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# LARGE DEVIATIONS AND RUIN PROBABILITIES FOR SOLUTIONS TO STOCHASTIC RECURRENCE EQUATIONS WITH HEAVY-TAILED INNOVATIONS

BY DIMITRIOS G. KONSTANTINIDES AND THOMAS MIKOSCH[1]

*University of the Aegean and University of Copenhagen*

In this paper we consider the stochastic recurrence equation $Y_t = A_t Y_{t-1} + B_t$ for an i.i.d. sequence of pairs $(A_t, B_t)$ of nonnegative random variables, where we assume that $B_t$ is regularly varying with index $\kappa > 0$ and $EA_t^\kappa < 1$. We show that the stationary solution $(Y_t)$ to this equation has regularly varying finite-dimensional distributions with index $\kappa$. This implies that the partial sums $S_n = Y_1 + \cdots + Y_n$ of this process are regularly varying. In particular, the relation $P(S_n > x) \sim c_1 n P(Y_1 > x)$ as $x \to \infty$ holds for some constant $c_1 > 0$. For $\kappa > 1$, we also study the large deviation probabilities $P(S_n - ES_n > x)$, $x \ge x_n$, for some sequence $x_n \to \infty$ whose growth depends on the heaviness of the tail of the distribution of $Y_1$. We show that the relation $P(S_n - ES_n > x) \sim c_2 n P(Y_1 > x)$ holds uniformly for $x \ge x_n$ and some constant $c_2 > 0$. Then we apply the large deviation results to derive bounds for the ruin probability $\psi(u) = P(\sup_{n\ge 1}((S_n - ES_n) - \mu n) > u)$ for any $\mu > 0$. We show that $\psi(u) \sim c_3 u P(Y_1 > u)\mu^{-1}(\kappa - 1)^{-1}$ for some constant $c_3 > 0$. In contrast to the case of i.i.d. regularly varying $Y_t$'s, when the above results hold with $c_1 = c_2 = c_3 = 1$, the constants $c_1$, $c_2$ and $c_3$ are different from 1.

## 1. Introduction.

The stochastic recurrence equation

$$Y_t = A_t Y_{t-1} + B_t, \qquad t \in \mathbb{Z}, \tag{1.1}$$

and its stationary solution have attracted much attention over the last years. Here $((A_t, B_t))$ is an i.i.d. sequence of pairs of nonnegative random variables $A_t$ and $B_t$. [In what follows, we write $A, B, Y, \dots,$ for generic elements of the stationary sequences $(A_t)$, $(B_t)$, $(Y_t)$, etc. We also write $c$ for any positive constant whose value is not of interest.]

Major applications of stochastic recurrence equations are in financial time series analysis. For example, the squares of the GARCH process can be embedded in

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a stochastic recurrence equation of type (1.1); we refer to Section 8.4 in [15] for an introduction to stochastic recurrence equations and [1] and [23] for recent surveys on the mathematics of GARCH models, their properties and relation with stochastic recurrence equations. The stochastic recurrence equation approach has also proved useful for the estimation of GARCH and related models; see [27, 37, 38]. In a financial or insurance context, the stochastic recurrence equation (1.1) has natural interpretations. For example, $B_t$ can be considered as annual payment and $A_t$ as a discount factor. The value $Y_t$ is then the aggregated value of past discounted payments. In a life insurance context, $(Y_t)$ is referred to as a perpetuity; see, for example, [14]. Stochastic recurrence equations have also been used to describe evolutions in biology; see [2] and the references therein.

It will be convenient to use the notation

$$\Pi_{s,t} = \begin{cases} A_s, \dots, A_t, & s \le t, \\ 1, & s > t, \end{cases} \quad \Pi_t = \Pi_{1,t}.$$

It is well known [5] that, under the assumptions $E \log^+ A < \infty$ and $E \log^+ B < \infty$, (1.1) has a unique strictly stationary ergodic causal solution $(Y_t)$ [i.e., $Y_t$ is a function only of $(A_s, B_s), s \le t$] if and only if

$$-\infty \le E \log A < 0. \tag{1.2}$$

In what follows, we always assume these conditions to be satisfied. The stationary solution has representation

$$Y_t = \sum_{i=-\infty}^{t} \Pi_{i+1,t} B_i = B_t + \sum_{i=-\infty}^{t-1} \Pi_{i+1,t} B_i, \qquad t \in \mathbb{Z}. \tag{1.3}$$

We say that any nonnegative random variable $Z$ and its distribution are regularly varying with index $\kappa$ if its right tail is of the form

$$P(Z > x) = \frac{L(x)}{x^\kappa}, \qquad x > 0,$$

for some $\kappa \ge 0$ and a slowly varying function $L$. A result of Kesten [21] shows that the stationary solution to the stochastic recurrence equation (1.1) has regularly varying distribution, under quite general conditions on $A$ and $B$. We cite this benchmark result for comparison with the results we obtain in this paper.

THEOREM 1.1 (Kesten [21]). *Assume that the following conditions hold*:

- *For some* $\epsilon > 0$, $EA^\epsilon < 1$.
- *The set*

  $$\{\log(a_n \cdots a_1)\colon n \ge 1, a_n \cdots a_1 > 0 \text{ and } a_n, \dots, a_1 \in \text{the support of } P_A\}$$

  *generates a dense group in* $\mathbb{R}$ *with respect to summation and the Euclidean topology. Here* $P_A$ *denotes the distribution of* $A$.

- *There exists* $\kappa_0 > 0$ *such that*

$$EA^{\kappa_0} \geq 1, \tag{1.4}$$

*and* $E(A^{\kappa_0} \log^+ A) < \infty$.

*Then the following statements hold*:

1. *There exists a unique solution* $\kappa \in (0, \kappa_0]$ *to the equation*

$$EA^{\kappa} = 1.$$

2. *If* $EB^{\kappa} < \infty$, *there exists a unique strictly stationary ergodic causal solution* $(Y_t)$ *to the stochastic recurrence equation* (1.1) *with representation* (1.3).
3. *If* $EB^{\kappa} < \infty$, *then* $Y$ *is regularly varying with index* $\kappa > 0$. *In particular, there exists* $c > 0$ *such that*

$$P(Y > x) \sim cx^{-\kappa}, \qquad x \to \infty.$$

Condition (1.4) is crucial. Goldie and Grübel [17] show that $P(Y > x)$ can decay exponentially fast to zero if (1.4) is not satisfied. Notice that (1.4) ensures that the support of $A$ is spread out sufficiently far.

The set-up of this paper is different from the one in Kesten's Theorem 1.1. The latter result is surprising insofar that a light-tailed distribution of $A$ (such as the exponential or the truncated normal distribution) can cause the stationary solution $(Y_t)$ to (1.1) to have a marginal distribution with Pareto-like tails. In this paper we consider the case when $B$ is regularly varying with index $\kappa$ and $A$ has a lighter right tail than $B$. In this case the conditions of Kesten's theorem are not met. In particular, we always assume that $EA^{\kappa} < 1$. The marginal distribution of the stationary solution $(Y_t)$ turns out to be regularly varying with the same index $\kappa$ as the innovations $B_t$.

It is the objective of this paper to study the interplay of the regular variation of $Y$ and the particular dependence structure of the $Y_t$'s with respect to the partial sums

$$S_n = Y_1 + \cdots + Y_n, \qquad n \geq 1.$$

Due to (multivariate) regular variation of the finite-dimensional distributions of $(Y_t)$, $S_n$ is regularly varying with index $\kappa$, and we establish the precise tail asymptotics for $P(S_n > x)$ for fixed $n$ and as $x \to \infty$. We will see that, in contrast to i.i.d. regularly varying random variables $Y_t$ (cf. Lemma 1.3.1 in [15]), the relation

$$\lim_{x\to\infty} \frac{P(S_n > x)}{nP(Y > x)} = 1, \qquad n \geq 2,$$

does not hold for the stationary solution $(Y_t)$ to (1.1), neither under the conditions of Kesten's theorem nor under the conditions imposed in this paper; see Section 3.3. We will show in Proposition 3.3 that

$$\begin{aligned}\lim_{x\to\infty}&\frac{P(S_n>x)}{P(Y>x)}\\ &=E\left(\sum_{i=1}^{n}\Pi_i\right)^{\kappa}+(1-EA^{\kappa})\sum_{t=0}^{n-1}E\left(\sum_{i=0}^{t}\Pi_i\right)^{\kappa},\qquad n\geq 2.\end{aligned}\tag{1.5}$$

A question which is closely related to (1.5) concerns the *large deviations* of the partial sum process $(S_n)$. In this case, one is interested in the asymptotic behavior of the tail $P(S_n > x_n)$ for real sequences $(x_n)$ increasing to infinity sufficiently fast. Classical results (see, e.g., [8, 29, 30]; cf. the surveys in Section 8.6 in [15] and [24]) say that, for i.i.d. $(Y_t)$ and thresholds $x_n \to \infty$, the relation

$$\begin{aligned}P(S_n>x_n)&\sim nP(Y>x_n)\\ &\sim P\bigl(\max(Y_1,\dots,Y_n)>x_n\bigr)\end{aligned}\tag{1.6}$$

holds. For reasons of comparison, we quote a general large deviation result for i.i.d. random variables.

THEOREM 1.2. *Assume that $B>0$ is regularly varying with index $\kappa>0$.*

1. ([29, 30]) *Assume that $\kappa>2$. Then*

$$P\left(\sum_{t=1}^{n}(B_t-EB)>x\right)=\overline{\Phi}(x/\sqrt{n})\bigl(1+o(1)\bigr)+nP(B>x)\bigl(1+o(1)\bigr),$$

*as $n\to\infty$ and uniformly for $x\geq\sqrt{n}$, where $\overline{\Phi}=1-\Phi$ is the right tail of the standard normal distribution function $\Phi$. In particular,*

$$P\left(\sum_{t=1}^{n}(B_t-EB)>x\right)=\overline{\Phi}(x/\sqrt{n})\bigl(1+o(1)\bigr)$$

*uniformly for $\sqrt{n}\leq x\leq\sqrt{an\log n}$ and $a<\kappa-2$, and*

$$P\left(\sum_{t=1}^{n}(B_t-EB)>x\right)=nP(B>x)\bigl(1+o(1)\bigr)$$

*uniformly for $x\geq\sqrt{an\log n}$ and $a>\kappa-2$.*

2. ([8]) *Assume that $\kappa\in(1,2)$. Then*

$$P\left(\sum_{t=1}^{n}(B_t-EB)>x\right)=nP(B>x)\bigl(1+o(1)\bigr),\tag{1.7}$$

*as $n\to\infty$ and uniformly for $x\geq a_nc_n$, where $(a_n)$ satisfies $nP(B>a_n)\sim 1$ and $(c_n)$ is any sequence satisfying $c_n\to\infty$.*

The uniformity of these large deviation results refers to the fact that the error bounds hold uniformly for the indicated $x$-regions. For example, in the case $\kappa \in (1,2)$, (1.7) means that

$$\lim_{n\to\infty} \sup_{x \ge x_n} \left| \frac{P(\sum_{t=1}^n (B_t - EB) > x)}{nP(B > x)} - 1 \right| = 0, \tag{1.8}$$

where $x_n = a_n c_n$.

We will show in Theorem 4.2 that the following analog to Theorem 1.2 holds, under the more restrictive condition that $(A_t)$ and $(B_t)$ are independent:

$$\lim_{n\to\infty} \sup_{x \ge x_n} \left| \frac{P(S_n - ES_n > x)}{nP(Y > x)} - (1 - EA^\kappa) E\left(\sum_{i=0}^\infty \Pi_i\right)^\kappa \right| = 0. \tag{1.9}$$

The question about large deviations is closely related to the *ruin probability* of the random walk $(S_n)$. Given that $EY < \infty$, this is the probability

$$\psi(u) = P\left(\sup_{n\ge 1}[(S_n - ES_n) - \mu n] > u\right), \qquad u, \mu > 0.$$

It is one of the very well studied objects of applied probability theory, starting with classical work by Cramér in the 1930s. For i.i.d. regularly varying and, more generally, subexponential $Y_t$'s, the asymptotic behavior of $\psi(u)$ as $u \to \infty$ was studied by various authors; see Chapter 1 in [15]. The following benchmark result is classical in the context of ruin for heavy-tailed distributions. We cite it here for comparison with the results of this paper.

THEOREM 1.3. *Assume that $B$ is regularly varying with index $\kappa > 1$. Then for any $\mu > 0$,*

$$P\left(\sup_{n\ge 1}\left(\sum_{t=1}^n (B_t - EB) - \mu n\right) > u\right) \sim \frac{1}{\mu}\frac{1}{\kappa - 1} u P(B > u), \qquad u \to \infty.$$

In Theorem 4.9 we prove an analogous result for $(Y_t)$:

$$\psi(u) \sim \frac{1}{\mu}\frac{1}{\kappa - 1}(1 - EA^\kappa) E\left(\sum_{i=0}^\infty \Pi_i\right)^\kappa u P(Y > u). \tag{1.10}$$

The results of this paper are derived by applications of the *heavy-tailed large deviations heuristics*. In the case of i.i.d. $Y_t$'s, this means that a large deviation of the random walk $S_n$ from its mean $ES_n$ must be due to exactly one unusually large value $Y_t$, whereas the $Y_s$'s for $s \ne t$ are small compared to $Y_t$. We refer to [35] for a review on these heuristics which can be exploited in the context of various applied probability models. For dependent $Y_t$'s, as considered in this paper, the large deviations heuristics has to be combined with the understanding of the dependence structure of the random walk $S_n$ exceeding high thresholds. In the proof of the ruin probability result, it turns out that the ruin probability of the random walk $(S_n)$ behaves very much like the ruin probability of the random walk $\sum_{t=1}^{n} B_t C_t$, where $C_t = \sum_{i=t}^{\infty} \Pi_{t+1,i}$, $t \in \mathbb{Z}$. This is another stationary sequence, but, under the conditions of this paper, its marginal distributions have tails less heavy than $(B_t)$. Since we assume independence of $(A_t)$ and $(B_t)$, hence, of $(C_t)$ and $(B_t)$, in Section 4.2, it is likely that a large value of $S_n$ is now caused by a large value $B_t C_t$, which in turn is caused by a large value of $B_t$. We make this intuition precise by showing (1.10).

The results (1.5) on the tail of $S_n$ for fixed $n$, (1.9) on the large deviations of $(S_n)$ and (1.10) on the ruin probability of $(S_n)$ and their analogs for i.i.d. $Y_t$'s illustrate some crucial differences between the behavior of a random walk with dependent and independent heavy-tailed step sizes far away from the origin. The constants on the right-hand sides of (1.5), (1.9) and (1.10), which differ from those in the case of i.i.d. regularly varying $Y_t$'s, can be considered as alternative measures of the extremal clustering behavior of the $Y_t$'s. Similar results were obtained only for a few classes of stationary processes $(Y_t)$. Those include results by Mikosch and Samorodnitsky [25, 26] on large deviations and ruin for random walks with step sizes which constitute a linear process with regularly varying innovations or a stationary ergodic stable process, and by Davis and Hsing [9] on large deviations for random walks with infinite variance regularly varying step sizes. So far the known results do not allow one to draw a general picture which would allow one to classify stationary sequences of regularly varying random variables $Y_t$ with respect to their extremal behavior of the random walk with negative drift $((S_n - ES_n) - \mu n)$. The cited results and also those of the present paper are steps in the search for appropriate measures of extremal dependence in a stationary sequence by studying the behavior of suitable functionals acting on the sequence.

The paper is organized as follows. In Section 2 we give conditions under which the stationary solution $(Y_t)$ to the stochastic recurrence equation (1.1) has regularly varying finite-dimensional distributions. In Section 3 we consider applications of this property to the weak convergence of related point processes, the central limit theorem of $(S_n)$ and the partial maxima of $(Y_t)$. In Section 4.1 we study the large deviations of $(S_n)$ and in Section 4.2 we give our main result on the asymptotic behavior of the ruin probability $\psi(u)$. Since the proofs of the main results are quite technical, we postpone them to particular sections at the end of the paper. The proof of Theorem 4.2 will be given in Section 5 and the one of Theorem 4.9 in Section 6.

## 2. Regular variation of the solution to the stochastic recurrence equation.

2.1. *Preliminaries.* We start with some auxiliary results in order to establish regular variation of $Y$. In what follows, we write $\overline{F}(x) = 1 - F(x)$ for the right tail of any distribution function $F$.

LEMMA 2.1 (Davis and Resnick [12]). *Let $F$ be a distribution function concentrated on $(0,\infty)$. Assume $Z_1,\dots,Z_n$ are independent nonnegative random variables satisfying*

$$\lim_{x\to\infty}\frac{P(Z_i>x)}{\overline{F}(x)}=c_i \tag{2.1}$$

*for some nonnegative finite values $c_i$, where $F(x)=P(Z_1\le x)$, and*

$$\lim_{x\to\infty}\frac{P(Z_i>x,Z_j>x)}{\overline{F}(x)}=0,\qquad i\neq j. \tag{2.2}$$

*Then*

$$\lim_{x\to\infty}\frac{P(Z_1+\cdots+Z_n>x)}{\overline{F}(x)}=c_1+\cdots+c_n.$$

We will frequently make use of the following elementary property which was proved by Breiman [7] in a special case. We refer to it as *Breiman's result* and prove a uniform version of it for further use.

LEMMA 2.2 (Breiman [7]). *Let $\xi,\eta$ be independent nonnegative nondegenerate random variables such that $\xi$ is regularly varying with index $\kappa>0$ and $E\eta^{\kappa+\varepsilon}<\infty$ for some $\varepsilon>0$. Then for any sequence $x_n\to\infty$,*

$$\lim_{n\to\infty}\sup_{x\ge x_n}\left|\frac{P(\xi\eta>x)}{P(\xi>x)}-E\eta^\kappa\right|=0.$$

This means that the product $\xi\eta$ inherits regular variation from $\xi$.

PROOF. Fix $M>0$. Then

$$\begin{aligned}\Delta(x)&=\frac{P(\xi\eta>x)}{P(\xi>x)}-E\eta^\kappa\\&=\int_{[0,M]}\left[\frac{P(\xi y>x)}{P(\xi>x)}-y^\kappa\right]dP(\eta\le y)\\&\quad-E\eta^\kappa I_{(M,\infty)}(\eta)+\int_{(M,\infty)}\frac{P(\xi y>x)}{P(\xi>x)}\,dP(\eta\le y)\\&=\Delta_1(x)-\Delta_2+\Delta_3(x).\end{aligned}$$

Obviously,

$$\lim_{M\to\infty}\Delta_2=0.$$

Moreover, the uniform convergence theorem for regularly varying functions (see [4]) implies that, for every fixed $M > 0$,

$$\begin{aligned}\sup_{x\ge x_n}|\Delta_1(x)| &\le \sup_{x\ge x_n}\int_{[0,M]}\left|\frac{P(\xi y > x)}{P(\xi > x)} - y^\kappa\right| dP(\eta \le y)\\ &\le \sup_{x\ge x_n}\sup_{y\le M}\left|\frac{P(\xi y > x)}{P(\xi > x)} - y^\kappa\right| \to 0.\end{aligned}$$

An application of the Potter bounds for regularly varying functions (see [4], page 25) yields, for $x, x/y \ge x_0$, for sufficiently large $x_0 > 0$ and all $y > M > 1$, that

$$\frac{P(\xi > x/y)}{P(\xi > x)} \le y^{\kappa+\varepsilon}.$$

Hence,

$$\begin{aligned}\sup_{x\ge x_n}|\Delta_3(x)| &\le \sup_{x\ge x_n}\int_{M<y\le x/x_0} y^{\kappa+\varepsilon}\, dP(\eta\le y) + \sup_{x\ge x_n}\frac{P(\eta > x/x_0)}{P(\xi > x)}\\ &\to 0\end{aligned}$$

by first letting $n \to \infty$ and then $M \to \infty$, since $E\eta^{\kappa+\varepsilon} < \infty$. This proves the lemma. $\square$

We now turn to the stochastic recurrence equation (1.1). After $n$ iterations, we obtain

$$Y_n = \Pi_n Y_0 + \sum_{t=1}^{n} \Pi_{t+1,n} B_t. \tag{2.3}$$

As in Section 1, we assume that $((A_t, B_t))$ is an i.i.d. sequence of pairs of nonnegative random variables $A_t$ and $B_t$. In addition, suppose that $B$ is regularly varying with index $\kappa > 0$ and $EA^{\kappa+\delta} < \infty$ for some $\delta > 0$. Then Breiman's result (Lemma 2.2) applies:

$$\frac{P(\Pi_{i-1}B_i > x)}{P(B > x)} \sim (EA^\kappa)^{i-1} \qquad \text{as } x \to \infty. \tag{2.4}$$

The following result will be crucial for the property of regular variation of the finite-dimensional distributions of the stationary solution $(Y_n)$ to (1.1). For its formulation, we assume that $Y_0 = c$ in (2.3) for some constant $c$. We use the same notation $(Y_n)$ in this case, slightly abusing notation since $(Y_n)$ is then not the stationary solution to (1.1).

PROPOSITION 2.3. *Assume $B$ is regularly varying with index $\kappa > 0$ and $EA^{\kappa+\delta} < \infty$ for some $\delta > 0$. Then the following relation holds for fixed $n \ge 1$*

*and $Y_n$ defined in* (2.3) *with $Y_0 = c$*:

$$P(Y_n > x) \sim P(B > x) \sum_{i=0}^{n-1} (EA^\kappa)^i \qquad as\ x \to \infty.$$

PROOF. We write

$$Z_0 = \Pi_n c, \qquad Z_t = \Pi_{t-1} B_t, \qquad t = 1, \dots, n.$$

Observe that

$$Y_n = \Pi_n c + \sum_{t=1}^n \Pi_{t+1,n} B_t \stackrel{d}{=} \Pi_n c + \sum_{t=1}^n \Pi_{t-1} B_t = \sum_{t=0}^n Z_t.$$

We have, for $1 \le i < j \le n$,

$$P(Z_i > x, Z_j > x) \le P\big(\Pi_{i-1} \min(B_i, \Pi_{i,j-1} B_j) > x\big).$$

Since $EA^{\kappa+\delta} < \infty$ and $B$ is regularly varying with index $\kappa$, we can find a function $g(x) \to \infty$ such that $g(x)/x \to 0$, and $P(\max(A_i, \Pi_i) > g(x)) = o(P(B > x))$. Hence, for $i < j$,

$$\begin{aligned}
&\frac{P(Z_i > x, Z_j > x)}{P(B > x)} \\
&\quad \le \frac{P(\Pi_{i-1} \min(B_i, \Pi_{i,j-1} B_j) > x, \max(A_i, \Pi_i) > g(x))}{P(B > x)} \\
&\qquad + \frac{P(\Pi_{i-1} \min(B_i, \Pi_{i,j-1} B_j) > x, \max(A_i, \Pi_i) \le g(x))}{P(B > x)} \\
&\quad \le \frac{P(\max(A_i, \Pi_i) > g(x))}{P(B > x)} + \frac{P(\Pi_{i-1} B_i > x, \Pi_{i+1,j-1} B_j > x/g(x))}{P(B > x)} \\
&\quad = o(1) + (EA^\kappa)^{j-2} P\big(B > x/g(x)\big)\big(1 + o(1)\big) \to 0.
\end{aligned}$$

In the last step we made multiple use of Breiman's result and the independence of $\Pi_{i-1} B_i$ and $\Pi_{i+1,j-1} B_j$. By Markov's inequality, we also have, for $1 \le i \le n$,

$$\frac{P(Z_0 > x, Z_i > x)}{P(B > x)} \le \frac{P(Z_0 > x)}{P(B > x)} \le c^n \frac{(EA^{\kappa+\delta})^n x^{-\kappa-\delta}}{P(B > x)} \to 0.$$

Hence, we are in the framework of Lemma 2.1 with $c_0 = 0$ and $c_i = (EA^\kappa)^{i-1}$, $i = 1, \dots, n$; see (2.4). This proves the proposition. $\square$

2.2. *Univariate regular variation of $Y$.* In this section we indicate regular variation of the marginal distribution of the stationary solution to the stochastic recurrence equation (1.1). From Proposition 2.3 and the representation (1.3) of the stationary solution $(Y_t)$, we conclude that

$$\liminf_{x\to\infty}\frac{P(Y>x)}{P(B>x)}\geq\lim_{x\to\infty}\frac{P(\sum_{i=1}^{n}\Pi_{i-1}B_i>x)}{P(B>x)}=\sum_{i=0}^{n-1}(EA^{\kappa})^i. \tag{2.5}$$

Letting $n\to\infty$ yields a lower bound for $P(Y>x)$. This relation suggests that

$$P(Y>x)\sim P(B>x)\sum_{i=0}^{\infty}(EA^{\kappa})^i,\qquad x\to\infty, \tag{2.6}$$

holds under the conditions that $B$ is regularly varying with index $\kappa>0$ and $EA^{\kappa}<1$. Obviously, only if the latter condition holds, relation (2.6) is meaningful. This also means that the conditions of Kesten's Theorem 1.1 cannot be satisfied. In that case, the index of regular variation $\kappa$ of $Y$ satisfies $EA^{\kappa}=1$ and $EB^{\kappa}<\infty$. Since in our case $B$ is assumed to be regularly varying with index $\kappa$, the moment condition on $B$ is not necessarily met either.

PROPOSITION 2.4 (Grey [18]). *Assume that $B$ is regularly varying with index $\kappa>0$, $EA^{\kappa+\delta}<\infty$ for some $\delta>0$ and $EA^{\kappa}<1$. Then a unique strictly stationary solution $(Y_t)$ to the stochastic recurrence equation* (1.1) *exists and satisfies*

$$P(Y>x)\sim P(B>x)(1-EA^{\kappa})^{-1}. \tag{2.7}$$

PROOF. The function $g(h)=EA^h$ satisfies $g(0)=1$, $g(\kappa)<1$ and it is continuous and convex in $[0,\kappa]$. Therefore, $g'(0+)=E\log A<0$ and (1.2) and $E\log^+A<\infty$ hold. Moreover, since $EB^{\gamma}<\infty$ for $\gamma<\kappa$, $E\log^+B<\infty$ is satisfied and, hence, a unique stationary solution $(Y_t)$ to (1.1) exists.

Relation (2.7) follows from Theorem 1 in [18]. □

2.3. *Regular variation of the finite-dimensional distributions of* $(Y_t)$. In what follows, we assume that the conditions of Proposition 2.4 are satisfied. The latter result states that the marginal distribution of the stationary sequence $(Y_n)$ is regularly varying with the same index $\kappa$ as the innovations $B_t$. It is the aim of this section to extend this result to the finite-dimensional distributions of the process $(Y_t)$.

For this reason, we introduce the notion of regular variation for an $m$-dimensional random vector: the vector $\mathbf{Y}\in\mathbb{R}^m$ is regularly varying with index $\kappa>0$ if there exists a nonnull Radon measure $\mu$ on the Borel $\sigma$-field $\mathcal{B}$ of $[0,\infty]^m\setminus\{\mathbf{0}\}$ such that

$$nP(a_n^{-1}\mathbf{Y}\in\cdot)\stackrel{v}{\to}\mu.$$

Here the sequence $(a_n)$ satisfies $P(|\mathbf{Y}|>a_n)\sim n^{-1}$, $\stackrel{v}{\to}$ denotes vague convergence in $\mathcal{B}$, and $\mu$ is a measure with the property $\mu(t\cdot)=t^{-\kappa}\mu(\cdot)$ for all $t>0$;

see [32] for an introduction to regular variation, related point process convergence and vague convergence. An equivalent way to characterize the limiting measure $\mu$ is via a presentation in spherical coordinates. This means that, for every fixed $t > 0$ and $(a_n)$ as above,

$$nP(|\mathbf{Y}| > ta_n, \mathbf{Y}/|\mathbf{Y}| \in \cdot) \stackrel{v}{\to} t^{-\kappa} P(\boldsymbol{\Theta} \in \cdot),$$

where $|\cdot|$ is any fixed norm, $\stackrel{v}{\to}$ refers to vague convergence on the Borel $\sigma$-field of the unit sphere $\mathbb{S}^{d-1}$ corresponding to this norm and $\boldsymbol{\Theta}$ is a vector with values in $\mathbb{S}^{d-1}$. Its distribution is referred to as the *spectral distribution* of $\mathbf{Y}$.

For fixed $m \ge 1$, we have

$$\begin{aligned}
\mathbf{Y}_m &= (Y_1, \dots, Y_m)' \\
&= (\Pi_1, \Pi_2, \dots, \Pi_m)' Y_0 + \left( B_1, B_2 + A_2 B_1, \dots, B_m + \sum_{i=1}^{m-1} \Pi_{i+1,m} B_i \right)' \\
&= \begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \\ \vdots \\ \Pi_{m-1} \\ \Pi_m \end{pmatrix} Y_0 + \begin{pmatrix} 1 \\ \Pi_{2,2} \\ \Pi_{2,3} \\ \vdots \\ \Pi_{2,m-1} \\ \Pi_{2,m} \end{pmatrix} B_1 + \begin{pmatrix} 0 \\ 1 \\ \Pi_{3,3} \\ \vdots \\ \Pi_{3,m-1} \\ \Pi_{3,m} \end{pmatrix} B_2 + \cdots + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} B_m \\
&=: \mathbf{A}_0 Y_0 + \mathbf{A}_1 B_1 + \cdots + \mathbf{A}_m B_m.
\end{aligned}$$

Notice that $\mathbf{A}_0$ and $Y_0$ are independent, and so are $\mathbf{A}_i$ and $B_i$ for every $i$. Since $E|\mathbf{A}_i|^{\kappa+\delta} < \infty$ for some $\delta > 0$ and $Y_0, B_1, \dots, B_m$ are independent and regularly varying with index $\kappa$, a multivariate version of Breiman's result (cf. [1, 33]) applies to conclude that each of the vectors $\mathbf{A}_0 Y_0, \mathbf{A}_1 B_1, \dots, \mathbf{A}_m B_m$ is regularly varying with index $\kappa$ with corresponding limiting measures $\mu_0, \dots, \mu_m$. We mention that the normalizing sequences for these vectors are of the same size since, by the one-dimensional Breiman result and Proposition 2.4, as $x \to \infty$,

$$\begin{aligned}
&P(|\mathbf{A}_0 Y_0| > x) \sim E|\mathbf{A}_0|^\kappa P(Y_0 > x) \sim E|\mathbf{A}_0|^\kappa (1 - EA^\kappa)^{-1} P(B > x), \\
&P(|\mathbf{A}_i B_i| > x) \sim E|\mathbf{A}_i|^\kappa P(B > x), \qquad i = 1, \dots, m.
\end{aligned}$$

We choose one normalizing sequence $(a_n)$ for all vectors such that $nP(|\mathbf{A}_0| Y_0 > a_n) \sim 1$. We can characterize $\mu_i$ via its spectral distribution. Indeed, by Breiman's result, we have, for any Borel set $S \subset \mathbb{S}^{d-1}$ whose boundary has mean zero with respect to the spectral distribution,

$$nP(|\mathbf{A}_i| B_i > ta_n, \mathbf{A}_i/|\mathbf{A}_i| \in S) \sim t^{-\kappa} nP(B > a_n) E[|\mathbf{A}_i|^\kappa I_S(\mathbf{A}_i/|\mathbf{A}_i|)],$$

and, therefore, the spectral distribution of $\mathbf{A}_i Y_i$ for these sets $S$ is given by

$$\frac{E[|\mathbf{A}_i|^\kappa I_S(\mathbf{A}_i/|\mathbf{A}_i|)]}{E|\mathbf{A}_i|^\kappa}.$$

Adapting the proof of Lemma 2.1 in [12] to the multivariate case, it follows that $\mathbf{Y}_m$ is regularly varying with index $\kappa$ and limiting measure

$$\mu(d\mathbf{x}) = \mu_0(d\mathbf{x}) + c_1\mu_1(d\mathbf{x}) + \cdots + c_m\mu_m(d\mathbf{x}), \tag{2.8}$$

where

$$c_i = \frac{E|\mathbf{A}_i|^\kappa}{E|\mathbf{A}_0|^\kappa}(1 - EA^\kappa),$$

provided that the following relations holds for any Borel sets $C_1, C_2 \subset [0,\infty]^m \setminus \{\mathbf{0}\}$ which are bounded away from zero:

$$nP(a_n^{-1}\mathbf{A}_i B_i \in C_1, a_n^{-1}\mathbf{A}_j B_j \in C_2) \to 0, \qquad 0 \le i < j \le m,$$

where we write $B_0 = Y_0$ for the sake of simplicity. Since $C_1$ and $C_2$ are bounded away from zero, there exists $M > 0$ such that $|\mathbf{x}| > M$ for all $\mathbf{x} \in C_1, C_2$. Therefore, for $i < j$ and any $\gamma > 0$,

$$\begin{aligned}
&\{a_n^{-1}\mathbf{A}_i B_i \in C_1, a_n^{-1}\mathbf{A}_j B_j \in C_2\}\\
&\quad \subset \{|\mathbf{A}_i|B_i > a_n M, |\mathbf{A}_j|B_j > a_n M\}\\
&\quad \subset \{\gamma B_i > Ma_n, \gamma B_j > Ma_n\}\\
&\qquad \cup \{\gamma B_i > Ma_n, |\mathbf{A}_j| I_{(\gamma,\infty)}(|\mathbf{A}_j|)B_j > Ma_n\}\\
&\qquad \cup \{\gamma B_j > Ma_n, |\mathbf{A}_i| I_{(\gamma,\infty)}(|\mathbf{A}_i|)B_i > Ma_n\}\\
&\qquad \cup \{|\mathbf{A}_i| I_{(\gamma,\infty)}(|\mathbf{A}_i|)B_i > Ma_n, |\mathbf{A}_j| I_{(\gamma,\infty)}(|\mathbf{A}_j|)B_j > Ma_n\}\\
&\quad = D_1 \cup \cdots \cup D_4.
\end{aligned}$$

By definition of $(a_n)$ and the independence of $B_i$ and $B_j$, it follows immediately that $nP(D_1) \to 0$. A similar approach applies to $D_2$ since $B_i$ is independent of $B_j\mathbf{A}_j$ and, by Breiman's result,

$$nP(D_2) \sim nP(\gamma B_i > Ma_n)E|\mathbf{A}_j|^\kappa I_{(\gamma,\infty)}(|\mathbf{A}_j|)P(B_j > Ma_n) \to 0.$$

Similarly,

$$\begin{aligned}
nP(D_3) &\le nP\big(|\mathbf{A}_i| I_{(\gamma,\infty)}(|\mathbf{A}_i|)B_i > Ma_n\big)\\
&\sim nE|\mathbf{A}_i|^\kappa I_{(\gamma,\infty)}(|\mathbf{A}_i|)P(B_i > Ma_n),
\end{aligned}$$

and by, the Lebesgue dominated convergence,

$$\lim_{\gamma\to\infty} \limsup_{n\to\infty} nP(D_3) = 0.$$

The relation $nP(D_4) \to 0$ can be proved in the same way.

We summarize our findings.

PROPOSITION 2.5. *If the conditions of Proposition* 2.4 *hold*, *then the finite-dimensional distributions of the stationary solution* $(Y_t)$ *to the stochastic recurrence equation* (1.1) *are regularly varying with index* $\kappa$ *and limiting measure given in* (2.8).

**3. Some applications of the regular variation property.** In this section we consider some applications of the property of regular variation of the solution $(Y_t)$ to the stochastic recurrence equation (1.1). In particular, we are interested in functionals of the $Y_t$'s and their limit behavior. The results include the central limit theorem for the partial sums of the sequence $(Y_t)$ and limit theory for its partial maxima.

3.1. *A remark about the strong mixing property of* $(Y_t)$. Recall that a stationary ergodic sequence $(Y_t)$ is said to be *strongly mixing* if

$$\alpha_k = \sup_{A\in\sigma(Y_s,s\le 0),B\in\sigma(Y_s,s\ge k)} |P(A\cap B)-P(A)P(B)|\to 0,$$

and it is said to be *strongly mixing with geometric rate* if there are constants $r\in(0,1)$ and $c>0$ such that $\alpha_k\le cr^k$ for all $k\ge 1$; see [34], compare [13]. Under general conditions, the latter property is satisfied for the stationary solution $(Y_t)$ of the stochastic recurrence equation (1.3).

PROPOSITION 3.1. *Assume $EA^{\varepsilon}<1$, $EB^{\varepsilon}<\infty$ for some $\varepsilon>0$. Then the stochastic recurrence equation* (1.1) *has a stationary ergodic solution* $(Y_t)$ *which is also strongly mixing with geometric rate if one of the following conditions holds*:

1. *The Markov chain $(Y_t)$ is $\mu$-irreducible, that is, there exists a measure $\mu$ such that, for any Borel set $R$ in the support* $\operatorname{supp}(Y)$ *of $Y$ with $\mu(R)>0$, the relation $\sum_{n=1}^{\infty} P(Y_n\in R|Y_0=x)>0$ holds.*
2. *$A_n=A(E_n)$ and $B_n=B(E_n)$, where $A(x)$ and $B(x)$ are polynomial functions of $x$ and $(E_n)$ are i.i.d. random variables. Moreover, $A(0)<1$ and $E_1$ has an a.e. positive Lebesgue density on* $[0,x_0]$ *for some* $0<x_0\le\infty$.

PROOF. Strong mixing of $(Y_t)$ with geometric rate under $\mu$-irreducibility follows from Theorem 2.8 in [1], using standard results on mixing Markov chains; see [22]. For polynomial $A_n$ and $B_n$, the mixing property follows from Theorem 4.5 in [27] or from Theorem 4.3 in [28].

REMARK 3.2. Squared GARCH processes satisfy a (in general multivariate) version of (1.1). They were found to be strongly mixing with geometric rate; see [6] who proved $\mu$-irreducibility with $\mu$ Lebesgue measure. A sufficient condition for $\mu$-irreducibility is that $\mu(R)>0$ for any $R\subset\operatorname{supp}(Y)$ implies $P(A_1x+B_1\in R)=P(A_1Y_0+B_1\in R|Y_0=x)>0$. This is satisfied if $A_1x+B_1$ has an a.e. positive density on $\operatorname{supp}(Y)$ with respect to Lebesgue measure $\mu$ for every $x\in\operatorname{supp}(Y)$. Alternatively, it suffices to show that $P(Y_n\in R|Y_0=x)>0$ for sufficiently large $n$ (possibly depending on $x$ and $R$). The latter condition is often more difficult to verify.

The relation $P(A_1x+B_1\in R|Y_0=x)>0$ also holds if $\mu(R)>0$ for $\mu$ Lebesgue measure and $A_t$ and $B_t$ have a joint independent multiplicative factor which has an a.e. positive density on $(0, \infty)$, that is, $A_t = F_t \widetilde{A}_t$ and $B_t = F_t \widetilde{B}_t$, where $(F_t)$ is an i.i.d. sequence and for every $t$, $F_t$ and $(\widetilde{A}_t, \widetilde{B}_t)$ are independent. The squared ARCH(1) process satisfies this condition if its innovations have a positive Lebesgue density on the real line; see [10] where the innovations of the ARCH(1) process were assumed to be i.i.d. Gaussian, but the same methodology can be used in the general case.

3.2. *The central limit theorem.* If the assumptions of Propositions 2.4 and 3.1 hold, we may conclude from Propositions 2.4, 2.5 and 3.1 that there exists a unique stationary solution $(Y_t)$ to the stochastic recurrence equation (1.1) which is strongly mixing with geometric rate and which has regularly varying finite-dimensional distributions with index $\kappa > 0$.

If $\kappa > 2$, a standard central limit theorem for stationary ergodic martingale difference sequences applies to $(Y_t)$ and no further mixing condition is needed. Indeed, we have

$$\begin{aligned}&n^{-1/2}(S_n - ES_n)\\&\qquad = n^{-1/2}\sum_{t=1}^{n}[(A_t - EA)Y_{t-1} + (B_t - EB)] + n^{-1/2} EA \sum_{t=1}^{n}(Y_{t-1} - EY).\end{aligned}$$

Hence,

$$n^{-1/2}(S_n - ES_n) = n^{-1/2}(1 - EA)^{-1}\sum_{t=1}^{n}[(A_t - EA)Y_{t-1} + (B_t - EB)] + o_P(1).$$

The sequence $[(A_t - EA)Y_{t-1} + (B_t - EB)]$ is a stationary ergodic martingale difference sequence with respect to the filtration $\mathcal{F}_t = \sigma((A_x, B_x), x \le t)$. Therefore, the central limit theorem from [3], Chapter 23, applies:

$$n^{-1/2}(S_n - ES_n) \xrightarrow{d} N(0, \sigma_Y^2),$$

where $\sigma_Y^2 = \operatorname{var}(Y)$. Notice that $EA < 1$ since $EA^\kappa < 1$, $\kappa > 2$ and $g(h) = EA^h$ is a convex function.

If $\kappa < 2$, infinite variance limits may occur for $(S_n)$; see [9, 10]. The proof relies on a point process argument for the lagged vectors $\mathbf{Y}_t(m) = (Y_t, \dots, Y_{t+m})'$ which is identical to the proof of Theorem 2.10 in [1] and requires regular variation of the finite-dimensional distributions and the strong mixing condition for $(Y_t)$ with geometric rate. It implies weak convergence of the point processes

$$N_n = \sum_{t=1}^{n} \varepsilon_{\mathbf{Y}_t(m)/a_n} \xrightarrow{d} N. \tag{3.1}$$

The limiting Poisson point process $N$ is described in [1] and $(a_n)$ is a sequence satisfying $nP(Y > a_n) \sim 1$.

The convergence result (3.1) and the arguments in [1, 9, 10] imply the weak convergence of the partial sums, sample autocovariances, sample autocorrelations and the partial maxima of the sequence $(Y_t)$. For details, we refer to the mentioned literature. For example, if $\kappa \in (0,2) \setminus \{1\}$,

$$a_n^{-1}(S_n - b_n) \stackrel{d}{\to} Z_\kappa,$$

where $Z_\kappa$ is totally skewed to the right infinite variance $\kappa$-stable random variable, $b_n = ES_n$ for $\kappa > 1$ and $b_n = 0$ for $\kappa < 1$. (We refer to [36] for an encyclopedic treatment of stable distributions and processes.) The proof of the weak convergence of the sample autocovariances and sample autocorrelations is identical to the one treated in [1] for solutions to the stochastic recurrence equation (1.1).

Moreover,

$$a_n^{-1} \max(Y_1, \dots, Y_n) \stackrel{d}{\to} R_\kappa(\theta),$$

where $P(R_\kappa \le x) = e^{-x^{-\kappa}}$, $x > 0$, is the Fréchet distribution function with shape parameter $\kappa$, $P(R_\kappa(\theta) \le x) = [P(R_\kappa \le x)]^\theta$ and $\theta \in (0,1)$ is the extremal index of the sequence $(Y_t)$. See [15] for an introduction to extreme value theory and, in particular, Section 8.1, where the notion of extremal index is treated. Extreme value theory for the solution $(Y_t)$ to (1.1), under the conditions of Kesten's Theorem 1.1, was studied in [19]. In their Theorem 2.1, they calculate

$$\theta = \int_1^\infty P\left(\max_{j \ge 1} \prod_{i=1}^{j} A_i \le y^{-1}\right) \kappa y^{\kappa - 1}\, dy.$$

We mention that the same proof as in [19] [with $n^{1/\kappa}$ replaced by $(a_n)$ as above] applies under the conditions of Proposition 2.4, when Kesten's result does not apply. Indeed, an inspection of their proof shows that it only requires the structure of the stochastic recurrence equation (1.1), the definition of $(a_n)$, the regular variation of $(Y_t)$ and the existence of some $h > 0$ such that $EA^h < 1$.

The definition of the extremal index $\theta$ implies that, for $x_n \ge a_n$,

$$P\big(\max(Y_1, \dots, Y_n) > x_n\big) \sim \theta n P(Y > x_n).$$

This is in contrast to i.i.d. $Y_t$'s, where this relation holds with $\theta = 1$. In the i.i.d. case we also know that $P(S_n - ES_n > x_n) \sim P(\max(Y_1, \dots, Y_n) > x_n)$ for suitable sequences $(x_n)$ with $x_n \to \infty$. The various results proved in this paper, including Proposition 3.3 and Theorem 4.2, show that the exceedances of the random walk $(S_n)$ and of the partial maxima $(\max(Y_1, \dots, Y_n))$ above high thresholds have different asymptotic behavior which is also different from the case of i.i.d. $Y_t$'s.

3.3. *Regular variation of sums.* In what follows we study the tail behavior of the sums

$$S_n = Y_1 + \cdots + Y_n$$

for fixed $n \geq 1$ under the assumptions of Proposition 2.5. It follows from Proposition 2.5 that all linear combinations of the lagged vector $\mathbf{Y}_m$ are regularly varying with index $\kappa$. In particular, $S_n$ is regularly varying with index $\kappa$. In this section we give a precise description of the tail asymptotics of $P(S_n > x)$ for fixed $n$ as $x \to \infty$.

We have

$$S_n = \sum_{i=1}^{n} \left( \Pi_i Y_0 + \sum_{t=1}^{i} \Pi_{t+1,i} B_t \right) = Y_0 \sum_{i=1}^{n} \Pi_i + \sum_{t=1}^{n} B_t \sum_{i=t}^{n} \Pi_{t+1,i}. \tag{3.2}$$

Write

$$Z_0 = Y_0 \sum_{i=1}^{n} \Pi_i \quad \text{and} \quad Z_t = B_t \sum_{i=t}^{n} \Pi_{t+1,i}, \qquad t = 1, \ldots, n.$$

Notice that $Y_0$ is independent of $\sum_{i=1}^{n} \Pi_i$ and $B_t$ is independent of $\sum_{i=t}^{n} \Pi_{t+1,i}$. Now an argument similar to the one in the proof of Proposition 2.3 shows that, for $0 \leq s < t \leq n$,

$$\frac{P(Z_t > x, Z_s > x)}{P(Z_0 > x)} \to 0, \qquad x \to \infty.$$

Also notice that the same result holds if $Y_0 = c$ is a constant initial value. An application of Lemma 2.1 yields the following result.

PROPOSITION 3.3. *Assume that the conditions of Proposition* 2.4 *hold. If* $(Y_n)$ *is the stationary solution to the stochastic recurrence equation* (1.1), *then*

$$\lim_{x \to \infty} \frac{P(S_n > x)}{P(B > x)} = (1 - EA^{\kappa})^{-1} E\left( \sum_{i=1}^{n} \Pi_i \right)^{\kappa} + \sum_{t=0}^{n-1} E\left( \sum_{i=0}^{t} \Pi_i \right)^{\kappa}. \tag{3.3}$$

*If* $(Y_n)$ *satisfies the stochastic recurrence equation* (1.1) *with* $Y_0 = c$ *for some constant* $c$, *then*

$$\lim_{x \to \infty} \frac{P(S_n > x)}{P(B > x)} = \sum_{t=0}^{n-1} E\left( \sum_{i=0}^{t} \Pi_i \right)^{\kappa}.$$

For comparison, assume for the moment that $(\widetilde{Y}_t)$ is an i.i.d. sequence $\widetilde{Y}_1 \stackrel{d}{=} Y$ and $Y$ has the stationary distribution given by (1.3). Then for every fixed $n \geq 1$,

$$\lim_{x \to \infty} \frac{P(\widetilde{Y}_1 + \cdots + \widetilde{Y}_n > x)}{n P(Y > x)} = 1. \tag{3.4}$$

This is the subexponential property of a regularly varying distribution; see [15], Section 1.3.2 and Appendix A3 for an extensive discussion of subexponential distributions. Property (3.4) does not remain valid for dependent stationary sequences with regularly varying finite-dimensional distributions. This was shown in [25] for the case of linear processes. In that case the limiting constant in (3.4) is, in general, different from 1 and depends on the coefficients of the linear process. Proposition 3.3 shows that a similar behavior can be expected for other nonlinear stationary processes. In particular, by Proposition 3.3, relation (3.3) can be re-written in the form

$$\lim_{x\to\infty}\frac{P(S_n>x)}{P(Y>x)}=E\left(\sum_{i=1}^{n}\Pi_i\right)^{\kappa}+(1-EA^{\kappa})\sum_{t=0}^{n-1}E\left(\sum_{i=0}^{t}\Pi_i\right)^{\kappa}. \tag{3.5}$$

It is interesting to observe that a similar relationship holds if the $(A_t, B_t)$'s satisfy the conditions of Kesten's Theorem 1.1. In that case, the condition $EA^{\kappa}=1$ is needed for regular variation of the stationary solution $(Y_t)$ to the stochastic recurrence equation (1.1) with index $\kappa>0$. Assume, in addition, that $EB^{\kappa+\delta}$ and $EA^{\kappa+\delta}$ are finite for some $\delta>0$. Then we may conclude from the representation (3.2), regular variation of $Y_0$ and Breiman's result that

$$\lim_{x\to\infty}\frac{P(S_n>x)}{P(Y>x)}=E\left(\sum_{i=1}^{n}\Pi_i\right)^{\kappa}.$$

In a sense, this is the limiting result in (3.5) for $EA^{\kappa}=1$.

## 4. Large deviations and ruin probabilities.

4.1. *Results on large deviations.* In this subsection we couple the increase of $x$ with $n$ to obtain probabilities of large deviations of the type

$$P(S_n-ES_n>x)\sim nEC^{\kappa}P(B>x)\qquad\text{uniformly for } x\ge x_n,$$

and appropriate sequences $x_n\to\infty$. Here $C$ is a generic element of the stationary sequence

$$C_t=\sum_{i=t}^{\infty}\Pi_{t+1,i},\qquad t\in\mathbb{Z}. \tag{4.1}$$

We start with an auxiliary result, where we collect some useful properties of this sequence.

LEMMA 4.1. *Assume that $(A_t)$ is an i.i.d. sequence and $EA^{\kappa}<1$ for some $\kappa>0$.*

1. *The sequence $(C_t)$ defined in* (4.1) *is well defined and strictly stationary.*

2. *The random variable $C$ has finite $p$th moment if and only if $EA^p < \infty$ for $p > 0$.*
3. *The sequences $(C_t)$ and $(D_t)$ given by* (4.2) *have the same finite-dimensional distributions. If $A$ has an a.e. positive Lebesgue density on $[0, x_0]$ for some $x_0 \le \infty$, then $(D_t)$ is strongly mixing with geometric rate.*

PROOF. 1. The sequence $(C_t)$ has the same distribution as the sequence

$$D_t = \sum_{i=-\infty}^{t} \Pi_{i+1,t}, \qquad t \in \mathbb{Z}. \tag{4.2}$$

The latter satisfies the stochastic recurrence equation

$$D_t = 1 + A_t \sum_{i=-\infty}^{t-1} \Pi_{i+1,t-1} = 1 + A_t D_{t-1}, \qquad t \in \mathbb{Z}. \tag{4.3}$$

It constitutes a unique strictly stationary sequence if $E \log A < 0$ and $E \log^+ A < \infty$, see (1.2), which is satisfied if $EA^\kappa < 1$ for some $\kappa > 0$.

2. From (4.3), the independence of $D_{t-1}$ and $A_t$ and the stationarity of $(D_t)$, we conclude that $D_t$ has finite $p$th moment if and only if $A_t$ has. Since $D \stackrel{d}{=} C$, the statement follows.

3. Follows from the second part of Proposition 3.1 with $A(x) = x$, $B(x) = 1$, $E_i = A_i$. □

In the following result we assume, in addition, that the sequences $(A_t)$ and $(B_t)$, hence, $(C_t)$ and $(B_t)$, are independent. Although we conjecture that this assumption can be avoided, we need the independence at various technical steps in the proof.

THEOREM 4.2. *Assume that $(A_t)$ and $(B_t)$ are independent i.i.d. sequences of nonnegative random variables, $B$ is regularly varying with index $\kappa > 1$, $EA^\kappa < 1$ and $EA^{2\kappa} < \infty$. Consider a sequence of positive numbers such that $nP(B > x_n) \to 0$ and, for every $c > 0$,*

$$\begin{aligned}\lim_{n\to\infty} \sup_{x \ge x_n} \Bigg| \Bigg( P\Bigg(\operatorname{var}\big(B I_{[0,x]}(B)\big) \sum_{t=1}^{n} C_t^2 > cx^2/\log x\Bigg) \\ + P\Bigg(\Bigg|\sum_{t=1}^{n}(C_t - EC)\Bigg| > cx\Bigg)\Bigg) \times \big(nP(B > x)\big)^{-1} \Bigg| \\ = 0.\end{aligned} \tag{4.4}$$

*Then the large deviation relations*

$$\lim_{n\to\infty} \sup_{x \ge x_n} \left| \frac{P(S_n - ES_n > x)}{nP(B > x)} - EC^\kappa \right| = 0 \tag{4.5}$$

*and*

$$\lim_{n\to\infty}\sup_{x\ge x_n}\frac{P(S_n-ES_n\le -x)}{nP(B>x)}=0 \tag{4.6}$$

*are satisfied.*

The proof of the theorem is rather technical and therefore postponed until Section 5.

REMARK 4.3. The validation of (4.4) is, in general, difficult. Sufficient conditions for (4.4) can be verified by assuming certain mixing conditions on $(C_t)$; see Lemma 4.6 below and Lemma 4.1 part 3.

REMARK 4.4. Theorem 4.2 is applicable for finite or infinite variance sequences $(B_t)$. The infinite variance comes into the picture in condition (4.4). For $\kappa>2$, $\operatorname{var}(BI_{[0,x]}(B))\to c$ for some finite $c>0$. Hence, condition (4.4) can be formulated without $\operatorname{var}(BI_{[0,x]}(B))$. If $\kappa<2$ or $\kappa=2$ and $\operatorname{var}(B)=\infty$, $\operatorname{var}(BI_{[0,x]}(B))\to\infty$. In particular, for $\kappa=2$, $\operatorname{var}(BI_{[0,x]}(B))$ is a slowly varying function which increases to infinity. If $\kappa\in(1,2)$, an application of Karamata's theorem yields, for some $c>0$, $\operatorname{var}(BI_{[0,x]}(B))\sim cx^2P(B>x)\to\infty$.

REMARK 4.5. The literature on large deviations for sums of stationary heavy-tailed random variables is rather sparse. The case of linear processes $Y_t=\sum_{j=-\infty}^{\infty}\varphi_jZ_{t-j}$ for i.i.d. regularly varying sequences $(Z_t)$ was treated in [25]. In this case, the limit of $(P(S_n-ES_n>x))/(nP(Y>x))$ is approximated uniformly for $x\ge cn$, any positive $c$. The limit depends in a complicated way on the coefficients $\varphi_j$ and on the coefficient of regular variation. Davis and Hsing [9] seems to be the only reference, where large deviation results were proved for general regularly varying stationary sequences, assuming certain mixing conditions and $\kappa<2$. They exploit point process convergence results and express the limit of the sequence $(P(S_n>x_n)/(nP(Y>x_n))$ in terms of the limiting point process, which is difficult to interpret. Unfortunately, their approach seems to work only in the case of infinite variance random variables.

We continue by giving some sufficient conditions for the validity of the relation (4.4).

LEMMA 4.6. *Assume $A$ has an a.e. positive Lebesgue density on its support $[0,x_0]$ for some $x_0\le\infty$, $B$ is regularly varying with index $\kappa$ and $EA^\kappa<1$ for some $\kappa>1$.*

1. *Assume*

$$\sup_{x\ge x_n} n^{(\kappa+\gamma)/2-1}\Big(x\Big/\sqrt{\operatorname{var}(BI_{[0,x]}(B))\log x}\,\Big)^{-(\gamma+\kappa)}\Big/P(B>x)\to 0 \tag{4.7}$$

*and $EA^{\kappa+\gamma}<\infty$ for some $\gamma$ such that $\kappa+\gamma>2$. Then relation* (4.4) *holds.*

2. *Assume that $C \le c$ a.s. for some constant $c > 0$ and for some $d, \varepsilon > 0$,*

$$\sup_{x \ge x_n} \frac{e^{-d(x/\sqrt{n})^2/(\log x \operatorname{var}(BI_{[0,x]}(B)))}}{nP(B > x)} + \sup_{x \ge x_n} \frac{e^{-(x/\sqrt{n})n^{-\varepsilon}}}{nP(B > x)} \to 0. \tag{4.8}$$

*Then relation* (4.4) *holds.*

REMARK 4.7. In particular, (4.4) holds for $(x_n)$ with (4.8) if $A \le c_0$ for some constant $c_0 < 1$ and $B$ is regularly varying with index $\kappa > 1$. Indeed, then $EA^d < 1$ for all $d > 0$ and $C \le \sum_{i=0}^{\infty} c_0^i = (1 - c_0)^{-1}$.

REMARK 4.8. We discuss the conditions on the $x$-regions where (4.4) holds. If $\kappa > 2$, $\operatorname{var}(B) < \infty$. Writing $P(B > x) = x^{-\kappa} L(x)$ for some slowly varying function $L$, (4.7) is satisfied if

$$[n^{(\kappa+\gamma)/2-1} x_n^{-\gamma}] \sup_{x \ge x_n} [(\log x)^{(\kappa+\gamma)/2} / L(x)] \to 0. \tag{4.9}$$

Since $(\log x)^{(\kappa+\gamma)/2}/L(x) \le x^{\varepsilon}$, for every $\varepsilon > 0$ and sufficiently large $x$, (4.9) holds if $x_n = n^{0.5+\delta}$ with $\delta > \gamma^{-1}(\kappa/2 - 1)$. This $\delta$ can be chosen the closer to zero the more moments of $A$ exist, that is, the larger $\gamma$ can be chosen. These growth rates are comparable to the case of i.i.d. $Y_t$'s for $\kappa > 2$, see Theorem 1.2, where one could choose $x_n = c\sqrt{n \log n}$ for some constant $c > 0$. Such precise results are hard to derive in the case of dependent $Y_t$'s.

If $\kappa \in (1, 2)$, a similar remark applies. Then $x_n$ can be chosen of the order $n^{(1/\kappa)+\delta}$ for some $\delta > 0$ which is in agreement with the order of magnitude of $(x_n)$ for i.i.d. sequences, see again Theorem 1.2.

Notice that, under the above conditions, $x_n = cn$ can be chosen in most cases of interest for $\kappa > 1$.

PROOF. By Lemma 4.1 part 3, the sequence $(D_t) \stackrel{d}{=} (C_t)$ is strongly mixing with geometric rate and so is $(N_t D_t)$, where the i.i.d. standard normal sequence $(N_t)$ is assumed to be independent of $(D_t)$. This follows by standard results on strong mixing; see, for example, [13].

By Markov's inequality, for every $y > 0$ and $\gamma > 0$ such that $EA^{\kappa+\gamma} < \infty$,

$$\begin{aligned} P\left(\sum_{t=1}^{n} C_t^2 > y\right) &\le y^{-(\gamma+\kappa)/2} E\left(\sum_{t=1}^{n} C_t^2\right)^{(\kappa+\gamma)/2} \\ &= y^{-(\gamma+\kappa)/2} E\left|\sum_{t=1}^{n} D_t N_t\right|^{\kappa+\gamma} \Big/ E|N|^{(\kappa+\gamma)/2} \\ &\le c(n/y)^{(\gamma+\kappa)/2}. \end{aligned} \tag{4.10}$$

In the last step we applied a moment estimate for sums of strongly mixing random variables with geometric rate and used the fact that $\gamma + \kappa > 2$; see [13], page 31. Applying (4.10) for $x > 1$, $d > 0$, we obtain

$$\frac{P(\operatorname{var}(BI_{[0,x]}(B)) \sum_{t=1}^n C_t^2 > dx^2/\log x)}{nP(B > x)}$$
$$\leq c \frac{(x/\sqrt{\log x \operatorname{var}(BI_{[0,x]}(B))}\,)^{-(\gamma+\kappa)} n^{(\kappa+\gamma)/2}}{nP(B > x)},$$

and the right-hand side converges to zero uniformly for $x \geq x_n$, by virtue of assumption (4.7).

Similarly, if $C \leq c$ a.s., applying an exponential Markov inequality for $h > 0$,

$$P\left(\sum_{t=1}^n C_t^2 > y\right) \leq e^{-(h^2/2)(y/n)} E e^{(h^2/2)n^{-1}\sum_{t=1}^n C_t^2}$$
$$= e^{-(h^2/2)(y/n)} E e^{hn^{-1/2}\sum_{t=1}^n D_t N_t}.$$

The central limit theorem for strongly mixing random variables with geometric rate (see [20]) yields

$$n^{-1/2} \sum_{t=1}^n D_t N_t \xrightarrow{d} N(0, \sigma^2),$$

where $\sigma^2 = \operatorname{var}(D)$. Moreover,

$$E e^{(h^2/2)n^{-1}\sum_{t=1}^n C_t^2} \leq E e^{(h^2/2)C_1^2} < \infty.$$

Applying a domination argument, the central limit theorem and assumption (4.8) prove that

$$\frac{P(\operatorname{var}(BI_{[0,x]}(B)) \sum_{t=1}^n C_t^2 > dx^2/\log x)}{nP(B > x)}$$
$$\leq c \frac{e^{-(h^2/2)d(x/\sqrt{n})^2/(\log x \operatorname{var}(BI_{[0,x]}(B)))}}{nP(B > x)} \to 0.$$

The estimates for $P(\sum_{t=1}^n (C_t - EC) > x)$ can be derived in a similar fashion. If $EA^{\kappa+\gamma} < \infty$, we have

$$P\left(\left|\sum_{t=1}^n (C_t - EC)\right| > x\right) \leq x^{-(\kappa+\gamma)} E\left|\sum_{t=1}^n (C_t - EC)\right|^{\kappa+\gamma}$$
$$\leq c x^{-(\kappa+\gamma)} n^{(\kappa+\gamma)/2}.$$

Now assume $C \le c$ a.s. Since $(C_t)$ is strongly mixing with geometric rate, the following exponential bound holds (see [13], page 34). For any $\varepsilon < 0.5$, there exists a constant $h > 0$ such that

$$P\left(\sum_{t=1}^{n}(C_t - EC) > x\right) \le e^{-h(x/\sqrt{n})n^{-\varepsilon}}.$$

This concludes the proof. $\square$

4.2. *Results on ruin probabilities.* In this subsection we study the ruin probability

$$\psi(u) = P\left(\sup_{n\ge 0}\big((S_n - ES_n) - \mu n\big) > u\right),$$

when the initial capital $u \to \infty$ and $\mu > 0$. Here $(Y_t)$ is the unique stationary ergodic solution to (1.3), $(A_t)$ and $(B_t)$ are independent and satisfy the conditions of Theorem 4.2. In particular, we assume that $\kappa > 1$. Then $EB < \infty$ and $EA < 1$ since $EA^\kappa < 1$. In particular, $EY = EB(1 - EA)^{-1} = EBEC$ is well defined. This choice and the strong law of large numbers ensure that the random walk $((S_n - ES_n) - \mu n)_{n\ge 0}$ has a negative drift.

THEOREM 4.9. *Assume that the conditions of Theorem* 4.2 *hold, that* $\kappa > 1$ *and* $x_n = cn$ *is a possible threshold sequence for every* $c > 0$. *Moreover, assume there exists* $\gamma > \kappa$ *such that* $EC^{\kappa+\gamma} < \infty$. *Assume that* $(C_t)$ *is strongly mixing with geometric rate. Then we have, for any* $\mu > 0$,

$$\lim_{u\to\infty} \frac{\psi(u)}{uP(B > u)} = EC^\kappa \frac{1}{\mu}\frac{1}{\kappa - 1}. \tag{4.11}$$

We postpone the proof of Theorem 4.9 to Section 6.

REMARK 4.10. The assumption that Theorem 4.2 holds for $x_n = cn$ is not really a strong restriction. Indeed, we discussed in Remark 4.8 that this condition is satisfied under very mild conditions.

REMARK 4.11. This result is similar to the case of i.i.d. $Y_t$'s; see Theorem 1.3 above. To compare with the latter one, we mention that (4.11) can be reformulated by using Proposition 2.4:

$$\lim_{u\to\infty} \frac{\psi(u)}{uP(Y > u)} = (1 - EA^\kappa)EC^\kappa \frac{1}{\mu}\frac{1}{\kappa - 1}.$$

**5. Proof of Theorem 4.2.** We will make use of the decomposition

$$
\begin{aligned}
S_n &= Y_0 \sum_{i=1}^{n} \Pi_i + \sum_{t=1}^{n} B_t \sum_{i=t}^{\infty} \Pi_{t+1,i} - \sum_{t=1}^{n} B_t \sum_{i=n+1}^{\infty} \Pi_{t+1,i} \\
&= S_{n,1} + S_{n,2} - S_{n,3}.
\end{aligned} \tag{5.1}
$$

PROOF OF (4.5). We start with an upper bound. Observe that, for small $\varepsilon > 0$,

$$
\begin{aligned}
&P(S_n - ES_n > x) \\
&\quad \le P(S_{n,1} - ES_{n,1} > x\varepsilon/2) \\
&\qquad + P\bigl(S_{n,2} - ES_{n,2} > x(1-\varepsilon)\bigr) + P(-S_{n,3} + ES_{n,3} > x\varepsilon/2) \\
&\quad = I_1(x) + I_2(x) + I_3(x).
\end{aligned}
$$

We bound the $I_j$'s in a series of lemmas.

LEMMA 5.1. *We have*

$$
\limsup_{n\to\infty} \sup_{x \ge x_n} \frac{I_j(x)}{nP(B > x)} = 0, \qquad j = 1, 3.
$$

PROOF. We start with $I_1$. The random variable $Y_0$ is regularly varying with index $\kappa$, by virtue of Proposition 2.4, and independent of $(\Pi_i)$. Moreover,

$$
\sum_{i=1}^{n} \Pi_i \uparrow \sum_{i=1}^{\infty} \Pi_i \stackrel{d}{=} C - 1.
$$

We also see that

$$
ES_{n,1} = EY \sum_{i=1}^{n} (EA)^i \uparrow \frac{EYEA}{1 - EA} = c'.
$$

The expectation $EA$ is smaller than one since $EA^\kappa < 1$ for some $\kappa > 1$ and $g(h) = EA^h$ is a convex function; see the discussion in the proof of Proposition 2.4. An application of Breiman's result (Lemma 2.2) and Proposition 2.4 yield that, for independent $C, Y$,

$$
\begin{aligned}
\sup_{x \ge x_n} \frac{I_1(x)}{nP(B > x)} &\le \sup_{x \ge x_n} \frac{P(|S_{n,1} - ES_{n,1}| > \varepsilon x/2)}{nP(B > x)} \\
&\le \sup_{x \ge x_n} \frac{P(Y(C-1) > \varepsilon x/2 - c')}{nP(B > x)} \\
&\le c \sup_{x \ge x_n} \frac{P(Y > \varepsilon x/2) E(C-1)^\kappa}{nP(B > x)} \to 0.
\end{aligned} \tag{5.2}
$$

For Breiman's result, one needs that $EC^{\kappa+\delta} < \infty$ for some $\delta > 0$. This condition is satisfied since $EA^{2\kappa} < \infty$, by virtue of Lemma 4.1 part 2.

Now we turn to $I_3$. We have

$$S_{n,3} = \sum_{t=1}^{n} B_t \Pi_{t+1,n+1} \sum_{i=n+1}^{\infty} \Pi_{n+2,i} \overset{d}{=} A_0 C_{n+1} \sum_{t=1}^{n} B_t \Pi_{t-1} \tag{5.3}$$

$$\overset{d}{\to} AC \sum_{t=1}^{\infty} B_t \Pi_{t-1} = ACY', \tag{5.4}$$

where $Y'$, $A$, $C$ are independent and $Y \overset{d}{=} Y'$. Similar arguments as for $I_1$ show that

$$\sup_{x \geq x_n} \frac{I_3(x)}{nP(B > x)} \leq \sup_{x \geq x_n} \frac{P(|S_{n,3} - ES_{n,3}| > x\varepsilon/2)}{nP(B > x)} \to 0. \tag{5.5}$$

This proves the lemma. $\square$

LEMMA 5.2. *We have*

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} \sup_{x \geq x_n} \left( \frac{I_2(x)}{nP(B > x)} - EC^{\kappa} \right) \leq 0.$$

PROOF. Write, for any $\delta > 0$,

$$Q_{n,1}(\delta) = \bigcup_{1 \leq t < s \leq n} \{B_t > \delta x, B_s > \delta x\},$$

$$Q_{n,2}(\delta) = \left\{ \max_{t \leq n} B_t \leq \delta x \right\},$$

$$Q_{n,3}(\delta) = \bigcup_{t=1}^{n} \{B_t > \delta x, B_s \leq \delta x, 1 \leq s \neq t \leq n\}.$$

Then

$$\begin{aligned} \frac{I_2(x)}{nP(B > x)} &= \frac{P(\{S_{n,2} - ES_{n,2} > x(1-\varepsilon)\} \cap Q_{n,1}(\delta))}{nP(B > x)} \\ &\quad + \frac{P(\{S_{n,2} - ES_{n,2} > x(1-\varepsilon)\} \cap Q_{n,2}(\delta))}{nP(B > x)} \\ &\quad + \frac{P(\{S_{n,2} - ES_{n,2} > x(1-\varepsilon)\} \cap Q_{n,3}(\delta))}{nP(B > x)} \\ &= I_{2,1}(x) + I_{2,2}(x) + I_{2,3}(x). \end{aligned}$$

Obviously, for any $\delta > 0$,

$$\lim_{n \to \infty} \sup_{x \geq x_n} I_{2,1}(x) = 0.$$

Writing for any $t \in \mathbb{Z}$, $x > 0$,

$$B_{t,x} = B_t I_{[0,x]}(B_t),$$

we obtain

$$\sup_{x \ge x_n} I_{2,2}(x) \le \sup_{x \ge x_n} \frac{P(\sum_{t=1}^n (B_{t,\delta x} C_t - E B_{1,\delta x} EC) > (1-\varepsilon)x)}{nP(B > x)}.$$

Notice that

$$\sup_{x \ge x_n} I_{2,2}(x) \le \sup_{x \ge x_n} \frac{P(E_1)}{nP(B > x)} + \sup_{x \ge x_n} \frac{P(E_2)}{nP(B > x)},$$

where

$$E_1 = \left\{ \sum_{t=1}^n (B_{t,\delta x} - E B_{1,\delta x}) C_t > 0.5(1-\varepsilon)x \right\},$$

$$E_2 = \left\{ E B_{1,\delta x} \sum_{t=1}^n (C_t - EC) > 0.5(1-\varepsilon)x \right\}.$$

Conditioning on $(C_t)$ and using the Fuk–Nagaev inequality (inequality (2.79) on page 78 in [31] with $p = 2\kappa$), we have, with $EC^{2\kappa} < \infty$,

$$\begin{aligned} &E[P(E_1 | (C_t))] \\ &\quad \le cE\Bigg( \big(0.5(1-\varepsilon)x\big)^{-2\kappa} \sum_{t=1}^n C_t^{2\kappa} \\ &\qquad\qquad + \exp\Bigg\{ -c\big(0.5(1-\varepsilon)x\big)^2 \Bigg[ \operatorname{var}(B_{1,\delta x}) \sum_{t=1}^n C_t^2 \Bigg]^{-1} \Bigg\} \Bigg) \\ &\quad \le c x^{-2\kappa} n E C^{2\kappa} \\ &\qquad + cE\Bigg( \exp\Bigg\{ -c\big(0.5(1-\varepsilon)x\big)^2 \Bigg[ \operatorname{var}(B_{1,\delta x}) \sum_{t=1}^n C_t^2 \Bigg]^{-1} \Bigg\} \\ &\qquad\qquad\qquad \times I_{\{\operatorname{var}(B_{1,\delta x}) \sum_{t=1}^n C_t^2 \le dx^2/\log x\}} \Bigg) \\ &\qquad + P\Bigg( \operatorname{var}(B_{1,\delta x}) \sum_{t=1}^n C_t^2 > dx^2/\log x \Bigg) \\ &\quad = J_1(x) + J_2(x) + J_3(x), \end{aligned} \tag{5.6}$$

where $d > 0$ is chosen small enough such that $d' = d'(d) = c[0.5(1-\varepsilon)]^2/d$ is large enough, implying

$$\sup_{x \ge x_n} \frac{J_2(x)}{nP(B > x)} \le \sup_{x \ge x_n} \frac{e^{-c(0.5(1-\varepsilon))^2 \log x/d}}{nP(B > x)} = \sup_{x \ge x_n} \frac{x^{-d'}}{nP(B > x)} \to 0.$$

We also have

$$\sup_{x \ge x_n} \frac{J_1(x)}{nP(B > x)} \to 0.$$

The relations

$$\sup_{x \ge x_n} \frac{J_3(x)}{nP(B > x)} \to 0 \quad \text{and} \quad \sup_{x \ge x_n} \frac{P(E_2)}{nP(B > x)} \to 0$$

follow by assumption (4.4). Collecting the above estimates, we proved, for every $\delta$,

$$\lim_{n\to\infty} \sup_{x \ge x_n} I_{2,2}(x) = 0.$$

Thus, it remains to show that

$$\lim_{\varepsilon\downarrow 0} \lim_{\delta\downarrow 0} \limsup_{n\to\infty} \sup_{x \ge x_n} \bigl(I_{2,3}(x) - EC^{\kappa}\bigr) \le 0. \tag{5.7}$$

We have

$$\begin{aligned}
I_{2,3}(x) \le \sum_{t=1}^{n} \Biggl( P\Biggl( & B_t C_t + \sum_{s=1, s\neq t}^{n} (B_s C_s - EBEC) > x(1-\varepsilon), \\
& \qquad\qquad B_t > \delta x, \max_{1 \le s \le n, s \neq t} B_s \le \delta x \Biggr)\Biggr) \\
& \times \bigl(nP(B > x)\bigr)^{-1} \\
\le{}& \sum_{t=1}^{n} \frac{P(B_t \min(C_t, \delta^{-1}(1-2\varepsilon)) > (1-2\varepsilon)x)}{nP(B > x)} \\
& + \sum_{t=1}^{n} \Biggl( P\Biggl( \sum_{s=1, s\neq t}^{n} (B_s C_s - EBEC) > x\varepsilon, \\
& \qquad\qquad B_t > \delta x, \max_{1 \le s \le n, s\neq t} B_s \le \delta x \Biggr)\Biggr) \times \bigl(nP(B > x)\bigr)^{-1} \\
\le{}& \frac{P(B_1 \min(C_1, \delta^{-1}(1-2\varepsilon)) > (1-2\varepsilon)x)}{P(B > x)} \\
& + \sum_{t=1}^{n} \Biggl( P\Biggl( \sum_{s=1, s\neq t}^{n} (B_s C_s - EBEC) > x\varepsilon, \\
& \qquad\qquad \max_{1 \le s \le n, s\neq t} B_s \le \delta x \Biggr) P(B > \delta x) \Biggr) \\
& \times \bigl(nP(B > x)\bigr)^{-1} \\
={}& L_1(x) + L_2(x).
\end{aligned}$$

By Breiman's result,

$$\lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} \limsup_{n \to \infty} \sup_{x \geq x_n} \biggl[ \biggl( L_1(x) - \frac{E[(\min(C, \delta^{-1}(1-2\varepsilon)))^\kappa]}{(1-2\varepsilon)^\kappa} \biggr) + \biggl( \frac{E[(\min(C, \delta^{-1}(1-2\varepsilon)))^\kappa]}{(1-2\varepsilon)^\kappa} - EC^\kappa \biggr) \biggr] = 0.$$

Similar calculations as for $I_{2,2}(x)$ yield that, for every $\delta, \varepsilon$,

$$\lim_{n \to \infty} \sup_{x \geq x_n} L_2(x) = 0.$$

We conclude that (5.7) holds. This completes the proof of the lemma. $\square$

Lemmas 5.1 and 5.2 prove that

$$\limsup_{n \to \infty} \sup_{x \geq x_n} \biggl( \frac{P(S_n - ES_n > x)}{nP(B > x)} - EC^\kappa \biggr) \leq 0. \tag{5.8}$$

We conclude the proof of (4.5) with the bound

$$\limsup_{n \to \infty} \sup_{x \geq x_n} \biggl( EC^\kappa - \frac{P(S_n - ES_n > x)}{nP(B > x)} \biggr) \leq 0. \tag{5.9}$$

Arguing as for (5.8), we see that, for any $\delta > 0$, uniformly for $x \geq x_n$,

$$\begin{aligned} \frac{P(S_n - ES_n > x)}{nP(B > x)} &\sim \frac{P(\{S_{n,2} - ES_{n,2} > x\} \cap Q_{n,2}(\delta))}{nP(B > x)} \\ &\quad + \frac{P(\{S_{n,2} - ES_{n,2} > x\} \cap Q_{n,3}(\delta))}{nP(B > x)} \\ &= K_1(x) + K_2(x). \end{aligned}$$

It follows by analogous arguments as for $I_{2,2}(x)$ that

$$\sup_{x \geq x_n} \frac{K_1(x)}{nP(B > x)} \to 0.$$

Write, for $\varepsilon > 0$,

$$L_t = \{ B_t \min(C_t, \delta^{-1}(1+\varepsilon)) > (1+\varepsilon)x \}, \qquad t \in \mathbb{Z}.$$

As regards $K_2(x)$, we have

$$\begin{aligned}
K_2(x) &= \sum_{t=1}^{n}\Bigg(P\Bigg(B_tC_t + \sum_{s=1,s\neq t}^{n} B_sC_s > x + nEBEC,\, B_t > \delta x, \\
&\qquad\qquad \max_{s\leq n, s\neq t} B_s \leq \delta x\Bigg)\Bigg) \\
&\qquad \times \big(nP(B>x)\big)^{-1} \\
&\geq [P(B_1 \leq \delta x)]^{n-1}\sum_{t=1}^{n}\frac{P(L_t)}{nP(B>x)} \\
&\quad - \sum_{t=1}^{n}\Bigg(P\Bigg(\Bigg\{\sum_{s=1,s\neq t}^{n}(B_sC_s - EBEC) < -\varepsilon x + EBEC\Bigg\} \\
&\qquad\qquad \cap L_t \cap \Big\{\max_{s\leq n,s\neq t} B_s \leq \delta x\Big\}\Bigg)\Bigg) \\
&\qquad \times \big(nP(B>x)\big)^{-1} \\
&= K_{2,1}(x) - K_{2,2}(x).
\end{aligned}$$

Since $nP(B > \delta x_n) \to 0$, we have

$$\sup_{x\geq x_n} |[P(B\leq \delta x)]^{n-1} - 1| \to 0.$$

Therefore and by regular variation of $B$,

$$\sup_{x\geq x_n}\big((1+\varepsilon)^{-\kappa}E\big[\min\big(C,\delta^{-1}(1+\varepsilon)\big)\big]^{\kappa} - K_{2,1}(x)\big) \to 0. \tag{5.10}$$

Write

$$T_{n,t} = \Bigg\{\sum_{s=1,s\neq t}^{n}(B_{s,\delta x}C_s - EBEC) \leq -\varepsilon x + EBEC\Bigg\}.$$

As regards $K_{2,2}(x)$, we have, for $0 < m < M < \infty$,

$$\begin{aligned}
&nP(B>x)K_{2,2}(x) \\
&\quad \leq \sum_{t=1}^{n} P(T_{n,t}\cap L_t \cap \{C_t \leq m\}) + \sum_{t=1}^{n} P(T_{n,t}\cap L_t\cap\{C_t > M\}) \\
&\qquad + \sum_{t=1}^{n} P\big(T_{n,t}\cap L_t\cap\{C_t \in (m,M]\}\big) \\
&\quad = K_{2,2,1}(x) + K_{2,2,2}(x) + K_{2,2,3}(x).
\end{aligned}$$

Then for small $\delta > 0$, by the uniform convergence theorem for regularly varying functions,

$$\lim_{m\to 0}\lim_{n\to\infty}\sup_{x\ge x_n}\frac{K_{2,2,1}(x)}{nP(B>x)} \le \lim_{m\to 0}\lim_{n\to\infty}\sup_{x\ge x_n}\frac{P(mB>(1+\varepsilon)x)}{P(B>x)}$$
$$= \lim_{m\to 0} m^{\kappa}(1+\varepsilon)^{-\kappa} = 0.$$

Moreover, by Breiman's result and Lebesgue dominated convergence,

$$\begin{aligned}
&\lim_{M\to\infty}\lim_{n\to\infty}\sup_{x\ge x_n}\frac{K_{2,2,2}(x)}{nP(B>x)}\\
&\quad= \lim_{M\to\infty}\lim_{n\to\infty}\sup_{x\ge x_n}\frac{P(CI_{\{C>M\}}B>(1+\varepsilon)x)}{P(B>x)}\\
&\quad= \lim_{M\to\infty} E\big(C^{\kappa}I_{(M,\infty)}(C)\big)(1+\varepsilon)^{-\kappa}\\
&\quad= 0.
\end{aligned}$$

Finally, using the same method of proof as for $I_{2,2}(x)$,

$$\begin{aligned}
&\sup_{x\ge x_n}\frac{K_{2,2,3}(x)}{nP(B>x)}\\
&\quad\le \sup_{x\ge x_n} n^{-1}c\sum_{t=1}^{n}P(T_{n,t})\\
&\quad\le \sup_{x\ge x_n} P\left(\sum_{s=1}^{n}(B_{s,\delta x}C_s - EBEC)\le -\varepsilon x/2\right) + \sup_{x\ge x_n}P(B_{1,\delta x}C > x\varepsilon/2)\\
&\quad\to 0.
\end{aligned}$$

Taking the above bounds and, in particular, (5.10) into account, we conclude that

$$\lim_{n\to\infty}\sup_{x\ge x_n}\big((1+\varepsilon)^{-\kappa}E\big[\min\big(C,\delta^{-1}(1+\varepsilon)\big)\big]^{\kappa} - K_2(x)\big) = 0,$$

and letting $\delta\downarrow 0$, $\varepsilon\downarrow 0$, (5.9) follows.

The proof of relation (4.5) is now complete. $\square$

PROOF OF (4.6). The proof is similar to the one for (4.5). It follows from relations (5.2) and (5.5) that it suffices to show

$$\sup_{x\ge x_n}\frac{P(S_{n,2}-ES_{n,2}\le -xr)}{nP(B>x)}\to 0$$

for any $r > 0$. We proceed similarly as for $I_2(x)$ and use the same notation. Then for any $\delta > 0$,

$$\sup_{x\ge x_n}\frac{P(\{S_{n,2}-ES_{n,2}\le -xr\}\cap Q_{n,1}(\delta))}{nP(B>x)} \le \sup_{x\ge x_n}\frac{P(Q_{n,1}(\delta))}{nP(B>x)}\to 0.$$

Moreover, by the uniform convergence theorem for regularly varying functions,

$$\begin{aligned}
\lim_{\delta\to\infty}\lim_{n\to\infty}\sup_{x\ge x_n}&\frac{P(\{S_{n,2}-ES_{n,2}\le -xr\}\cap Q_{n,3}(\delta))}{nP(B>x)}\\
&\le \lim_{\delta\to\infty}\lim_{n\to\infty}\sup_{x\ge x_n}\frac{P(B>\delta x)}{P(B>x)}\\
&=\lim_{\delta\to\infty}\delta^{-\kappa}=0.
\end{aligned}$$

Finally, uniformly for $x\ge x_n$, sufficiently large $n$,

$$\begin{aligned}
\Sigma &= \frac{P(\{S_{n,2}-ES_{n,2}\le -xr\}\cap Q_{n,2}(\delta))}{nP(B>x)}\\
&\le \frac{P(\sum_{t=1}^n(B_{t,\delta x}C_t-EB_{1,\delta x}EC)\le -xr+nECE(BI_{(\delta x,\infty)}(B)))}{nP(B>x)}\\
&\le \frac{P(\sum_{t=1}^n(B_{t,\delta x}C_t-EB_{1,\delta x}EC)\le -xr/2)}{nP(B>x)}.
\end{aligned}$$

Here we used the fact that, by Karamata's theorem, since $x\ge x_n$ and $nP(B>x_n)\to 0$,

$$nECE\big(BI_{(\delta x,\infty)}(B)\big)\le cnxP(B>x)\le cnxP(B>x_n)=o(x).$$

Hence,

$$\begin{aligned}
\Sigma &\le \frac{P(\sum_{t=1}^n(B_{t,\delta x}-EB_{1,\delta x})C_t\le -xr/4)}{nP(B>x)}\\
&\quad+\frac{P(EB_{1,\delta x}\sum_{t=1}^n(C_t-EC)\le -xr/4)}{nP(B>x)}\\
&=\Sigma_1(x)+\Sigma_2(x).
\end{aligned}$$

The relation $\sup_{x\ge x_n}\Sigma_2(x)\to 0$ follows from assumption (4.4). The relation $\sup_{x\ge x_n}\Sigma_1(x)\to 0$ follows by another application of the Fuk–Nagaev inequality in the same way as for $P(E_1)$ in combination with assumption (4.4). □

## 6. Proof of Theorem 4.9.

We will use the notation

$$T_0=0,\quad T_n=(Y_1-EY)+\cdots+(Y_n-EY),\qquad n\ge 1.$$

*Proof of the upper bound.* First, we show the relation

$$\limsup_{u\to\infty}\frac{\psi(u)}{uP(B>u)}\le EC^\kappa\frac{1}{\mu}\frac{1}{\kappa-1}, \tag{6.1}$$

by a series of auxiliary results. Before we proceed with them, we give some intuition on the steps of the proof:

- In Lemmas 6.1 and 6.2 we show that the event $\{\sup_{n\le u/M}(T_n-\mu n)>u\}$ does not contribute to the order of $\psi(u)$ for sufficiently large $u$ and $M$.
- In Lemma 6.3 we show that the order of $\psi(u)$ is essentially determined by the event $D(u)=\{\sup_{n\ge u/M}(\sum_{t=[u/M]}^n(B_t-EB)C_t-\mu n)>u\}$.
- In Lemma 6.4 we show that it is unlikely that $D(u)$ is caused by more than one large value $B_t>\theta t$ for any $\theta>0$.
- In Lemma 6.5 we show that it is unlikely that $D(u)$ occurs if all $B_t$'s in the sum $\sum_{t=[u/M]}^n(B_t-EB)C_t$ are bounded by $\theta(t+u)$.
- In Lemma 6.6 we finally show that $D(u)$ is essentially caused by exactly one unusually large value $B_t>\delta(\mu t+u)$, whereas all other values $B_s$, $s\ne t$, are of smaller order. This lemma also gives the desired upper bound (6.1) of $\psi(u)$.

LEMMA 6.1. *For any* $\mu>0$,

$$\lim_{M\to\infty}\limsup_{u\to\infty}\frac{P(\sup_{n\le u/M}(T_n-\mu n)>u)}{uP(B>u)}=0.$$

PROOF. We have

$$P\Big(\sup_{n\le u/M}(T_n-\mu n)>u\Big)\le P\big(T_{[u/M]}>u-EY[u/M]\big). \tag{6.2}$$

For sufficiently large $M$, $(1-EY/M)>0$. Then an application of the large deviation result of Theorem 4.2 yields that the right-hand side in (6.2) is of the order

$$\sim c[u/M](1-EY/M)^{-\kappa}P(B>u),\qquad u\to\infty.$$

The latter estimate implies the statement of the lemma by letting $M\to\infty$. □

LEMMA 6.2. *We have, for any* $\mu>0$,

$$\lim_{M\to\infty}\limsup_{u\to\infty}\frac{P(\sup_{n\ge u/M}(T_{[u/M]}-\mu n)>u)}{uP(B>u)}=0.$$

PROOF. We have, by virtue of the large deviation results,

$$\begin{aligned}
&\frac{P(\sup_{n\ge u/M}(T_{[u/M]}-\mu n)>u)}{uP(B>u)}\\
&\qquad\le\frac{P(T_{[u/M]}>u+\mu[u/M])}{uP(B>u)}\\
&\qquad\sim c\frac{[u/M]P(B>u(1+\mu/M))}{uP(B>u)},\qquad u\to\infty,\\
&\qquad\sim cM^{-1}(1+\mu/M)^{-\kappa}\to0,\qquad M\to\infty.
\end{aligned}$$

□

In the light of the two lemmas, it suffices to bound the probability

$$J(u) = P\Bigl(\sup_{n \ge u/M} \bigl[\bigl(T_n - T_{[u/M]}\bigr) - (1-\varepsilon)\mu n\bigr] > (1-\varepsilon)u\Bigr)$$

for fixed $M > 0$ and any small $\varepsilon > 0$. By (5.1) and by virtue of Breiman's result, for large $u$,

$$\begin{aligned}
J(u) \le P\Biggl(&Y_0 \sum_{i=1}^{\infty} \Pi_i \\
&+ \sup_{n \ge u/M} \Biggl(\sum_{t=[u/M]+1}^{n} (B_t C_t - EBEC) - (1-\varepsilon)\mu n\Biggr) > (1-2\varepsilon)u\Biggr) \\
\le\ & P\Biggl(Y_0 \sum_{i=1}^{\infty} \Pi_i > \varepsilon u\Biggr) \\
&+ P\Biggl(\sup_{n \ge u/M} \Biggl(\sum_{t=[u/M]+1}^{n} (B_t C_t - EBEC) - (1-\varepsilon)\mu n\Biggr) > (1-3\varepsilon)u\Biggr) \\
\sim\ & \varepsilon^{-\kappa} P(Y > u) E(C-1)^{\kappa} \\
&+ P\Biggl(\sup_{n \ge u/M} \Biggl(\sum_{t=[u/M]+1}^{n} (B_t C_t - EBEC) - (1-\varepsilon)\mu n\Biggr) > (1-3\varepsilon)u\Biggr) \\
\le\ & cP(Y > u) \\
&+ P\Biggl(\sup_{n \ge u/M} \Biggl(\sum_{t=[u/M]+1}^{n} (B_t - EB)C_t - (1-\varepsilon/2)\mu n\Biggr) > (1-4\varepsilon)u\Biggr) \\
&+ P\Biggl(\sup_{n \ge u/M} \Biggl(EB \sum_{t=[u/M]+1}^{n} (C_t - EC) - \varepsilon\mu n/2\Biggr) > \varepsilon u\Biggr) \\
=\ & J_1(u) + J_2(u) + J_3(u).
\end{aligned}$$

We show that

$$J_3(u) = o\bigl(uP(Y > u)\bigr).$$

LEMMA 6.3. *Assume $(C_t)$ is strongly mixing with geometric rate and $EC^{\kappa+\gamma} < \infty$ for some $\gamma > \kappa$. Then for any $M, \mu > 0$,*

$$\lim_{u \to \infty} \frac{P(\sup_{n \ge u/M}(\sum_{t=[u/M]+1}^{n}(C_t - EC) - \mu n) > u)}{uP(B > u)} = 0.$$

PROOF. We have, by Markov's inequality,

$$\begin{aligned}
&P\biggl(\sup_{n\ge u/M}\biggl(\sum_{t=[u/M]+1}^{n}(C_t-EC)-\mu n\biggr)>u\biggr)\\
&\quad\le\sum_{n=[u/M]}^{\infty}P\biggl(\sum_{t=[u/M]+1}^{n}(C_t-EC)>\mu n+u\biggr)\\
&\quad\le\sum_{n=[u/M]}^{\infty}(\mu n+u)^{-(\kappa+\gamma)}E\biggl|\sum_{t=[u/M]+1}^{n}(C_t-EC)\biggr|^{\kappa+\gamma}\\
&\quad\le c\sum_{n=[u/M]}^{\infty}(n+u)^{-(\kappa+\gamma)}n^{(\kappa+\gamma)/2}.
\end{aligned}\tag{6.3}$$

In the last step we applied the moment estimate

$$E\biggl|n^{-1/2}\sum_{t=1}^{n}(C_t-EC)\biggr|^{\kappa+\gamma}\le c,$$

which is valid for strongly mixing sequences with geometric rate if $\gamma>\kappa$ and $EC^{\gamma+\kappa}<\infty$, see, e.g., [13], page 31. An application of Karamata's theorem shows that (6.3) is of the order

$$\sim cu^{1-(\kappa+\gamma)/2}=o\bigl(uP(B>u)\bigr),$$

for $\gamma>\kappa$. $\square$

Thus, it remains to estimate $J_2(u)$. We proceed by a series of lemmas.

LEMMA 6.4. *For every* $\theta>0$,

$$P(B_t>\theta t \textit{ for at least two } t\ge u)=o\bigl(uP(B>u)\bigr).$$

PROOF. We have, by Karamata's theorem,

$$\begin{aligned}
&P(B_t>\theta t\text{ for at least two }t\ge u)\\
&\quad\le\sum_{t=[u]}^{\infty}P(B_t>\theta t,B_j>\theta j\text{ for some }j\ne t)\\
&\quad\le\sum_{t=[u]}^{\infty}P(B>\theta t)\sum_{j=[u],j\ne t}^{\infty}P(B>\theta j)\\
&\quad\sim c[uP(B>u)]^2,
\end{aligned}$$

from which the statement of the lemma follows. $\square$

LEMMA 6.5. *Assume* $(C_t)$ *is strongly mixing with geometric rate,* $EC^{\kappa+\gamma} < \infty$ *for some* $\gamma > \kappa$. *Then for every* $M, \mu, \theta > 0$,

$$\widetilde{J}(u) = P(A_u) = o\big(uP(B > u)\big),$$

*where*

$$A_u = \bigcup_{n \ge [u/M]} \Bigg\{ \sum_{t=[u/M]+1}^{n} (B_t - EB)C_t > (1-4\varepsilon)(\mu n + u), \\ B_j \le \theta(j+u) \text{ for all } j = [u/M]+1, \dots, n \Bigg\}.$$

PROOF. We have

$$\begin{aligned}\widetilde{J}(u) &\le \sum_{n=[u/M]}^{\infty} P\Bigg( \sum_{t=[u/M]+1}^{n} (B_t - EB)C_t > (1-4\varepsilon)(\mu n + u), \\ &\qquad\qquad \max_{j=[u/M]+1,\dots,n} B_j \le \theta(n+u) \Bigg) \\ &\le \sum_{n=[u/M]}^{\infty} P\Bigg( \sum_{t=[u/M]+1}^{n} \big(B_{t,\theta(n+u)} - EB_{1,\theta(n+u)}\big)C_t > (1-4\varepsilon)(\mu n + u) \Bigg),\end{aligned}$$

where

$$B_{t,x} = B_t I_{[0,x]}(B_t), \qquad x > 0.$$

Analogously to (5.6), an application of the Fuk–Nagaev inequality, conditionally on $(C_t)$, yields, for $d > 0$ and $d' = d'(d) > 0$,

$$\begin{aligned}\widetilde{J}(u) &\le c \sum_{n=[u/M]}^{\infty} n(n+u)^{-2\kappa} + c \sum_{n=[u/M]}^{\infty} (n+u)^{-d'} \\ &\quad + c \sum_{n=[u/M]}^{\infty} P\Bigg( \operatorname{var}\big(B_{1,\theta(n+u)}\big) \\ &\qquad\qquad \times \sum_{t=1}^{n} C_t^2 > d[(1-4\varepsilon)(\mu n + u)]^2 / \log\big((1-4\varepsilon)(\mu n + u)\big) \Bigg) \\ &= \widetilde{J}_1(u) + \widetilde{J}_2(u) + \widetilde{J}_3(u).\end{aligned}$$

Choosing $d > 0$ sufficiently small such that $d'$ becomes sufficiently large, an application of Karamata's theorem yields

$$\widetilde{J}_1(u) \le cu^{2-2\kappa} = o\big(uP(B > u)\big)$$

and

$$\widetilde{J}_2(u) \le c u^{1-d'} = o\big(u P(B > u)\big).$$

An application of (4.10) yields, for $\gamma > \kappa$,

$$\widetilde{J}_3(u) \le c \sum_{n=[u/M]}^{\infty} n^{(\kappa+\gamma)/2} \biggl( \frac{(\mu n + u)^2}{\log(\mu n + u) \operatorname{var}(B_{1,\theta(n+u)})} \biggr)^{-(\kappa+\gamma)/2}. \tag{6.4}$$

If $\kappa \ge 2$, $\operatorname{var}(B_{1,x})$ is slowly varying and if $\kappa \in (1,2)$, $\operatorname{var}(B_{1,x}) \sim c x^2 P(B > x)$. This follows by Karamata's theorem. These facts and (6.4) ensure that $\widetilde{J}_3(u) = o(u P(B > u))$. This proves the lemma. $\square$

Finally, we bound $J_2(u)$ and obtain the desired upper bound (6.1) in the theorem.

LEMMA 6.6. *The following result holds*:

$$\lim_{\varepsilon \downarrow 0} \limsup_{u \to \infty} \frac{J_2(u)}{u P(B > u)} \le E C^{\kappa} \frac{1}{\mu} \frac{1}{\kappa - 1}.$$

PROOF. By virtue of Lemmas 6.4 and 6.5,

$$\begin{aligned} &\limsup_{u \to \infty} \frac{J_2(u)}{u P(B > u)} \\ &\quad \le \limsup_{u \to \infty} \frac{P(\bigcup_{n \ge u/M} \{\sum_{t=[u/M]+1}^{n} (B_t - EB) C_t > (1 - 4\varepsilon)(\mu n + u)\} \cap A_\delta)}{u P(B > u)}, \end{aligned}$$

where, for any $\delta > 0$,

$$A_\delta = \bigcup_{t=[u/M]}^{\infty} \{B_t > \delta(\mu t + u), B_s \le \delta(\mu s + u) \text{ for all } s \ge [u/M], s \ne t\}.$$

Hence,

$$\begin{aligned} &\limsup_{u \to \infty} \frac{J_2(u)}{u P(B > u)} \\ &\quad \le \limsup_{u \to \infty} \frac{\sum_{t=1}^{\infty} P(B_1 \min(C_1, \delta^{-1}(1 - 5\varepsilon)) > (1 - 5\varepsilon)(\mu t + u))}{u P(B > u)} \\ &\qquad + \limsup_{u \to \infty} \sum_{t=[u/M]}^{\infty} \frac{P(B > \delta(\mu t + u))}{u P(B > u)} \\ &\qquad\qquad \times P\biggl( \bigcup_{t > n \ge [u/M]} \biggl\{ \sum_{s=[u/M]+1}^{n} (B_s - EB) C_s > (1 - 4\varepsilon)(\mu n + u) \biggr\} \end{aligned}$$

$$\cup \bigcup_{n\ge t}\left\{\sum_{s=[u/M]+1, s\ne t}^{n}(B_s - EB)C_s > \varepsilon(\mu n + u)\right\}$$

$$\cap \{B_s \le \delta(\mu s + u), \text{all } s \ne t\}\Bigg)$$

$$= \limsup_{u\to\infty} K_1(u) + \limsup_{u\to\infty} K_2(u).$$

Similar arguments as for $\widetilde{J}(u)$ above show that

$$K_2(u) = o(1)\sum_{t=[u/M]}^{\infty}\frac{P(B > \delta(\mu t + u))}{uP(B > u)} = o(1).$$

An application of Breiman's result and Karamata's theorem yields

$$K_1(u) \sim (1-5\varepsilon)^{-\kappa} E\big[\min\big(C_1, \delta^{-1}(1-5\varepsilon)\big)\big]^{\kappa}\frac{1}{\mu}\frac{1}{\kappa-1}.$$

Noticing that

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}(1-5\varepsilon)^{-\kappa}E\big[\min\big(C_1, \delta^{-1}(1-5\varepsilon)\big)\big]^{\kappa} = EC^{\kappa},$$

the lemma is proved. $\square$

*Proof of the lower bound.* Now we want to prove that

$$\liminf_{u\to\infty}\frac{\psi(u)}{uP(B > u)} \ge EC^{\kappa}\frac{1}{\mu}\frac{1}{\kappa-1}. \tag{6.5}$$

Again, we proceed by a series of auxiliary results. We start with a short outline of the steps in the proof:

- In Lemmas 6.7 and 6.8 we show that the order of $\psi(u)$ is essentially determined by the event

$$\widetilde{D}(u) = \left\{\sup_{n\ge u/M}\left(\sum_{t=[u/M]+1}^{n}(B_tC_t - EBEC) - \mu n\right) > u\right\}.$$

- In Lemma 6.9 we complete the lower bound (6.5) of $\psi(u)$ by first showing that $\widetilde{D}(u)$ is essentially determined by the event

$$D(u) = \left\{\sup_{n\ge u/M}\left(\sum_{t=[u/M]+1}^{n}(B_t - EB)C_t - \mu n\right) > u\right\}.$$

  The probability of $D(u)$ is bounded from below by intersecting $D(u)$ with the union of the events $\{B_t > \delta(\mu t + u), B_s \le \delta(\mu t + u), \text{for all } s \ne t\}$, that is, $B_t$ is unusually large, whereas all the other $B_s$'s are smaller.

LEMMA 6.7. *For every $\varepsilon, M, \mu > 0$,*

$$\psi(u) \ge L_1(u) + o\bigl(uP(B > u)\bigr),$$

*where*

$$L_1(u) = P\biggl(\sup_{n \ge [u/M]} (T_n - T_{[u/M]} - \mu n) > u(1+\varepsilon)\biggr).$$

PROOF. We have

$$\begin{aligned}
\psi(u) &\ge P\biggl(\sup_{n \ge [u/M]} \bigl(T_n - T_{[u/M]} - \mu n\bigr) + T_{[u/M]} > u\biggr) \\
&\ge P\biggl(\sup_{n \ge [u/M]} \bigl(T_n - T_{[u/M]} - \mu n\bigr) > (1+\varepsilon)u, T_{[u/M]} \ge -\varepsilon u\biggr) \\
&\ge P\biggl(\sup_{n \ge [u/M]} \bigl(T_n - T_{[u/M]} - \mu n\bigr) > (1+\varepsilon)u\biggr) - P\bigl(T_{[u/M]} \le -\varepsilon u\bigr),
\end{aligned}$$

but, by (4.6),

$$P\bigl(T_{[u/M]} \le -\varepsilon u\bigr) = o\bigl(uP(B > u)\bigr).$$

This concludes the proof. $\square$

LEMMA 6.8. *We have, for any $\varepsilon, \mu, M > 0$, $k \ge 1$ and some $c > 0$,*

$$\begin{aligned}
L_1(u) \ge P\Biggl(\sup_{n \ge [u/M]} \Biggl(\sum_{t=[u/M]+1}^{n-k} (B_t C_t - EBEC) - (1+\varepsilon)\mu n\Biggr) > (1+3\varepsilon)u\Biggr) \\
- c(EA^\kappa)^k u P(B > u).
\end{aligned}$$

PROOF. Using the decomposition (5.1) and writing

$$R_1(k,u) = \sup_{n \ge [u/M]} \Biggl(\sum_{t=[u/M]+1}^{n-k} (B_t C_t - EBEC) - (1+\varepsilon)\mu n\Biggr),$$

$$R_2(k,u) = \sup_{n \ge [u/M]} \Biggl(\sum_{t=1}^{n-k} B_t \sum_{i=n+1}^{\infty} \Pi_{t+1,i} - \varepsilon\mu n\Biggr),$$

we have, for large $u$,

$$\begin{aligned}
L_1(u) &\ge P\bigl(R_1(k,u) - R_2(k,u) > (1+2\varepsilon)n\bigr) \\
&\ge P\bigl(R_1(k,u) > (1+3\varepsilon)u, -R_2(k,u) > -\varepsilon u\bigr) \\
&\ge P\bigl(R_1(k,u) > (1+3\varepsilon)u\bigr) - P\bigl(R_2(k,u) \ge \varepsilon u\bigr) \\
&= L_2(u) - L_3(u).
\end{aligned}$$

We show that

$$L_3(u) \le c(EA^\kappa)^k u P(B > u).$$

We have, for $k \ge 1$,

$$\begin{aligned}
L_3(u) &\le P\Biggl(\sup_{n \ge [u/M]}\Biggl(\sum_{t=1}^{[u/M]} B_t \Pi_{t+1,n+1} C_{n+1} - \varepsilon\mu n/2\Biggr) \ge \varepsilon u/2\Biggr) \\
&\quad + P\Biggl(\sup_{n \ge [u/M]}\Biggl(\sum_{t=[u/M]+1}^{n-k} B_t \Pi_{t+1,n+1} C_{n+1} - \varepsilon\mu n/2\Biggr) \ge \varepsilon u/2\Biggr) \\
&= L_{3,1}(u) + L_{3,2}(u).
\end{aligned}$$

Then, by (5.3) and Markov's inequality, for $0 < \delta < 1$,

$$\begin{aligned}
L_{3,1}(u) &\le \sum_{n=[u/M]}^{\infty} P\Biggl(\sum_{t=1}^{[u/M]} B_t \Pi_{t+1,[u/M]} \Pi_{[u/M]+1,n+1} C_{n+1} > (\varepsilon/2)(\mu n + u)\Biggr) \\
&\le \sum_{n=[u/M]}^{\infty} P\bigl(Y_0 \Pi_{[u/M]+1,n+1} C_{n+1} > (\varepsilon/2)(\mu n + u)\bigr) \\
&\le c \sum_{n=[u/M]}^{\infty} (EA^{\kappa-\delta})^{n-[u/M]} (n+u)^{-\kappa+\delta} \\
&\le c u^{-\kappa+\delta} = o\bigl(uP(B > u)\bigr).
\end{aligned}$$

Moreover, by (5.3) and Breiman's result,

$$\begin{aligned}
L_{3,2}(u) &\le \sum_{n=[u/M]}^{\infty} P\Biggl(\sum_{t=[u/M]+1}^{n-k} B_t \Pi_{t+1,n-k} \Pi_{n-k+1,n+1} C_{n+1} > (\varepsilon/2)(\mu n + u)\Biggr) \\
&\le \sum_{n=[u/M]}^{\infty} P\bigl(Y_0 \Pi_{n-k+1,n+1} C_{n+1} > (\varepsilon/2)(\mu n + u)\bigr) \\
&\le c(EA^\kappa)^k u P(B > u). \qquad \square
\end{aligned}$$

Next we bound $L_2$.

LEMMA 6.9. *We have, for every $k \ge 1$,*

$$\lim_{\varepsilon \downarrow 0} \liminf_{u \to \infty} \frac{L_2(u)}{uP(B > u)} \ge EC^\kappa \frac{1}{\mu} \frac{1}{\kappa - 1}.$$

PROOF. Writing

$$R_1(k,u)=\sup_{n\ge[u/M]}\Biggl(\sum_{t=[u/M]+1}^{n-k}(B_t-EB)C_t-(1+2\varepsilon)\mu n\Biggr),$$

$$R_2(k,u)=\inf_{u\ge[u/M]}\Biggl(EB\sum_{t=[u/M]+1}^{n-k}(C_t-EC)+\varepsilon\mu n\Biggr),$$

we have

$$\begin{aligned}L_2(u)&\ge P\bigl(R_1(k,u)+R_2(k,u)>(1+3\varepsilon)u\bigr)\\&\ge P\bigl(R_1(k,u)>(1+4\varepsilon)u,R_2(k,u)>-\varepsilon u\bigr)\\&\ge P\bigl(R_1(k,u)>(1+4\varepsilon)u\bigr)-P\bigl(R_2(k,u)\le-\varepsilon u\bigr)\\&=L_4(u)-L_5(u).\end{aligned}$$

Lemma 6.3 and its proof show that

$$L_5(u)=o\bigl(uP(B>u)\bigr).$$

Now we turn to $L_4$. Writing

$$\begin{aligned}D_t(\delta,u)&=\bigl\{B_s\le\delta(\mu s+u)\text{ for all }s\in\bigl[[u/M],\infty\bigr)\setminus\{t\}\bigr\},\\E_t(\delta,u)&=\bigl\{B_t\min\bigl(C_t,\delta^{-1}(1+5\varepsilon)\bigr)>(1+5\varepsilon)(\mu t+u)\bigr\},\end{aligned}$$

we have, for small $\delta>0$,

$$\begin{aligned}L_4(u)\ge\sum_{t=[u/M]}^{\infty}P\Biggl(&\{B_t>\delta(\mu t+u)\}\cap D_t(\delta,u)\\&\cap\Biggl\{\sup_{n\ge t}\Biggl(\sum_{r=[u/M]+1}^{n-k}(B_r-EB)C_r-(1+4\varepsilon)\mu n\Biggr)\\&\qquad>(1+4\varepsilon)u\Biggr\}\Biggr)\\\ge\sum_{t=[u/M]}^{\infty}&P\bigl(E_t(\delta,u)\cap D_t(\delta,u)\bigr)\\-\sum_{t=[u/M]}^{\infty}&P\Biggl(E_t(\delta,u)\cap D_t(\delta,u)\\&\cap\Biggl\{\sup_{n\ge t}\Biggl(\sum_{r=[u/M]+1}^{n-k}(B_r-EB)C_r-(1+4\varepsilon)\mu n\Biggr)\end{aligned}$$

$$\leq (1+4\varepsilon)u\Bigg\}\Bigg)$$

$$\begin{aligned}
&\geq \sum_{t=[u/M]}^{\infty} P\big(E_1(\delta,u)\big)P\big(B_s \leq \delta(\mu s+u) \text{ for all } s \geq [u/M]\big)\\
&\quad - \sum_{t=[u/M]}^{\infty} P\Bigg(E_t(\delta,u)\cap D_t(u,\delta)\\
&\qquad\qquad \cap \Bigg\{\sup_{n\geq t}\Bigg(\sum_{r=[u/M]+1, r\neq t}^{n-k}(B_r - EB)C_r - (1+4\varepsilon)\mu n\Bigg)\\
&\qquad\qquad\qquad \leq (1+4\varepsilon)u - (1+5\varepsilon)(\mu t+u) + EBC_t\Bigg\}\Bigg)\\
&= L_{4,1}(u) - L_{4,2}(u).
\end{aligned}$$

By Breiman's result and Karamata's theorem, as $u \to \infty$,

$$\begin{aligned}
L_{4,1}(u) &\sim \sum_{t=[u/M]}^{\infty} \big[(1+5\varepsilon)^{-\kappa} E\big[\min\big(C_1, \delta^{-1}(1+5\varepsilon)\big)\big]^{\kappa} P(B > \mu t+u)\big]\\
&\qquad \times P\big(B_s \leq \delta(\mu s+u) \text{ for all } s \geq [u/M]\big)\\
&\geq (1+6\varepsilon)^{-\kappa} E\big[\min\big(C_1, \delta^{-1}(1+5\varepsilon)\big)\big]^{\kappa} \sum_{t=[u/M]}^{\infty} P(B > \mu t+u)\\
&\sim (1+6\varepsilon)^{-\kappa} E\big[\min\big(C_1, \delta^{-1}(1+5\varepsilon)\big)\big]^{\kappa} \frac{1}{\mu}\frac{1}{\kappa-1} P(B > u).
\end{aligned}$$

We conclude that

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}\liminf_{u\to\infty} \frac{I_{4,1}(u)}{uP(B>u)} \geq EC^{\kappa}\frac{1}{\mu}\frac{1}{\kappa-1}.$$

As regards $L_{4,2}(u)$, we have

$$\begin{aligned}
L_{4,2}(u) &\leq c \sum_{t=[u/M]}^{\infty} P(B > t+u)\\
&\qquad \times P\Bigg(\sup_{n\geq t}\Bigg(\sum_{r=[u/M]+1, r\neq t}^{n-k}(B_r - EB)C_r - (1+4\varepsilon)\mu n\Bigg)\\
&\qquad\qquad \leq -\varepsilon u - (1+5\varepsilon)\mu t + EBC_t\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&\le c \sum_{t=[u/M]}^{\infty} P(B > t + u) \\
&\qquad \times P\Biggl(\sup_{n \ge t}\Biggl(\sum_{r=[u/M]+1, r\ne t}^{n-k} (B_r - EB)C_r - (1+4\varepsilon)\mu n\Biggr) \\
&\qquad\qquad \le -\varepsilon u - (1+5\varepsilon)\mu t + EBM\Biggr) \\
&\quad + c \sum_{t=[u/M]}^{\infty} P(B > t + u) P(C > M) \\
&= I_{4,2,1}(u) + I_{4,2,2}(u).
\end{aligned}
$$

We have

$$
\lim_{M\to\infty} \limsup_{u\to\infty} \frac{I_{4,2,2}(u)}{uP(B > u)} \le c \lim_{M\to\infty} P(C > M) = 0.
$$

Observe that, for large $u$,

$$
\begin{aligned}
&P\Biggl(\sup_{n \ge t}\Biggl(\sum_{r=[u/M]+1, r\ne t}^{n-k} (B_r - EB)C_r - (1+4\varepsilon)\mu n\Biggr) \\
&\qquad \le -\varepsilon u - (1+5\varepsilon)\mu t + EBM\Biggr) \\
&\quad \le P\Biggl(\Biggl(\sum_{r=[u/M]+1, r\ne t}^{[u/M]+u-k} (B_r - EB)C_r - (1+4\varepsilon)\mu([u/M] + u)\Biggr) \le -\varepsilon u\Biggr).
\end{aligned}
$$

Now an argument similar to the one for Theorem 4.2 shows that

$$
I_{4,2,1}(u) = o\bigl(uP(B > u)\bigr).
$$

This proves the lemma. $\square$

Now a combination of the above lemmas shows that the lower bound (6.5) holds. Indeed, we have, for any $k \ge 1$,

$$
\liminf_{u\to\infty} \frac{\psi(u)}{uP(B > u)} \ge \liminf_{u\to\infty} \frac{L_1(u)}{uP(B > u)} \ge \liminf_{u\to\infty} \frac{L_2(u)}{uP(B > u)} - c(EA^\kappa)^k.
$$

Now, observing that $EA^\kappa < 1$, let $k \to \infty$, $\varepsilon \downarrow 0$. This proves the theorem. $\square$

*Extensions.* A careful study of the proofs in the previous sections shows that the particular structure of the sequence $(Y_t)$ was inessential for the proofs. Indeed, we made extensive use of the fact that the random walk $(S_n)$ can be approximated by the random walk $\widetilde{S}_n = \sum_{t=1}^n B_t C_t$. It is not difficult to see that the results of Theorems 4.2 and 4.9 remain valid if $S_n$ is replaced by $\widetilde{S}_n$ and the following conditions on any stationary sequence $(C_t)$ hold: $(B_t)$ is independent of $(C_t)$, $(C_t)$ is strongly mixing with geometric rate, $EC^{\kappa+\gamma} < \infty$ for some $\gamma > \kappa$ and (4.4) holds. Moreover, the assertion of Lemma 4.6 remains valid. A stationary sequence $X_t = B_t C_t$ for $(B_t)$ and $(C_t)$ independent is called a *stochastic volatility model* in the econometrics literature; see [11] for some theory and further references.

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DEPARTMENT OF STATISTICS
AND ACTUARIAL SCIENCE
UNIVERSITY OF THE AEGEAN
KARLOVASSI
GR-83 200 SAMOS
GREECE
E-MAIL: konstant@aegean.gr
URL: www.samos.aegean.gr/math/konstant

LABORATORY OF ACTUARIAL MATHEMATICS
UNIVERSITY OF COPENHAGEN
UNIVERSITETSPARKEN 5
DK-2100 COPENHAGEN
DENMARK
AND
MAPHYSTO
THE DANISH RESEARCH NETWORK
IN MATHEMATICAL PHYSICS AND STOCHASTICS
E-MAIL: mikosch@math.ku.dk
URL: www.math.ku.dk/~mikosch