+\eta_{k-1})^{\alpha+\epsilon} - \mathbb{E}(\eta_{k-1}^{\alpha+\epsilon})\big) + \mathbb{E}A^{\alpha+\epsilon}\mathbb{E}\eta_{k-1}^{\alpha+\epsilon}.\end{aligned}$$

Thus we get the recursive relation

$$\begin{aligned}\mathbb{E}\eta_k^{\alpha+\epsilon} &= \sum_{i=1}^{k}(\mathbb{E}A^{\alpha+\epsilon})^{k-i+1}\big(\mathbb{E}(1+\eta_{i-1})^{\alpha+\epsilon} - \mathbb{E}(\eta_{i-1}^{\alpha+\epsilon})\big) \\ &\le c\sum_{i=1}^{k}(\mathbb{E}A^{\alpha+\epsilon})^{k-i+1} \le c\,\frac{(\mathbb{E}A^{\alpha+\epsilon})^k}{\mathbb{E}A^{\alpha+\epsilon}-1}.\end{aligned}\tag{B.3}$$

Indeed, we will prove that if $\epsilon < 1$ then there is a constant c such that for $i \ge 1$,

$$\mathbb{E}(1+\eta_i)^{\alpha+\epsilon} - \mathbb{E}\eta_i^{\alpha+\epsilon} \le c\,.$$

If $\alpha+\epsilon \le 1$ then this follows from the concavity of the function $f(x) = x^{\alpha+\epsilon}$, $x > 0$. If $\alpha+\epsilon > 1$ we use the mean value theorem to obtain

$$\mathbb{E}(1+\eta_i)^{\alpha+\epsilon} - \mathbb{E}\eta_i^{\alpha+\epsilon} \le (\alpha+\epsilon)\,\mathbb{E}(1+\eta_i)^{\alpha+\epsilon-1} \le (\alpha+\epsilon)\mathbb{E}\eta_\infty^{\alpha+\epsilon-1} < \infty.$$

Now we choose $\epsilon = k^{-0.5}$. Then by (B.2), (B.3) and Lemma 3.3,

$$\mathbb{E}\eta_k^{\alpha}\mathbf{1}_{\{\eta_k > x\}} \le c\,\frac{(\mathbb{E}A^{\alpha+\epsilon})^k}{\mathbb{E}A^{\alpha+\epsilon}-1}x^{-\epsilon} \le c\,\frac{e^{\rho n_1/\sqrt{k} - \log x/\sqrt{k}}}{\mathbb{E}A^{\alpha+\epsilon}-1} \le c\,\frac{e^{-\rho m/\sqrt{k}}}{\mathbb{E}A^{\alpha+\epsilon}-1}\,.$$

In the last step we used that $k \le n_1 = n_0 - m$, where $n_0 = [\rho^{-1}\log x]$. Moreover, since $m = [(\log x)^{0.5+\sigma}]$, $m/\sqrt{k} \ge 2\,c_1(\log x)^\sigma$ for some $c_1 > 0$. On the other hand, setting $\gamma = \epsilon = k^{-0.5}$ in (3.11), we obtain $\mathbb{E}A^{\alpha+\epsilon} - 1 \ge \rho k^{-0.5}/2$. Combining the bounds above, we finally arrive at

$$\sup_{r_n \le k \le n_1, b_n \le x} k^{-1}\mathbb{E}\eta_k^{\alpha}\mathbf{1}_{\{\eta_k > x\}} \le c\,e^{-c_1\,(\log x)^\sigma}$$

for constants $c, c_1 > 0$. This estimate yields the desired relation (B.1) and thus completes the proof of the first part of the lemma when $c_\infty^+ > 0$.

If $c_\infty^+ = 0$ we proceed in the same way, observing that for any $\delta, z > 0$ and sufficiently large x,

$$\frac{\mathbb{P}\{Y > x/z\}}{\mathbb{P}\{|Y| > x\}} < \delta z^\alpha$$

and hence I_1 converges to 0 as n goes to infinity. This proves the lemma.

Acknowledgment. We would like to thank the referee whose comments helped us to improve the presentation of this paper.

References

[1] Bartkiewicz, K., Jakubowski, A., Mikosch, T. and Wintenberger, O. (2011) Infinite variance stable limits for sums of dependent random variables. *Probab. Th. Rel. Fields* **150** 337-372.

[2] Bingham, N.H., Goldie, C.M. and Teugels, J.L. (1987) *Regular Variation.* Cambridge University Press, Cambridge.

[3] Cline, D.B.H. and Hsing, T. 1998. Large deviation probabilities for sums of random variables with heavy or subexponential tails, Technical Report, Texas A& M University.

[4] Davis, R.A. and Hsing, T. (1995) Point process and partial sum convergence for weakly dependent random variables with infinite variance. *Ann. Prob.* **23** 879–917.

[5] Denisov, D., Dieker, A.B. and Shneer, V. (2008) Large deviations for random walks under subexponentiality: the big-jump domain. *Ann. Probab.* **36**, 1946–1991.

[6] Embrechts, P., Klüppelberg, C. and Mikosch, T. (1997) *Modelling Extremal Events for Insurance and Finance.* Springer, Berlin.

[7] Embrechts, P. and Veraverbeke, N. (1982) Estimates for the probability of ruin with special emphasis on the possibility of large claims. *Insurance Math. Econom.* **1**, 55–72.

[8] Gantert, N. (2000) A note on logarithmic tail asymptotics and mixing. *Statist. Probab. Lett.* **49**, 113-118.

[9] Gihman, I.I and Skorohod, A.V. (1974) *The Theory of Stochastic Processes I.* Springer, Berlin.

[10] Goldie, C.M. (1991) Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Probab.* **1**, 126–166.

[11] Gnedenko, B.V. (1943) Sur la distribution limité du terme d'une série aléatoire. *Ann. Math.* **44**, 423–453.

[12] Haan, L. de, Resnick, S.I., Rootzén, H. and Vries, C.G. de (1989) Extremal behaviour of solutions to a stochastic difference equation with applications to ARCH processes. *Stoch. Proc. Appl.* **32**, 213–224.

[13] Hult, H., Lindskog, F., Mikosch, T. and Samorodnitsky, G. (2005) Functional large deviations for multivariate regularly varying random walks. *Ann. Appl. Probab.* **15**, 2651-2680.

[14] Jakubowski, A. (1993) Minimal conditions in p-stable limit theorems. *Stoch. Proc. Appl.* **44**, 291–327.

[15] Jakubowski, A. (1997) Minimal conditions in p-stable limit theorems - II. *Stoch. Proc. Appl.* **68**, 1–20.

[16] Jessen, A.H. and Mikosch, T. (2006) Regularly varying functions. *Publ. Inst. Math.* Nouvelle Série **80(94)**, 171–192.

[17] Kesten, H. (1973) Random difference equations and renewal theory for products of random matrices. *Acta Math.* **131**, 207–248.

[18] Konstantinindes, D. and Mikosch, T. (2005) Large deviations and ruin probabilities for solutions to stochastic recurrence equations with heavy-tailed innovations. *Ann. Probab.* **33**, 1992–2035.

[19] Mikosch, T. and Samorodnitsky, G. (2000) The supremum of a negative drift random walk with dependent heavy-tailed steps. *Ann. Appl. Probab.* **10**, 1025–1064.

[20] Mikosch, T. and Wintenberger, O. (2011) Precise large deviations for dependent regularly varying sequences. Work in progress.

[21] Nagaev, A.V. (1969) Integral limit theorems for large deviations when Cramér's condition is not fulfilled I,II. *Theory Probab. Appl.* **14**, 51–64 and 193–208.

[22] Nagaev, S.V. (1979) Large deviations of sums of independent random variables. *Ann. Probab.* **7**, 745–789.

[23] Petrov, V.V. *Limit Theorems of Probability Theory. Sequences of Independent Random Variables.* Oxford Studies in Probability, 4. Oxford University Press, New York, 1995.

D. Buraczewski, E. Damek, and J. Zienkiewicz, Instytut Matematyczny, Uniwersytet Wroclawski, 50-384 Wroclaw, pl. Grunwaldzki 2/4, Poland

E-mail address: dbura@math.uni.wroc.pl, edamek@math.uni.wroc.pl, zenek@math.uni.wroc.pl

T.Mikosch, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen, Denmark

E-mail address: mikosch@math.ku.dk