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Supplement to "Applications of Distance Correlation to Time Series"

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This supplement provides the omitted technical details to the proofs of Theorem 4.2 and Lemma 4.1, in Davis et al. (2016). The setting, notation, equation reference numbers are retained from the main paper. For convenience, the corresponding results in Davis et al. (2016) using the same reference numbers are stated before the proofs. The Remarks 3.4 and 3.9 from the main paper have also been duplicated here.

Proof of Theorem 4.2

For the convenience of the reader, we recall two remarks from Davis et al. (2016) which are mentioned in the proof.

Remark 3.4. *If (X_j) and (Y_j) are two independent iid sequences then the statement of Theorem 3.2 (1) remains valid if for some $\alpha \in (0,2]$, $\mathbb{E}[|X|^\alpha] + \mathbb{E}[|Y|^\alpha] < \infty$ and*

$$\int_{\mathbb{R}^{p+q}} (1 \wedge |s|^\alpha)(1 \wedge |t|^\alpha)\, \mu(ds, dt) < \infty\,. \tag{3.8}$$

Remark 3.9. *From the proof of Theorem 3.2 (the central limit theorem for the multivariate empirical characteristic function) it follows that G_h has covariance function*

$$\begin{aligned}\Gamma((s,t),(s',t')) &= \operatorname{cov}(G_h(s,t), G_h(s',t')) \\ &= \sum_{j\in\mathbb{Z}} \mathbb{E}\big[\big(\mathrm{e}^{\,i\,\langle s, X_0\rangle} - \varphi_X(s)\big)\big(\mathrm{e}^{\,i\,\langle t, Y_h\rangle} - \varphi_Y(t)\big) \\ &\qquad \times \big(\mathrm{e}^{-i\,\langle s', X_j\rangle} - \varphi_X(-s')\big)\big(\mathrm{e}^{-i\,\langle t', Y_{j+h}\rangle} - \varphi_Y(-t')\big)\big]\,. \end{aligned} \tag{3.9}$$

In the special case when (X_t) and (Y_t) are independent sequences G_h is the same across all h with covariance function

$$\Gamma((s,t),(s',t')) = \big(\varphi_X(s-s') - \varphi_X(s)\varphi_X(s')\big)\big(\varphi_Y(t-t') - \varphi_Y(t)\varphi_Y(t')\big)\,.$$

Since G_h is centered Gaussian its squared L^2-norm $\|G_h\|_\mu^2$ has a weighted χ^2-distribution; see Kuo (1975), Chapter 1. The distribution of $\|G_h\|_\mu^2$ is not tractable and therefore one needs resampling methods for determining its quantiles.

Theorem 4.2. *Consider a causal $\mathrm{AR}(p)$ process with iid noise (Z_t). Assume*

$$\int_{\mathbb{R}^2} \big[(1\wedge |s|^2)\,(1\wedge |t|^2)\,\mu(ds,dt) + (s^2+t^2)\,\mathbf{1}(|s|\wedge|t|>1)\mu(ds,dt) < \infty. \tag{4.7}$$

1. *If $\sigma^2 = \mathrm{Var}(Z) < \infty$, then*

$$n\,T_{n,\mu}^{\widehat{Z}}(h) \stackrel{d}{\to} \|G_h + \xi_h\|_\mu^2\,, \qquad \text{and} \qquad n\,R_{n,\mu}^{\widehat{Z}}(h) \stackrel{d}{\to} \frac{\|G_h+\xi_h\|_\mu^2}{T_\mu^Z(0)}\,, \tag{4.8}$$

where (G_h,ξ_h) are jointly Gaussian limit random fields on $\mathbb{R}^2$. The covariance structure of G_h is specified in Remark 3.9 above for the sequence $((Z_t,Z_{t+h}))$, ξ_h and the joint limit structure of (G_h,ξ_h) are given in Lemma 4.1 below.

2. *If Z is in the domain of attraction of a stable law of index $\alpha\in(0,2)$, i.e., $\mathbb{P}(|Z|>x) = x^{-\alpha}L(x)$ for $x>0$ and $L(\cdot)$ is a slowly varying function at ∞, and*

$$\frac{\mathbb{P}(Z>x)}{\mathbb{P}(|Z|>x)} \to p \quad \text{and} \quad \frac{\mathbb{P}(Z<-x)}{\mathbb{P}(|Z|>x)} \to 1-p$$

as $x\to\infty$ for some $p\in[0,1]$ (Feller (1971), p. 313). Then we have

$$n\,T_{n,\mu}^{\widehat{Z}}(h) \stackrel{d}{\to} \|G_h\|_\mu^2 \qquad \text{and} \qquad n\,R_{n,\mu}^{\widehat{Z}}(h) \stackrel{d}{\to} \frac{\|G_h\|_\mu^2}{T_\mu^Z(0)}\,, \tag{4.9}$$

where G_h is a Gaussian limit random field on $\mathbb{R}^2$. The covariance structure of G_h is specified in Remark 3.9 for the sequence $((Z_t,Z_{t+h}))$.

We proof the result for the residuals calculated from least square estimates (LSEs). One may show that the same result holds for maximum likelihood and Yule-Walker estimates. We recall the relevant asymptotic results for the least squares estimator from Section 4 in Davis et al. (2016); we use the same reference numbers for mathematical formulæ as in Davis et al. (2016). The least-squares estimator $\widehat{\boldsymbol{\phi}}$ of $\boldsymbol{\phi}$ satisfies the relation

$$\widehat{\boldsymbol{\phi}} - \boldsymbol{\phi} \;=\; \Gamma_{n,p}^{-1}\,\frac{1}{n}\sum_{t=p+1}^{n} \mathbf{X}_{t-1}\,Z_t\,,$$

where

$$\Gamma_{n,p} \quad = \quad \frac{1}{n} \sum_{t=p+1}^{n} \mathbf{X}_{t-1}^T \mathbf{X}_{t-1} \,.$$

If $\sigma^2 = \mathrm{var}(Z_t) < \infty$, we have by the ergodic theorem,

$$\Gamma_{n,p} \stackrel{\text{a.s.}}{\to} \Gamma_p = \big(\gamma_X(j-k)\big)_{1\le j,k\le p}\,, \qquad \text{where } \gamma_X(h) = \mathrm{cov}(X_0, X_h)\,, h \in \mathbb{Z}\,. \tag{4.1}$$

Causality of the process implies that the partial sum $\sum_{t=p+1}^{n} \mathbf{X}_{t-1}\, Z_t$ is a martingale and applying the martingale central limit theorem yields

$$\sqrt{n}\,\big(\widehat{\boldsymbol{\phi}} - \boldsymbol{\phi}\big) \stackrel{d}{\to} \mathbf{Q}\,, \tag{4.2}$$

where $\mathbf{Q}$ is $N(\mathbf{0}, \sigma^2 \Gamma_p^{-1})$ distributed.

Keeping this in mind, we start with a joint central limit theorem for C_n^Z and $\widehat{\boldsymbol{\phi}}$.

Lemma 4.1. *Consider an iid sequence (Z_t) with finite variance.*

1. *For every $h \ge 0$,*
$$\sqrt{n}\,\big(C_n^Z, \widehat{\boldsymbol{\phi}} - \boldsymbol{\phi}\big) \stackrel{d}{\to} (G_h, \mathbf{Q})\,,$$
where the convergence is in $\mathcal{C}(K) \times \mathbb{R}^p$, $K \subset \mathbb{R}^2$ is a compact set, G_h is the limit process of C_n^Z with covariance structure specified in Remark 3.9 for the sequence $((Z_t, Z_{t+h}))$, $\mathbf{Q}$ is the limit in (4.2), $(G_h, \mathbf{Q})$ are mean-zero and jointly Gaussian with covariance matrix
$$\mathrm{cov}(G_h(s,t), \mathbf{Q}) = -\varphi_Z'(s)\, \varphi_Z'(t)\, \Gamma_p^{-1} \Psi_h\,, \qquad s,t \in \mathbb{R}\,, \tag{4.4}$$
where $\Psi_h = (\psi_{h-j})_{j=1,\dots,p}$ and φ_Z' is the first derivative of φ_Z.
2. *For every $h \ge 0$,*
$$\sqrt{n}\,\big(C_n^Z, C_n^{\widehat{Z}} - C_n^Z\big) \stackrel{d}{\to} (G_h, \xi_h)\,,$$
where $(G_h, \mathbf{Q})$ are specified in (4.4) and
$$\xi_h(s,t) = t\varphi_Z(t)\, \varphi_Z'(s) \Psi_h^T \mathbf{Q}, \qquad (s,t) \in K\,, \tag{4.5}$$
the convergence is in $\mathcal{C}(K, \mathbb{R}^2)$, $K \subset \mathbb{R}^2$ is a compact set. In particular, we have
$$\sqrt{n}\, C_n^{\widehat{Z}} \stackrel{d}{\to} G_h + \xi_h\,. \tag{4.6}$$

Proof of part (1). We observe that, uniformly for $(s,t)\in K$,

$$\begin{aligned} C_n^Z(s,t) &= \frac{1}{n}\sum_{j=1}^{n-h} \mathrm{e}^{\,isZ_j+itZ_{j+h}} - \frac{1}{n}\sum_{j=1}^{n-h} \mathrm{e}^{\,isZ_j}\,\frac{1}{n}\sum_{j=1}^{n-h} \mathrm{e}^{\,itZ_{j+h}} \\ &= \frac{1}{n}\sum_{j=1}^{n} \big(\mathrm{e}^{\,isZ_j}-\varphi_Z(s)\big)\big(\mathrm{e}^{\,itZ_{j+h}}-\varphi_Z(t)\big) \\ &\quad -\frac{1}{n}\sum_{j=1}^{n} \big(\mathrm{e}^{\,isZ_j}-\varphi_Z(s)\big)\frac{1}{n}\sum_{j=1}^{n} \big(\mathrm{e}^{\,itZ_j}-\varphi_Z(t)\big) + O_{\mathbb{P}}(n^{-1})\,. \end{aligned}$$

In view of the functional central limit theorem for the empirical characteristic function of an iid sequence (see Csörgő (1981a,1981b)) we have uniformly for $(s,t)\in K$,

$$\begin{aligned} \sqrt{n}\,C_n^Z(s,t) &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n} \big(\mathrm{e}^{\,isZ_j}-\varphi_Z(s)\big)\big(\mathrm{e}^{\,itZ_{j+h}}-\varphi_Z(t)\big) + O_{\mathbb{P}}(n^{-1/2}) \\ &= I_n(s,t) + O_{\mathbb{P}}(n^{-1/2})\,. \end{aligned}$$

Therefore it suffices to study the convergence of the finite-dimensional distributions of $\big(I_n,\sqrt{n}\,(\widehat{\boldsymbol\phi}-\boldsymbol\phi)\big)$. In view of (4.1) it suffices to show the convergence of the finite-dimensional distributions of $\big(I_n,(1/\sqrt{n})\sum_{j=1}^n \mathbf{X}_{j-1}Z_j\big)$. This convergence follows by an application of the martingale central limit theorem and the Cramér-Wold device. It remains to determine the limiting covariance structure, taking into account the causality of the process (X_t). We have

$$\mathrm{cov}\Big(I_n,\frac{1}{\sqrt{n}}\sum_{j=1}^n \mathbf{X}_{j-1}Z_j\Big) = \frac{1}{n}\mathbb{E}\Big[\sum_{j=1}^n\sum_{k=1}^n \big(\mathrm{e}^{\,isZ_j}-\varphi_Z(s)\big)\big(\mathrm{e}^{\,itZ_{j+h}}-\varphi_Z(t)\big)\mathbf{X}_{k-1}Z_k\Big]\,.$$

By causality, X_k and Z_j are independent for $k<j$. Hence $\mathbb{E}[(\mathrm{e}^{\,isZ_j}-\varphi_Z(s))(\mathrm{e}^{\,itZ_{j+h}}-\varphi_Z(t))X_{l-k}Z_l]$ is non-zero if and only if $l=j+h$ and $k\le h$, resulting in

$$\begin{aligned} &\mathbb{E}\big[\big(\mathrm{e}^{\,isZ_j}-\varphi_Z(s)\big)\big(\mathrm{e}^{\,itZ_{j+h}}-\varphi_Z(t)\big)\,X_{l-k}\,Z_l\big] \\ &= \mathbb{E}\big[X_{j+h-k}\big(\mathrm{e}^{\,isZ_j}-\varphi_Z(s)\big)\big]\,\mathbb{E}\big[Z_{j+h}\big(\mathrm{e}^{\,itZ_{j+h}}-\varphi_Z(t)\big)\big] \\ &= \psi_{h-k}\,\mathbb{E}\big[Z\big(\mathrm{e}^{\,isZ}-\varphi_Z(s)\big)\big]\,\mathbb{E}\big[Z\big(\mathrm{e}^{\,itZ}-\varphi_Z(t)\big)\big] \\ &= -\psi_{h-k}\,i\mathbb{E}\big[Z\mathrm{e}^{\,isZ}\big]\,i\mathbb{E}\big[Z\mathrm{e}^{\,itZ}\big] \\ &= -\psi_{h-k}\,\varphi_Z'(s)\,\varphi_Z'(t)\,. \end{aligned}$$

This implies (4.4). □

Proof of part (2). We observe that, uniformly for $(s,t)\in K$,

$$\begin{aligned} & C_n^{\widehat{Z}}(s,t) - C_n^Z(s,t) && \text{(S.1)}\\ &= \frac{1}{n}\sum_{j=1}^n \mathrm{e}^{\,isZ_j+itZ_{j+h}}\left(\mathrm{e}^{\,i\,(\boldsymbol{\phi}-\widehat{\boldsymbol{\phi}})^T(s\mathbf{X}_{j-1}+t\mathbf{X}_{j+h-1})}-1\right)\\ &\quad+\frac{1}{n}\sum_{j=1}^n\left(1-\mathrm{e}^{\,i(\boldsymbol{\phi}-\widehat{\boldsymbol{\phi}})^T s\mathbf{X}_{j-1}}\right)\mathrm{e}^{\,isZ_j}\,\frac{1}{n}\sum_{j=1}^n \mathrm{e}^{\,itZ_{j+h}}\\ &\quad+\frac{1}{n}\sum_{j=1}^n \mathrm{e}^{\,i(\boldsymbol{\phi}-\widehat{\boldsymbol{\phi}})^T s\mathbf{X}_{j-1}+isZ_j}\,\frac{1}{n}\sum_{j=1}^n\left(1-\mathrm{e}^{\,i(\boldsymbol{\phi}-\widehat{\boldsymbol{\phi}})^T t\mathbf{X}_{j+h-1}}\right)\mathrm{e}^{\,itZ_{j+h}}+O_{\mathbb{P}}(n^{-1})\\ &= E_{n1}(s,t)+E_{n2}(s,t)+E_{n3}(s,t)+O_{\mathbb{P}}(n^{-1})\,. && \text{(S.2)}\end{aligned}$$

Write

$$\widetilde{E}_{n1}(s,t) \;=\; i\,(\boldsymbol{\phi}-\widehat{\boldsymbol{\phi}})^T\,\frac{1}{n}\sum_{j=1}^n (s\mathbf{X}_{j-1}+t\,\mathbf{X}_{j+h-1})\,\mathrm{e}^{\,isZ_j+itZ_{j+h}}\,.$$

In view of the uniform ergodic theorem, (4.2) and the causality of (X_t) we have

$$\begin{aligned}\sqrt{n}\widetilde{E}_{n1}(s,t) &\stackrel{d}{\to} -i\mathbf{Q}^T\mathbb{E}\big[(s\mathbf{X}_0+t\mathbf{X}_h)\,\mathrm{e}^{\,i(sZ_1+tZ_{h+1})}\big] && \text{(S.3)}\\ &= -t\varphi_Z(t)\varphi_Z'(s)\Psi_h^T\mathbf{Q}=\xi_h(s,t)\,,\end{aligned}$$

where the convergence is in $\mathcal{C}(K)$. By virtue of part (1) and the mapping theorem we have the joint convergence $\sqrt{n}(C_n^Z,\widetilde{E}_{n1})\stackrel{d}{\to}(G_h,\xi_h)$ in $\mathcal{C}(K,\mathbb{R}^2)$. Denoting the sup-norm in $\mathcal{C}(K)$ by $\|\cdot\|$, it remains to show that $\sqrt{n}\big(\|E_{n2}\|+\|E_{n3}\|+\|E_{n1}-\widetilde{E}_{n1}\|\big)\stackrel{\mathbb{P}}{\to}0$. The proof for E_{n2} and E_{n3} is analogous to (S.3) by observing that the limiting expectation is zero. We have by a Taylor expansion for some positive constant c,

$$\sqrt{n}\|E_{n1}(s,t)-\widetilde{E}_{n1}(s,t)\| \;\le\; c\,\big|\sqrt{n}(\boldsymbol{\phi}-\widehat{\boldsymbol{\phi}})\big|^2\sup_{(s,t)\in K}\frac{1}{n^{3/2}}\sum_{j=1}^n\big|s\mathbf{X}_{j-1}+t\mathbf{X}_{j+h-1}\big|^2\stackrel{\mathbb{P}}{\to}0\,.$$

In the last step we used the uniform ergodic theorem and (4.2). □

Proof of Theorem 4.2(1). We proceed as in the proof of Theorem 3.2. By virtue of (4.6) and the continuous mapping theorem we have

$$\int_{K_\delta}|\sqrt{n}\,C_n^{\widehat{Z}}(s,t)|^2\,\mu(ds,dt)\stackrel{d}{\to}\int_{K_\delta}|G(s,t)+\xi_h(s,t)|^2\,\mu(ds,dt)\,,\qquad n\to\infty\,.$$

Thus it remains to show that

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\mathbb{P}\Big(\int_{K_\delta^c}|\sqrt{n}C_n^{\hat{Z}}(s,t)|^2\mu(ds,dt)>\varepsilon\Big)=0\,,\qquad \varepsilon>0\,. \tag{S.4}$$

Following the lines of the proof of Theorem 3.2, we have

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\int_{K_\delta^c}\mathbb{E}[|\sqrt{n}C_n^Z(s,t)|^2]\,\mu(ds,dt)=0\,;$$

see also Remark 3.4. Thus it suffices to show

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\mathbb{P}\Big(\int_{K_\delta^c}|\sqrt{n}(C_n^{\hat Z}(s,t)-C_n^Z(s,t))|^2\mu(ds,dt)>\varepsilon\Big)=0\,,\qquad \varepsilon>0\,.$$

For convenience we redefine

$$C_n^Z=\frac{1}{n}\sum_{j=p+1}^{n-h}\mathrm{e}^{\,isZ_j+itZ_{j+h}}-\frac{1}{n}\sum_{j=p+1}^{n-h}\mathrm{e}^{\,isZ_j}\,\frac{1}{n}\sum_{j=p+1}^{n-h}\mathrm{e}^{\,itZ_{j+h}}\,.$$

This version does not change previous results for C_n^Z.

Using telescoping sums, we have for $\bar n=n-p-h$,

$$\begin{aligned}
&\frac{\bar n}{n}(C_n^{\hat Z}(s,t)-C_n^Z(s,t))\\
&=\frac{1}{\bar n}\sum_{j=p+1}^{n-h}A_jB_j-\frac{1}{\bar n}\sum_{j=p+1}^{n-h}A_j\frac{1}{\bar n}\sum_{j=p+1}^{n-h}B_j-\frac{1}{\bar n}\sum_{j=p+1}^{n-h}U_j\sum_{j=p+1}^{n-h}B_j-\frac{1}{\bar n}\sum_{j=p+1}^{n-h}V_j\sum_{j=p+1}^{n-h}A_j\\
&\quad+\frac{1}{\bar n}\sum_{j=p+1}^{n-h}U_jB_j+\frac{1}{\bar n}\sum_{j=p+1}^{n-h}V_jA_j=:\sum_{j=1}^{6}I_{nj}(s,t),
\end{aligned}$$

where, suppressing the dependence on s,t in the notation,

$$\begin{aligned}
U_j&=\mathrm{e}^{\,isZ_j}-\varphi_Z(s)\,,\qquad V_j=\mathrm{e}^{\,itZ_{j+h}}-\varphi_Z(t)\,,\\
A_j&=\mathrm{e}^{\,isZ_j}\,(\mathrm{e}^{is(\boldsymbol\phi-\widehat{\boldsymbol\phi})'X_{j-1}}-1)\,,\qquad B_j=\mathrm{e}^{\,itZ_{j+h}}\,(\mathrm{e}^{is(\boldsymbol\phi-\widehat{\boldsymbol\phi})'X_{j+h-1}}-1).
\end{aligned}$$

Write $K_n=|\sqrt{n}(\boldsymbol\phi-\widehat{\boldsymbol\phi})|$ and $c>0$ for any positive constant which may differ from line to line. By Taylor expansions we have

$$\begin{aligned}
&n\,|I_{n1}(s,t)|^2\le\Big(\frac{\sqrt{n}}{\bar n}\sum_{j=p+1}^{n-h}|A_jB_j|\Big)^2\\
&\le c\Big(\frac{\sqrt{n}}{\bar n}\sum_{j=p+1}^{n-h}(1\wedge|s|\,|\boldsymbol\phi-\widehat{\boldsymbol\phi}|\,|X_{j-1}|)\,(1\wedge|t|\,|\boldsymbol\phi-\widehat{\boldsymbol\phi}||X_{j+h-1}|)\Big)^2\\
&\le c\Big(\min\big(|s\,t|\,K_n^2\,\frac{1}{\bar n^{3/2}}\sum_{j=p+1}^{n-h}|X_{j-1}\,X_{j+h-1}|\,,\ |s|\,K_n\,\frac{1}{\bar n}\sum_{j=p+1}^{n-h}|X_{j-1}|\,,\\
&\qquad\qquad |t|\,K_n\,\frac{1}{\bar n}\sum_{j=p+1}^{n-h}|X_{j+h-1}|\big)\Big)^2.
\end{aligned}$$

The quantities K_n are stochastically bounded. From ergodic theory, $n^{-1}\sum_{j=1}^n |X_j| = O_{\mathbb{P}}(1)$ and $n^{-3/2}\sum_{j=1}^n |X_j\,X_{j+h}| = o_{\mathbb{P}}(1)$. Hence

$$n\,|I_{n1}(s,t)|^2 \;\le\; \min(s^2,t^2,(st)^2)\,O_{\mathbb{P}}(1) \;\le\; \big((1\wedge s^2)\,(1\wedge t^2)+(s^2+t^2)\mathbf{1}(|s|\wedge|t|\ge 1)\big)\;O_{\mathbb{P}}(1),$$

where the term $O_{\mathbb{P}}(1)$ does not depend on s and t. Thus we conclude for $k=1$ that

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\mathbb{P}\Big(n\int_{K_\delta^c}|I_{nk}(s,t)|^2\,\mu(ds,dt)>\varepsilon\Big)=0\,,\qquad \varepsilon>0\,. \tag{S.5}$$

A similar argument yields

$$\begin{aligned}
& n\,|I_{n2}(s,t)|^2\\
\le\;& \Bigg(\frac{\sqrt{n}}{\bar n^2}\sum_{j,k=p+1}^{n-h}|A_j|\,|B_k|\Bigg)^2\\
\le\;& \Bigg(\frac{\sqrt{n}}{\bar n^2}\sum_{j,k=p+1}^{n-h}(1\wedge|s|\,|\boldsymbol{\phi}-\widehat{\boldsymbol{\phi}}|\,|X_{j-1}|)(1\wedge|t|\,|\boldsymbol{\phi}-\widehat{\boldsymbol{\phi}}|\,|X_{k+h-1}|)\Bigg)^2\\
\le\;& c\Bigg(\min\Big(|st|\,K_n^2\,\frac{1}{\bar n^{5/2}}\sum_{j,k=p+1}^{n-h}|X_{j-1}\,X_{k+h-1}|,\\
&\qquad |s|\,K_n\,\frac{1}{\bar n}\sum_{j=p+1}^{n-h}|X_{j-1}|\,,\,|t|\,K_n\,\frac{1}{\bar n}\sum_{k=p+1}^{n-h}|X_{k+h-1}|\Big)\Bigg)^2\\
\le\;& \min(s^2,t^2,(st)^2)\,O_{\mathbb{P}}(1).
\end{aligned}$$

Then (S.5) holds for $k=2$. Taylor expansions also yield

$$\begin{aligned}
n\,|I_{n3}(s,t)|^2 \;&\le\; \Bigg(\frac{\sqrt{n}}{\bar n^2}\sum_{j,k=p+1}^{n-h}|U_j|\,|B_k|\Bigg)^2\\
&\le\; c\,\Bigg(\frac{\sqrt{n}}{\bar n^2}\sum_{j,k=p+1}^{n-h}(1\wedge\frac{1}{2}|s|\,(|Z_j|+\mathbb{E}|Z|))(1\wedge|t|\,|\boldsymbol{\phi}-\widehat{\boldsymbol{\phi}}|\,|X_{k+h-1}|)\Bigg)^2\\
&\le\; \min(t^2,(st)^2)\,O_{\mathbb{P}}(1).
\end{aligned}$$

This proves (S.5) for $k=3$. By a symmetry argument but with the corresponding bound

$\min(s^2,(st)^2)\,O_{\mathbb{P}}(1)$, (S.5) for $k=4$ follows as well. By Taylor expansion, we also have

$$\begin{aligned} n\,|I_{n5}(s,t)|^2 &\le \Big(\frac{\sqrt{n}}{\bar n}\sum_{j=p+1}^{n-h}|U_j|\,|B_j|\Big)^2 \\ &\le c\,\Big(\frac{\sqrt{n}}{\bar n}\sum_{j=p+1}^{n-h}(1\wedge\frac{1}{2}|s|\,(|Z_j|+\mathbb{E}|Z|))(1\wedge|t|\,|\boldsymbol{\phi}-\widehat{\boldsymbol{\phi}}|\,|X_{j+h-1}|)\Big)^2 \\ &\le \min(t^2,(st)^2)\,O_{\mathbb{P}}(1). \end{aligned}$$

We may conclude that (S.5) holds for $k=5$. The case $k=6$ follows in a similar way with the corresponding bound $\min(s^2,(st)^2)\,O_{\mathbb{P}}(1)$. □

Proof of Theorem 4.2(2). We follow the proof of Theorem 4.2(1) by first showing that

$$\sqrt{n}\,C_n^{\widehat{Z}}\xrightarrow{d}G_h \tag{S.6}$$

in $\mathcal{C}(K)$ for $K\subset\mathbb{R}^2$ compact, and then (S.4). The convergence $\sqrt{n}\,C_n^Z\xrightarrow{d}G_h$ in $\mathcal{C}(K)$ continues to hold as in the proof of Theorem 4.2(1) since the conditions in Csörgő (1981a,1981b) are satisfied if some moment of Z is finite. For (S.6) it suffices to show that

$$\sqrt{n}\,(C_n^{\widehat{Z}}-C_n^Z)\xrightarrow{p}0 \tag{S.7}$$

in $\mathcal{C}(K)$. Recalling the decomposition (S.2), we now can show directly that $\sup_{|s|,|t|\le M}\sqrt{n}|E_{ni}(s,t)|\xrightarrow{p}0$ for any $M>0$ and $i=1,2,3$, which implies (S.7). We focus only on the case $i=1$ to illustrate the method; the cases $i=2,3$ are analogous. We observe that for $\delta>0$,

$$\begin{aligned} \sup_{|s|,|t|\le M}\sqrt{n}|E_{n1}(s,t)| &\le \sup_{|s|,|t|\le M}\sqrt{n}|\boldsymbol{\phi}-\widehat{\boldsymbol{\phi}}|\,\frac{1}{n}\sum_{j=p+1}^{n-h}|s\mathbf{X}_{j-1}+t\,\mathbf{X}_{j+h-1}| \\ &\le M\,n^{\frac{1}{\delta}}|\boldsymbol{\phi}-\widehat{\boldsymbol{\phi}}|\;\;n^{-\frac{1}{\delta}-\frac{1}{2}}\sum_{j=1}^{n}|\mathbf{X}_j|\,. \end{aligned} \tag{S.8}$$

On the other hand, under the conditions of Theorem 4.2(2) Hannan and Kanter (1977) showed for $\delta>\alpha$,

$$n^{1/\delta}\,(\boldsymbol{\phi}-\widehat{\boldsymbol{\phi}})\xrightarrow{\text{a.s.}}0.$$

For $\alpha\in(1,2)$, $\mathbb{E}[|\mathbf{X}|]<\infty$ and since we can choose $\delta=2$ such that $1/\delta+1/2=1$. The ergodic theorem finally yields that the right-hand side in (S.8) converges to zero a.s. As regards the case $\alpha\in(0,1]$, we have $\mathbb{E}[|\mathbf{X}|^{\alpha-\gamma}]<\infty$ for any small γ and

$$\mathbb{E}\Big[\Big|n^{-1/\delta-1/2}\sum_{j=1}^{n}|\mathbf{X}_j|\Big|^{\alpha-\gamma}\Big]\le n^{-(\alpha-\gamma)(1/\delta+1/2)+1}\,\mathbb{E}[|\mathbf{X}|^{\alpha-\gamma}]\to 0.$$

If we choose δ close to α and γ close to zero the right-hand side in (S.8) converges to zero in probability.

Using the same bounds as in part (1), but writing this time $K_n = n^{1/\delta}|\boldsymbol{\phi} - \widehat{\boldsymbol{\phi}}|$, we have

$$\begin{aligned} n\,|I_{n1}(s,t)|^2 &\leq c\Big(\min\big(|s\,t|\,K_n^2\,n^{-1/2-2/\delta}\sum_{j=1}^n |X_{j-1}X_{j+h-1}|\,,|s|\,K_n\,n^{-1/\delta-1/2}\sum_{j=0}^n |X_j|, \\ &\qquad |t|\,K_n\,n^{-1/\delta-1/2}\sum_{j=0}^n |X_j|\big)\Big)^2 \\ &\leq c\,\min(\,|s\,t|^2,\,|s|^2,\,|t|^2\,)\,\max\Big(K_n^2\,n^{-1/2-2/\delta}\sum_{j=1}^n |X_{j-1}X_{j+h-1}|, \\ &\qquad K_n\,n^{-1/\delta-1/2}\sum_{j=0}^n |X_j|\Big)^2. \end{aligned}$$

The same argument as above shows that $n^{-1/\delta-1/2}\sum_{j=0}^n |X_j| = O_{\mathbb{P}}(1)$ for δ close to α. Since $2|X_{j-1}X_{j+h-1}| \leq X_{j-1}^2 + X_{j+h-1}^2$ a similar argument shows that $n^{-1/2-2/\delta}\sum_{j=1}^n |X_{j-1}X_{j+h-1}| = O_{\mathbb{P}}(1)$. These facts establish (S.5) for $k=1$. The same arguments show that bounds analogous to part (1) can be derived for $n\,|I_{nk}(s,t)|^2$ for $k=2,\dots,6$. We omit further details. □

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