EXACT SIMULATION OF BROWN-RESNICK RANDOM FIELDS

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Abstract. We propose an exact simulation method for Brown-Resnick random fields, building on new representations for these stationary max-stable fields. The main idea is to apply suitable changes of measure.

1. Introduction

Max-stable random fields are fundamental building blocks for spatial extremes. These models have been coined by de Haan [8], and have recently found applications to extreme meteorological events such as rainfall modeling and extreme temperatures (Buishand et al. [2], de Haan and Zhou [9], Dombry et al. [5], Davis et al. [3], Huser and Davison [10]). These processes are considered the analogs of Gaussian processes in the world of extremes. There are three different kinds of max-stable processes, with Gumbel, Fréchet, and Weibull marginals, respectively. In what follows, we restrict ourselves to max-stable processes with Gumbel marginals; corresponding results for Fréchet and Weibull marginals can be obtained by a monotone transformation of the Gumbel case.

This paper studies a particular class of max-stable random fields known as *Brown-Resnick random fields*. Simulation of these and related processes is complicated, and an extensive literature has been devoted to approximate simulation; see for example Schlather [17], Oesting et al. [14], Engelke et al. [7], Oesting and Schlather [15], Dombry et al. [5].

This paper is the first to devise an *exact* simulation method for Brown-Resnick random fields. The key ingredient is a new representation for Brown-Resnick random fields, which is of independent interest. In fact, we show that there is an uncountable *family* of representations. At the heart of our derivation of these representations lies a change of measure argument.

We now describe the results in this paper in more detail. For some index set $T \subset \mathbb{R}^d$, the process $(Y(t))_{t\in T}$ of non-negative random variables is *max-stable* (with Gumbel marginals) if for a sequence of iid copies $(Y^{(i)}(t))_{t\in T}$, $i = 1, 2, \ldots$, of $(Y(t))_{t\in T}$ the following relation holds

$$\big(\max_{i=1,\ldots,n} Y^{(i)}(t) - \log n \big)_{t\in T} \stackrel{d}{=} (Y(t))_{t\in T}\,, \quad n \geq 1\,,$$

where this relation is interpreted in the sense of equality of the finite-dimensional distributions. Then, in particular, all one-dimensional marginals of the process $(Y(t))_{t\in T}$ are Gumbel distributed, i.e., $Y(t)$ has distribution function $\Lambda(x-c(t)) = \exp(-\mathrm{e}^{-(x-c(t))})$, $x \in \mathbb{R}$, for some function $c(t) \in \mathbb{R}$, $t \in T$. Throughout this paper, we work with $T = \mathbb{R}^d$.

In this paper, we consider a class of max-stable processes with representation

$$\eta(t) = \sup_{i\geq 1} \big(V_i + W_i(t) - \sigma^2(t)/2 \big)\,, \qquad t \in \mathbb{R}^d\,, \tag{1.1}$$

where $\sigma^2(t) = \mathbb{V}\mathrm{ar}(W_1(t))$, $t \in \mathbb{R}^d$, (W_i) is a sequence of iid centered Gaussian processes with stationary increments on $\mathbb{R}^d$, and (V_i) are the points of a Poisson process on $\mathbb{R}$ with intensity measure $\mathrm{e}^{-x}\,dx$. In the case of a Brownian motion, the process (1.1) was considered by Brown and

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Resnick [1] and shown to be stationary. It is common to refer to the more general model (1.1) as *Brown-Resnick random field* as well.

The representation (1.1) is not particularly suitable for exact sampling. Although (V_i) and (W_i) are easily simulated, it turns out that the naive simulation approach of replacing $\sup_{i\geq 1}$ by $\sup_{i\leq N}$ for some large N, may fail. For example, assume that W_i is standard Brownian motion on $\mathbb{R}$. Then, in view of the law of the iterated logarithm, each of the processes $W_i(t)$ drifts to $-\infty$ a.s. as $t\to\infty$. In turn, the process $\sup_{i\leq N}\left(V_i + W_i(t) - \sigma^2(t)/2\right)$ drifts to $-\infty$ as $t\to\infty$ as well. In particular, the simulation of η requires an increasing number N if one aims at a sample path of the process on a larger interval. More importantly, it is unclear how N should be chosen.

Using our new representations, we obtain an exact sampling method for η at the points $t_1,\ldots,t_n\in\mathbb{R}^d$, meaning that the output of the method has the same distribution as $(\eta(t_1),\ldots,\eta(t_n))$. In our method, it is no longer problematic that the processes $Z_i(t)$ drift away to $-\infty$.

Several properties of Brown-Resnick processes readily follow from our representations, although they are not straightforward to see from (1.1). For instance, the process η is stationary in the sense that η has the same distribution as $\eta(\cdot+c)$ for any choice of $c\in\mathbb{R}^d$. Moreover, the process η also has Gumbel Λ marginals. In deriving our representations from (1.1), σ^2 drops out and we recover the known fact that the law of η only depends on the *variogram*

$$\gamma(t) = \frac{1}{2}\mathbb{E}(W(t)-W(0))^2\,, \qquad t\in\mathbb{R}^d.$$

These properties were proved in Kabluchko et al. [11] with arguably more elaborate techniques.

Given an exact simulation method for the Brown-Resnick process with Gumbel marginals, we also have an exact simulation method for this process with Fréchet or Weibull marginals. For example, the processes $\mathrm{e}^{\,\eta}$ and $-\mathrm{e}^{\,-\eta}$ have Fréchet $\Phi_1(x)=\mathrm{e}^{\,-x^{-1}}$, $x>0$, and Weibull $\Psi_1(x)=\mathrm{e}^{\,-|x|}$, $x<0$, marginals, respectively.

Notation. We use the symbols $W_1, W_2,\ldots$ for iid centered Gaussian random fields with stationary increments, variance function σ^2, and variogram γ. We use the symbols $Z_1, Z_2,\ldots$ for iid Gaussian random fields with stationary increments, mean function $-\gamma$, variance function 2γ, and variogram γ. A generic copy of these fields is denoted by W and Z, respectively.

2. Representations

In this section we provide new representations for the Brown-Resnick random field η given in (1.1). These representations arise from a change of measure. We make the same assumptions on the stationary Brown-Resnick process as in the previous section. All proofs for this section are in Section 5. We fix the functions σ^2 and γ throughout this section.

The following theorem is the main result of this section.

Theorem 2.1. *Suppose we are given an arbitrary probability measure μ on $\mathbb{R}^d$. Consider*

$$\zeta(t) = \sup_{i\geq 1}\Big(V_i + Z_i(t-T_i) - \log\big(\int_{\mathbb{R}^d}\exp\left(Z_i(s-T_i)\right)\mu(ds)\big)\Big)\,, \qquad t\in\mathbb{R}^d\,,$$

where $\big((V_i,T_i)\big)_{i\geq 1}$ are the points of a Poisson process on $\mathbb{R}\times\mathbb{R}^d$ with intensity measure $\mathrm{e}^{\,-x}dx\times\mu(ds)$. Then the random fields $(\eta(t))_{t\in\mathbb{R}^d}$ and $(\zeta(t))_{t\in\mathbb{R}^d}$ have the same distribution.

Remark 2.2. There is a continuum of random fields with the same distribution as η, one for each measure μ.

Remark 2.3. Oesting et al. [14] provided various alternative point process representations of the Brown-Resnick process. In particular, they proposed to introduce random time shifts of the processes W_i and used this idea to derive approximate sampling methods for η. Their representations are different from ours.

Remark 2.4. If $\sigma^2/2 = \gamma$, then $(W_i(t - T_i))$ has the same distribution as $(W_i(t) - W_i(T_i))$. The term $W_i(T_i)$ drops out of the expression for ζ, so in that case the random field

$$\sup_{i\geq 1}\Big(V_i + W_i(t) - \gamma(t - T_i) - \log\big(\int_{\mathbb{R}^d} \exp\big(W_i(s) - \gamma(s - T_i)\big)\mu(ds)\big)\Big), \qquad t \in \mathbb{R}^d,$$

also has the same distribution as $(\eta(t))_{t\in\mathbb{R}^d}$.

Theorem 2.1 leads to the following three well known facts proved in Kabluchko et al. [11].

Corollary 2.5. *The field η is stationary.*

Proof. Let μ be a Dirac point mass at some arbitrary $t^* \in \mathbb{R}^d$. Theorem 2.1 implies that the random field $(\sup_{i\geq 1}(V_i + Z_i(t - t^*)))_{t\in\mathbb{R}^d}$ has the same distribution as $(\eta(t))_{t\in\mathbb{R}^d}$. In particular, the distribution does not depend on t^*. □

Corollary 2.6. *The one-dimensional marginals of $(\eta(t))_{t\in\mathbb{R}^d}$ have the Gumbel distribution.*

Proof. If we let μ be a point mass as in the proof of the preceding corollary, then we find that $\eta(t)$ has the same distribution as $\sup_{i\geq 1} V_i$ for every $t \in \mathbb{R}^d$. □

Corollary 2.7. *The distribution of $(\eta(t))_{t\in\mathbb{R}^d}$ only depends on the variogram γ.*

Proof. Since the processes Z_i are completely determined by γ, the law of $(\zeta(t))_{t\in\mathbb{R}^d}$ depends only on γ. Theorem 2.1 therefore immediately yields the claim. □

There are some interesting connections between Brown-Resnick random fields and familiar quantities in extreme value theory, which simply follow from the known finite-dimensional distribution functions of such fields. Details on this distribution function can be found in Section 5 (specifically Lemma 5.1); for now, we note that if η is stochastically continuous we have, for any $N > 0$,

$$\mathbb{P}\Big(\sup_{t\in[0,N]^d}\eta(t) \leq x\Big) \;=\; \exp\Big(-\mathrm{e}^{-x}\,\mathbb{E}\exp\big(\sup_{t\in[0,N]^d} Z(t)\big)\Big), \qquad x \in \mathbb{R},$$

and therefore

$$\mathbb{P}\Big(\sup_{t\in[0,N]^d}\eta(t) - d\log N \leq x\Big) \;=\; \exp\Big(-\mathrm{e}^{-x}\,N^{-d}\,\mathbb{E}\exp\big(\sup_{t\in[0,N]^d} Z(t)\big)\Big), \qquad x \in \mathbb{R}.$$

Dieker and Yakir [4, Cor. 1], show that the set function

$$f(A) = \mathbb{E}\exp\big(\sup_{t\in A} Z(t)\big), \qquad A \subset \mathbb{R}^d$$

is translation invariant: $f(A) = f(t+A)$ for $t \in \mathbb{R}^d$. (They only write out the one-dimensional case, but the multidimensional case follows from exactly the same arguments; it is based on Lemma 5.2 below.) Moreover, f is subadditive in the sense that $f(A_1 \cup A_2) \leq f(A_1) + f(A_2)$ for disjoint subsets $A_1, A_2 \subset \mathbb{R}^d$. A basic fact about such functions (e.g., Xanh [18]) is that $f(A)$ grows like the Lebesgue measure of A for large sets A. In particular, this result implies that the limit

$$\lim_{N\to\infty} N^{-d}\,\mathbb{E}\exp\Big(\sup_{t\in[0,N]^d} Z(t)\Big)$$

exists. This quantity is known as *Pickands's constant*. The numerical determination of this constant and the simulation of the Brown-Resnick process η suffer from the same problems mentioned in the Introduction. Dieker and Yakir [4] proposed a Monte Carlo method for determining the Pickands constant.

The discrete analogs of Pickands's constant are connected to extremal indices of the Brown-Resnick processes. Assume $d = 1$ and consider a Brown-Resnick process $(\eta(t))_{t\in\mathbb{R}}$. Its restriction to the integers yields a strictly stationary time series $(\eta(i))_{i\in\mathbb{Z}}$. For $x \in \mathbb{R}$ we have

$$\mathbb{P}\Big(\max_{i=1,\dots,n}\eta(i) - \log n \leq x\Big) = \exp\Big(-\mathrm{e}^{-x}n^{-1}\mathbb{E}\Big[\max_{i=1,\dots,n}\mathrm{e}^{Z(i)}\Big]\Big).$$

This leads to the limit relation

$$\lim_{n\to\infty} \mathbb{P}\Big(\max_{i=1,\ldots,n} \eta(i) - \log n \le x \Big) = \Lambda^\theta(x)\,, \qquad x \in \mathbb{R}\,,$$

where the limit

$$\theta = \lim_{n\to\infty} n^{-1} \mathbb{E}\Big[\max_{i=1,\ldots,n} \mathrm{e}^{\,Z(t)} \Big]$$

exists by subadditivity and translation invariance as in the continuous case. It is well known (see Leadbetter et al. [13], cf. Section 8.1 in Embrechts et al. [6]) that θ is a number in $[0,1]$. The quantity θ is the *extremal index* of the stationary sequence $(\eta(i))_{i\in\mathbb{Z}}$. The reciprocal of this quantity is often interpreted as the expected value of the cluster size of high-level exceedances of the sequence (X_i); see for example [13]; cf. Section 8.1 in [6]. The constant θ appears in Dieker and Yakir [4] as a special case of the constants $\eta\mathcal{H}_\alpha^\eta$; see Proposition 3 in [4] for a characterization alternative to the extremal index. Although we do not have a proof that θ is smaller than Pickands's constant in the continuous-time case, simulation evidence indicates that this fact is true.

3. A simulation algorithm

This section presents a simulation algorithm for Brown-Resnick random fields on a discrete set of points $t_1, \ldots, t_n \in \mathbb{R}^d$. We may assume that $\sigma^2/2 = \gamma$ in this section. Since Theorem 2.1 gives a different representation for each choice of μ, it would be interesting to know which choice leads to the fastest algorithm. Here we simply let μ be uniform on $\{t_1, \ldots, t_n\}$.

Remark 2.4 shows that the vector $(N(t_1), \ldots, N(t_n))$ with, for $j = 1, \ldots, n$,

$$N(t_j) = \sup_{i\ge 1} \left(V_i + W_i(t_j) - \gamma(t_j - T_i) - \log\Big(n^{-1} \sum_{\ell=1}^n \exp(W_i(t_\ell) - \gamma(t_\ell - T_i)) \Big) \right)$$

has the same distribution as $(\eta(t_1), \ldots, \eta(t_n))$, where $\big((V_i, T_i)\big)_{i\ge1}$ belong to a Poisson process on $\mathbb{R} \times \{t_1, \ldots, t_n\}$ with intensity measure $\mathrm{e}^{-x}dx \times \big(n^{-1} \sum_{i=1}^n \delta_{t_i}(dy)\big)$. We slightly rewrite the above display as

$$N(t_j) = \sup_{i\ge 1} \left(V_i + \log n + W_i(t_j) - \gamma(t_j - T_i) - \log\Big(\sum_{\ell=1}^n \exp(W_i(t_\ell) - \gamma(t_\ell - T_i)) \Big) \right).$$

This is the representation we use for our simulation algorithm.

A point V_i on $\mathbb{R}$ gives rise to a 'cluster' of points $\{C_i(t_j) : j = 1, \ldots, n\}$ with

$$C_i(t_j) = (V_i + \log n) + W_i(t_j) - \gamma(t_j - t_{T_i}) - \log\Big(\sum_{\ell=1}^n \exp(W_i(t_\ell) - \gamma(t_\ell - t_{T_i})) \Big).$$

These cluster points can be visualized by interpreting them as belonging to different levels depending on the value of j; see Figure 1. The variable $N(t_j) = \sup_{i\ge1} C_i(t_j)$ is then the maximum of all cluster points on the j-th level. The crucial insight is that *only a finite number of points/cluster pairs* (V_i, C_i) *need to be generated, since* $C_i(t_j) \le V_i + \log n$ *and we seek* $\sup_{i\ge1} C_i(t_j)$ *for* $j = 1, \ldots, n$. The algorithm generates points/cluster pairs $(V + \log n, C)$ in decreasing order of $(V + \log n)$-value, until the next $(V + \log n)$-value is smaller than the current maximum over the cluster points on each level. For instance, in Figure 1, after $V_3 + \log 4$ has been generated, none of the remaining cluster points can change the values of $(N(t_1), \ldots, N(t_4))$, which have been given a different color.

We remark that this algorithm is suitable for parallelization. Indeed, several points of the V-process can be generated simultaneously instead of one at the time, with corresponding clusters being computed on different processors.

To get a sense of how many points of V will be generated, let us consider the (degenerate) case where $t_1 = \cdots = t_n = t$. We then have $C_i(t_j) = V_i$ for $j = 1, \ldots, n$, so the algorithm terminates

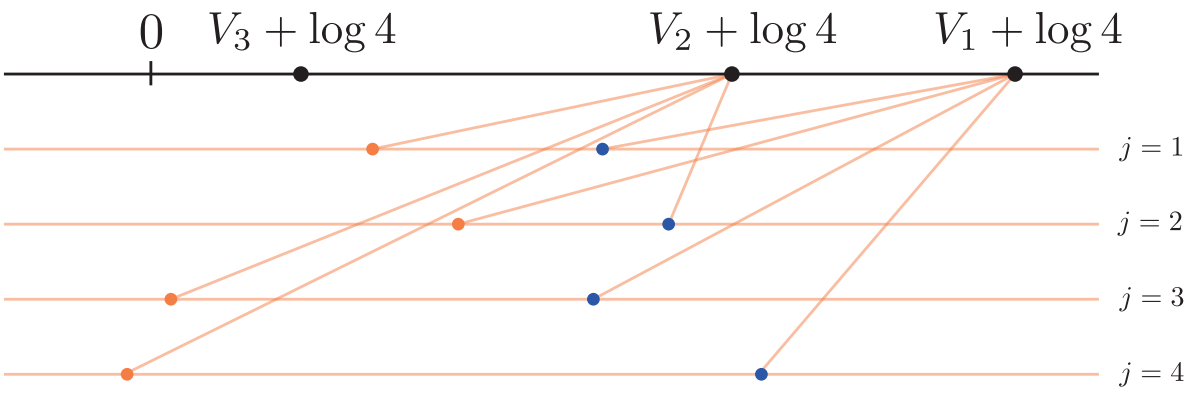

FIGURE 1. Illustration of our algorithm for $n = 4$. The points $V_i + \log 4$ are generated in decreasing order. Each 'level' below the axis represents a value of j, and each $(V_i + \log 4)$-point is connected to its cluster points $C_i(t_j)$. The cluster points $C_i(t_j)$ always lie to the left of $V_i + \log 4$.

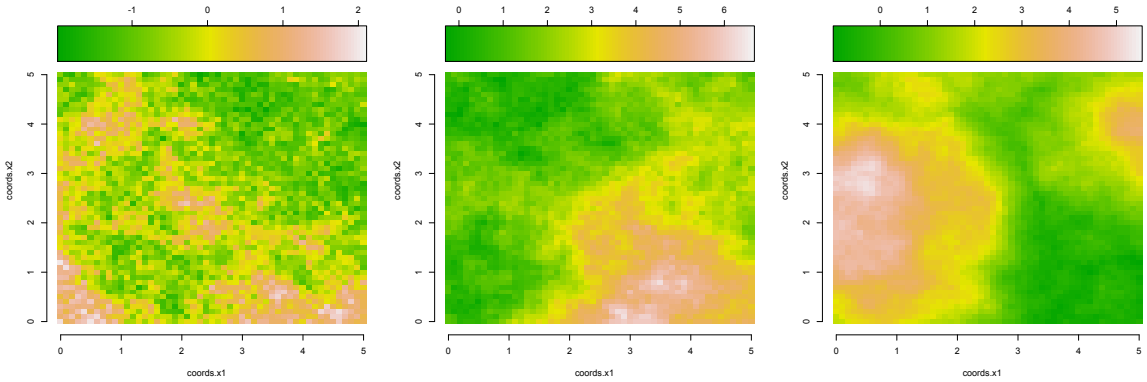

FIGURE 2. Sample of a Brown-Resnick random field on $[0, 5]^2$ with variogram $\gamma(t) = |t|^\alpha/2$ for $\alpha = 1/2$, $\alpha = 1$, $\alpha = 3/2$ from left to right, respectively. The grid mesh is 0.1.

after generating $\inf\{M : V_M + \log n < V_1\}$ points of V. For large n, this implies that the number of points is of order n.

4. Numerical experiments

This section reports on several simulation experiments we have carried out in order to validate our algorithm and to test its performance in terms of speed. Throughout, we work with Brown-Resnick random fields with variogram $\gamma(t) = |t|^\alpha/2$ for some $\alpha \in (0, 2]$.

Smoothness. We have implemented the algorithm in R (see [16]) in order to leverage the existing toolkit to generate the Gaussian random fields that are needed in our algorithm. We use the R package `RandomFields` by Schlather et al., which is available through R's package manager. Three representative samples of Brown-Resnick random fields are given in Figure 2, with various levels of a smoothing parameter α. We see that the paths become rougher as α decreases, as it should be. The random field is the maximum of random 'mountains' (given by quadratic forms) if $\alpha = 2$, and our replication for $\alpha = 3/2$ exhibits similar behavior in the sense that two mountains can be distinguished.

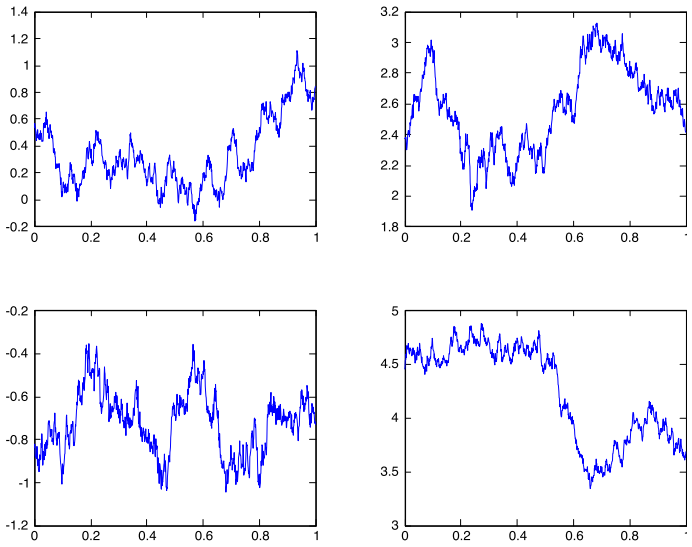

FIGURE 3. Representative samples of a Brown-Resnick process on $[0,1]$ with variogram $\gamma(t) = |t|/2$.

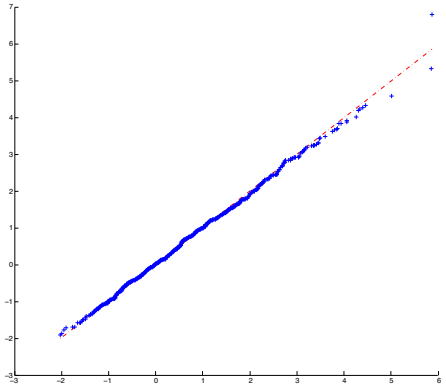

FIGURE 4. A representative Q–Q plot illustrating the equality of the marginal distributions.

Marginal distributions. The experiments in the rest of this section are for the one-dimensional case $d = 1$ for computational ease. We generate the Gaussian processes with the recent Matlab implementation by Kroese and Botev [12]. Appendix A has the details.

Figure 3 depicts some representative samples for $\alpha = 1$. Note that it indeed appears that these are realizations of a stationarity process even though our algorithm does not require truncating the number of Gaussian random field samples if one aims at a sample path of the process on a larger interval.

We have generated 1000 iid replications of the vector $(\eta(0), \eta(1/1024), \ldots, \eta(1 - 1/1024))$ for $\alpha = 1$. Figure 4 compares the empirical quantiles of $\eta(0)$ and $\eta(1 - 1/1024)$. Note that the distribution of both these variables is the standard Gumbel distribution, and the Q–Q plot supports that the simulated distributions are indeed equal.

Max-stability. The max-stable property of η implies that $(\eta_1)_{t\in\mathbb{R}^d}$ and $(\max(\eta_2, \eta_3) - \log(2))_{t\in\mathbb{R}^d}$ have the same distribution. Interpreting η as a vector, we verify the equality of the distribution of

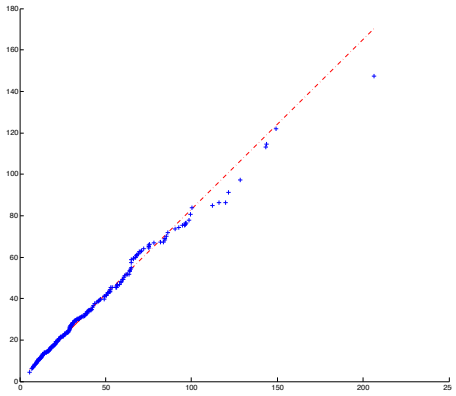

FIGURE 5. Q–Q plot illustrating the max-stable property of our simulation output.

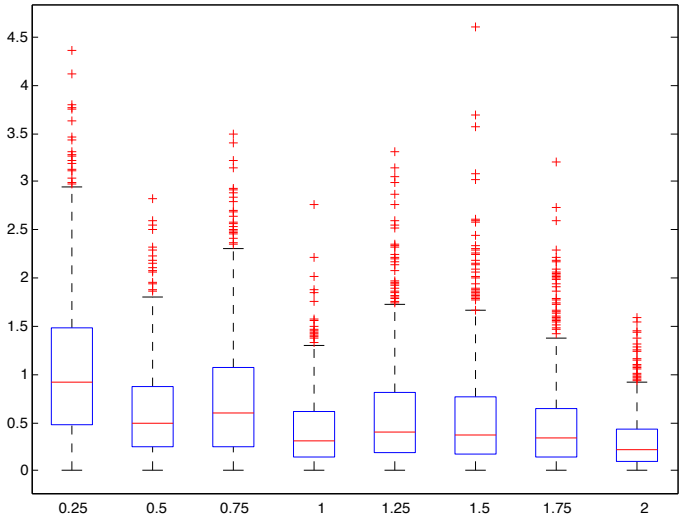

FIGURE 6. Box plot showing the dependence of α on the algorithm's running time (in seconds). The edges of the box are the 25th and 75th percentiles.

$\|\eta_1\|$ and $\|\max(\eta_2, \eta_3) - \log(2)\|$, where η_1, η_2, η_3 are iid and the maximum is taken component-wise. Using our 1000 replications, we get a total of 333 samples from $\|\eta_1\|$ and 333 from $\|\max(\eta_2, \eta_3) - \log(2)\|$. The resulting Q–Q plot is given in Figure 5.

Speed. An interesting question is the relationship between the smoothness parameter α and the speed of the computer code. The computational effort needed to generate a sample from the underlying Gaussian process is independent of α in our implementation. However, the dependency structure may influence the number of points V_i that need to be generated for a single replication of the Brown-Resnick process, thereby impacting the speed of our algorithm.

To investigate the influence of α on the algorithm speed, we generated 1000 replications of $(\eta(0), \eta(1/1024), \ldots, \eta(1-1/1024))$ on a single core of a 2.7 GHz Intel Core i7 processor for various values of α. A sample is generated in the order of seconds regardless the value of α. The running time depends linearly on the number of points V_i that are generated by the algorithm, which is different for different samples. The results are summarized in the box plot in Figure 6. The data gives evidence that rougher paths are harder to simulate, which suggests that the order n bound derived in Section 3 is in fact a lower bound on the number of V_i points that need to be generated.

In the code used for Figure 6, we preprocess some of the computations required for sampling the W_i. This results in significant savings. We have not included this code in Appendix A for expository reasons.

5. Proofs

This section presents the proof of Theorem 2.1 We fix the functions σ^2 and γ throughout this section. Contrary to the preceding two sections, we do not assume that $\gamma = \sigma^2/2$ but we shall see that the function σ^2 vanishes from our calculations.

We start with an auxiliary lemma. The proof uses standard arguments, but we include it for completeness; see also de Haan [8] and Kabluchko et al. [11].

Lemma 5.1. *Let (X_i) be iid copies of some random field X on $\mathbb{R}^d$ and (V_i) the points of a Poisson process on $\mathbb{R}$ with intensity measure $\mathrm{e}^{-x}\,dx$. If we write*

$$\xi(t) = \sup_{i\geq 1} (V_i + X_i(t))\,, \qquad t \in \mathbb{R}^d\,,$$

then we have for $y_j \in \mathbb{R}\,, t_j \in \mathbb{R}^d\,, i = 1, \ldots, n,$

$$\mathbb{P}(\xi(t_1) \leq y_1, \ldots, \xi(t_n) \leq y_n) = \exp\Big(- \mathbb{E} \exp\Big(\max_{j=1,\ldots,n} (X(t_j) - y_j)\Big)\Big)\,.$$

Proof. We observe that

$$\mathbb{P}(\xi(t_1) \leq y_1, \ldots, \xi(t_n) \leq y_n) = \mathbb{P}\Big(\max_{j=1,\ldots,n} \sup_{i\geq 1} \Gamma_i^{-1} \exp\big(X_i(t_j) - y_j\big) \leq 1\Big)\,,$$

where $0 < \Gamma_1 < \Gamma_2 < \cdots$ are the ordered points of a unit rate Poisson process on $(0,\infty)$ independent of (X_i). We write $N(t) = \#\{i \geq 1 : \Gamma_i \leq t\}$, $t > 0$. Then an application of the order statistics property of N implies that

$$\begin{aligned}
&\mathbb{P}(\xi(t_1) \leq y_1, \ldots, \xi(t_n) \leq y_n)\\
=\ & \lim_{t\to\infty} \mathbb{E}\Big[\mathbb{P}\Big(\max_{j=1,\ldots,n} \sup_{i\leq N(t)} \Gamma_i^{-1} \exp\big(X_i(t_j) - y_j\big) \leq 1 \mid N(t)\Big)\Big]\\
=\ & \lim_{t\to\infty} \mathbb{E}\Big[\mathbb{P}\Big(\sup_{i\leq N(t)} (t\beta_i)^{-1} \max_{j=1,\ldots,n} \exp\big(X_i(t_j) - y_j\big) \leq 1 \mid N(t)\Big)\Big]\\
=\ & \lim_{t\to\infty} \mathbb{E}\Big[\Big(\mathbb{P}\Big((t\beta_1)^{-1} \max_{j=1,\ldots,n} \exp\Big(X_i(t_j) - y_j\Big) \leq 1\Big)\Big)^{N(t)}\Big]\,,
\end{aligned}$$

where (β_i) is an iid sequence of uniform random variables on $(0,1)$ independent of (X_i) and N. Now direct calculation shows that the right-hand side equals

$$\begin{aligned}
&\lim_{t\to\infty} \exp\Big(- \int_0^t \mathbb{P}\Big(\max_{j=1,\ldots,n} \exp\Big(X_i(t_j) - y_j\Big) > z\Big)\, dz\Big)\\
&\quad = \exp\Big(- \int_0^\infty \mathbb{P}\Big(\max_{j=1,\ldots,n} \exp\Big(X_i(t_j) - y_j\Big) > z\Big)\, dz\Big)\,,
\end{aligned}$$

as claimed. □

The following change of measure lemma plays a key role in our argument, and shows why the variance function σ^2 vanishes from the calculations. It is a variant of Lemma 1 in Dieker and Yakir [4]. We only sketch the key idea of the proof insofar as it highlights the differences with [4], since the lemma follows from the same arguments as given there.

Lemma 5.2. *Fix $t \in \mathbb{R}^d$. For a measurable functional F on $(\mathbb{R}^d)^{\mathbb{R}}$ that is invariant under addition of a constant function, we have*

$$\mathbb{E} e^{W(t) - \sigma^2(t)/2} F(W - \sigma^2/2) = \mathbb{E} F(\theta_t Z),$$

where the shift θ_t is defined through $(\theta_t Z)(s) = Z(s-t)$.

Proof sketch. Set $\mathbb{Q}(A) = \mathbb{E}[e^{W(t)-\sigma^2(t)/2}1_A]$, and write $\mathbb{E}^{\mathbb{Q}}$ for the expectation operator with respect to $\mathbb{Q}$. In this sketch, we first show that $W(s) - \sigma^2(s)/2$ under $\mathbb{Q}$ has the same distribution as $W(s)-\gamma(s-t)+\sigma^2(t)/2$ under $\mathbb{P}$. The full proof requires doing this calculation for finite-dimensional distributions to conclude that $(W(s)-\sigma^2(s)/2)_{s\in\mathbb{R}^d}$ under $\mathbb{Q}$ has the same distribution as $(W(s)-\gamma(s-t)+\sigma^2(t)/2)_{s\in\mathbb{R}^d}$ under $\mathbb{P}$, but doesn't require additional insights. We compare generating functions: for any $\beta\in\mathbb{R}$,

$$\begin{aligned}
&\log \mathbb{E}^{\mathbb{Q}} \exp\left(\beta(W(s)-\sigma^2(s)/2)\right)\\
&= -\frac{1}{2}\sigma^2(t) - \frac{\beta}{2}\sigma^2(s) + \frac{1}{2}\mathbb{V}\mathrm{ar}\left[W(t)+\beta W(s)\right]\\
&= -\frac{\beta}{2}\sigma^2(s) + \beta\mathbb{C}\mathrm{ov}(W(t),W(s)) + \frac{1}{2}\mathbb{V}\mathrm{ar}\left[\beta W(s)\right]\\
&= \beta\left[\frac{1}{2}\sigma^2(t)-\gamma(s-t)\right] + \frac{1}{2}\mathbb{V}\mathrm{ar}\left[\beta W(s)\right]\\
&= \beta\mathbb{E}\left[W(s)-\gamma(s-t)+\frac{1}{2}\sigma^2(t)\right] + \frac{\beta^2}{2}\mathbb{V}\mathrm{ar}\left[W(s)-\gamma(s-t)+\frac{1}{2}\sigma^2(t)\right].
\end{aligned}$$

Since F is translation invariant, the F-value of $(W(s)-\gamma(s-t)+\sigma^2(t)/2)_{s\in\mathbb{R}^d}$ must be the same as the F-value of $(W(s)-W(t)-\gamma(s-t))_{s\in\mathbb{R}^d}$. The latter has the same distribution as $(Z(s-t))_{s\in\mathbb{R}^d}$, which yields the claim. □

Proof of Theorem 2.1. Let $t_i\in\mathbb{R}^d, i=1,\dots,n$ and $y_i\in\mathbb{R}, i=1,\dots,n$ be arbitrary. From Lemma 5.1 with $X_i = W_i - \sigma^2/2$ we deduce that

$$\mathbb{P}(\eta(t_1)\le y_1,\dots,\eta(t_n)\le y_n) = \exp\Big(-\mathbb{E}\exp\Big(\max_{j=1,\dots,n}(W(t_j)-\sigma^2(t_j)/2-y_j)\Big)\Big).$$

Suppose that μ is an arbitrary probability measure on $\mathbb{R}^d$. Applying Lemma 5.2 with

$$F(x) = \frac{\max_{j=1\dots n}\exp(x(t_j)-y_j)}{\int_{\mathbb{R}^d}\exp(x(s))\mu(ds)},$$

we find that

$$\begin{aligned}
&\mathbb{E}\exp\Big(\max_{j=1,\dots,n}(W(t_j)-\sigma^2(t_j)/2-y_j)\Big)\\
&= \int_{\mathbb{R}^d}\mathbb{E}\Big[\exp(W(t)-\sigma^2(t)/2)\frac{\exp\Big(\max_{j=1,\dots,n}(W(t_j)-\sigma^2(t_j)/2-y_j)\Big)}{\int_{\mathbb{R}^d}\exp(W(s)-\sigma^2(s)/2)\,\mu(ds)}\Big]\,\mu(dt)\\
&= \int_{\mathbb{R}^d}\mathbb{E}\Big[\frac{\exp\left(\max_{j=1,\dots,n}(Z(t_j-t)-y_j)\right)}{\int_{\mathbb{R}^d}\exp(Z(s-t))\,\mu(ds)}\Big]\,\mu(dt)\\
&= \mathbb{E}\Big[\frac{\exp\left(\max_{j=1,\dots,n}(Z(t_j-T)-y_j)\right)}{\int_{\mathbb{R}^d}\exp(Z(s-T))\,\mu(ds)}\Big],
\end{aligned}$$

where T has distribution μ and is independent of Z. Applying Lemma 5.1 with

$$X_i(t) = Z_i(t-T_i) - \log\Big(\int_{\mathbb{R}^d}\exp(Z_i(s-T_i))\,\mu(ds)\Big)$$

shows that

$$\mathbb{P}(\eta(t_1)\le y_1,\dots,\eta(t_n)\le y_n) = \mathbb{P}(\zeta(t_1)\le y_1,\dots,\zeta(t_n)\le y_n).$$

This yields the claim of Theorem 2.1. □

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Appendix A. Computer code

This Matlab code is for 1-dimensional parameter spaces, but it is almost immediately adaptable for use with random fields due to Matlab's capabilities to work with multidimensional arrays. We present the Matlab code here since it can be read as pseudo-code, while reading the R code requires some knowledge of R objects designed for spatial data.

```
function res = generate_cluster(n,V)
    T = floor(n*rand());
    W = generateWwithdriftandcenter(T);
    res = V + W - log(sum(exp(W)));
end


function supremum = maxstable(n)
    supremum = -Inf(n,1);
    expminusV = -log(rand())/n;
    C = generate_cluster(n,-log(expminusV));

    while ( min(max(supremum, C)) < -log(expminusV) )
        supremum = max(supremum, C);
        expminusV = expminusV - log(rand())/n;
        C = generate_cluster(n,-log(expminusV));
    end

    supremum = max(supremum, C);
end
```

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