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GENERAL INVERSE PROBLEMS FOR REGULAR VARIATION

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Abstract

Regular variation of distributional tails is known to be preserved by various linear transformations of some random structures. An inverse problem for regular variation aims at understanding whether the regular variation of a transformed random object is caused by regular variation of components of the original random structure. In this paper we build up on previous work and derive results in the multivariate case and in situations where regular variation is not restricted to one particular direction or quadrant.

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1. Introduction

The four authors of this paper are very much honored to contribute to this special issue of one of the oldest journals in applied probability. We wish happy birthday and a very long life to this excellent journal. Two of us, Thomas Mikosch and Gennady Samorodnitsky, were inivited to contribute short papers to this special issue. With the permission of the editors, we merged efforts leading to this longer and more substantial paper.

In this paper we study regular variation of the tails of measures on $\mathbb{R}^d$, most importantly probability measures. Stated somewhat vaguely, it is well known that regular variation tends to be preserved by various linear operations (such as linear transformations of the space, convolutions, integrals, etc.) We would like to understand to what degree the inverse statements are valid. That is, if the result of a linear operation on a measure is regularly varying in the appropriate space, was the original measure necessarily regularly varying as well?

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This type of questions is often referred to as *inverse problems for regular variation,* and in the previous paper Jacobsen et al. (2008) a fairly complete answer to this problem for certain non-negative linear transformations of one-dimensional measures was given. Our aims in this paper are to treat the inverse problem in the multivariate case and to get rid of the non-negativity assumption on the linear transformations. We are fairly successful in our latter task, but only partially in the former one.

Now we will be more precise. Let $\mathbb{R}_0^d = \mathbb{R}^d \backslash \{\mathbf{0}\}$ and $\overline{\mathbb{R}}_0^d = \overline{\mathbb{R}}^d \backslash \{\mathbf{0}\}$, where $\overline{\mathbb{R}} = [-\infty, \infty]$. Recall that a random vector $\mathbf{X}$ with values in $\mathbb{R}^d$ is said to be *regularly varying* if there exists a non-null Radon measure $\mu_{\mathbf{X}}$ on the Borel σ-field of $\overline{\mathbb{R}}_0^d$ (that does not charge the set of infinite points) such that

$$\frac{P(s^{-1}\mathbf{X} \in \cdot)}{P(\|\mathbf{X}\| > s)} \stackrel{v}{\to} \mu_{\mathbf{X}}\,, \qquad \text{as } s \to \infty,$$

where $\stackrel{v}{\to}$ stands for vague convergence on the Borel σ-field of $\overline{\mathbb{R}}_0^d$; see e.g. Kallenberg (1983) or Resnick (1987). Recall that, in this context, a Borel measure is Radon if it is finite outside of any ball of positive radius centered at the origin. The measure $\mu_{\mathbf{X}}$ necessarily satisfies the relation $\mu_{\mathbf{X}}(tA) = t^{-\alpha}\mu_{\mathbf{X}}(A)$, $t > 0$, for all Borel sets A, some $\alpha > 0$. We will refer to α as the *index of regular variation* and $\mu_{\mathbf{X}}$ as the *tail measure.* The notion of regular variation applies equally well to σ-finite Borel measures on $\mathbb{R}^d$ that are finite outside of any ball of a positive radius centered at the origin. Specifically, any such measure ν is said to be *regularly varying* if, as above, there is a non-null Radon measure μ on $\overline{\mathbb{R}}_0^d$ that does not charge the set of infinite points such that

$$\frac{\nu(s\cdot)}{\nu(\{\mathbf{y} : \|\mathbf{y}\| > s\})} \stackrel{v}{\to} \mu\,, \qquad \text{as } s \to \infty.$$

As in the case of probability measures, the limiting measure μ scales with index $\alpha > 0$. We will write $\nu \in \mathrm{RV}(\alpha, \mu)$. Of course, this language allows the measure ν to be the law of a random vector $\mathbf{X}$, but in the case of random vectors it is even more common to simply write $\mathbf{X} \in \mathrm{RV}(\alpha, \mu_{\mathbf{X}})$.

To give a taste of linear operations on regularly varying measures we have in mind, we proceed with examples. The reader will notice that these examples are more general versions of the corresponding examples in Jacobsen et al. (2008).

Example 1. (*Weighted sums*). Let Ψ_j, $j = 1, 2, \ldots$ be (non-random) $d \times m$ matrices and $(\mathbf{Z}^{(j)})$ an iid sequence of regularly varying $\mathbb{R}^m$-valued random (column) vectors with a generic element $\mathbf{Z} \in \mathrm{RV}(\alpha, \mu_{\mathbf{Z}})$. Then under appropriate size conditions on the matrices (Ψ_j), the series $\mathbf{X} = \sum_{j=1}^{\infty} \Psi_j\, \mathbf{Z}^{(j)}$ converges with probability 1, and $\mathbf{X}$ is regularly varying with index α and

$$\frac{P(s^{-1}\mathbf{X} \in \cdot)}{P(\|\mathbf{Z}\| > s)} \stackrel{v}{\to} \sum_{j=1}^{\infty} \mu_{\mathbf{Z}} \circ \Psi_j^{-1}\,, \qquad \text{as } s \to \infty, \tag{1.1}$$

assuming that the right-hand side does not vanish; see Hult and Samorodnitsky (2008, 2010). This statement is always true if the sum is finite; see Resnick and Willekens (1991), Basrak et al. (2002b).

Is the converse statement true? That is, if $\mathbf{X}$ is regularly varying, does it follow that the iid vectors $\mathbf{Z}_i$ are regularly varying as well? In Jacobsen et al. (2008) this problem was solved for iid *real-valued* Z_i and *non-negative* scalars $\Psi_j = \psi_j$. (Here and in what follows, we use the symbol ψ_j for scalars instead of genuine matrices Ψ_j.) It was shown that (under appropriate size conditions in the case of an infinite sum), Z_i inherits regular variation with index α from X if the condition

$$\sum_{j=1}^{\infty} \psi_j^{\alpha+i\theta} \neq 0\,, \quad \text{for all} \quad \theta \in \mathbb{R}\,, \tag{1.2}$$

holds. Moreover, if (1.2) fails, then one can find iid (Z_i) which are not regularly varying but X is regularly varying. In this paper we want to extend the result to the multivariate case and/or drop the assumption of non-negative coefficients.

Example 2. (*Products*). Let $\mathbf{Z} \in \mathrm{RV}(\alpha, \mu_{\mathbf{Z}})$ be a random (column) vector in $\mathbb{R}^m$ and $\mathbf{A}$ be a random $d \times m$ matrix, independent of $\mathbf{Z}$ such that its matrix norm satisfies $E\|\mathbf{A}\|^{\alpha+\varepsilon} < \infty$ for some $\varepsilon > 0$. Then $\mathbf{X} = \mathbf{AZ}$ is regularly varying with index α in $\mathbb{R}^d$, and

$$\frac{P(s^{-1}\mathbf{X} \in \cdot)}{P(\|\mathbf{Z}\| > s)} \stackrel{v}{\to} E\Big[\mu_{\mathbf{Z}} \circ \mathbf{A}^{-1}\Big] \qquad \text{as } s \to \infty, \tag{1.3}$$

provided the measure on the right-hand side does not vanish; see Basrak et al. (2002a). Once again, is the converse statement true? That is, if $\mathbf{X}$ is regularly varying, does it follow that the vector $\mathbf{Z}$ is regularly varying (assuming that the random matrix $\mathbf{A}$ is suitably small)? In Jacobsen et al. (2008) it was shown that, if A and Z are real-valued and $A > 0$, then Z inherits regular variation with index α from X if and only if

$$E\big[A^{\alpha+i\theta}\big] \neq 0\,, \quad \text{for all} \quad \theta \in \mathbb{R}\,. \tag{1.4}$$

We would like to remove the restriction to one dimension and the assumption of non-negativity.

As in the one-dimensional non-negative case, these questions turn out to be related to a certain cancellation property of measures, which we address in Section 2. The proof of the cancellation property requires some abstract Fourier analytic arguments. The reader interested in applications of these results in the spirit of Examples 1 and 2 is referred to Section 3–5. In Section 3 we study the inverse problem for weighted sums of a multivariate iid sequence. In Section 4 we consider the corresponding problem for matrix products, where the random matrix has diagonal structure. Some examples in the case of non-diagonal deterministic matrices are given in Section 5. While the results in Section 3 yield a rather complete picture for weighted sums, the results in the remaining sections are of example-type leaving space for further investigations.

2. The generalized cancellation property

Let ρ and ν be σ-finite measures on $\mathbb{R}^d$. We define the *multiplicative convolution* of ν and ρ as a (not necessarily σ-finite) measure on $\mathbb{R}^d$ given by

$$(\nu \circledast \rho)\,(B) = \int_{\mathbb{R}^d} \nu\big(T_{\mathbf{x}}^{-1}(B)\big)\,\rho(d\mathbf{x}), \quad \text{any Borel set } B \subset \mathbb{R}^d\,,$$

where $T_{\mathbf{x}} = \mathrm{diag}(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^d$.

We start with the following result that will motivate the cancellation property discussion in the sequel.

Theorem 1. *Assume $\alpha > 0$ and let ρ, ν be σ-finite measures, such that ρ is not concentrated on any proper coordinate subspace of $\mathbb{R}^d$, that is,*

$$\inf_{j=1,\dots,d} \rho\big(\{\mathbf{x} : x_j \neq 0\}\big) > 0\,, \tag{2.1}$$

$\nu \circledast \rho \in \mathrm{RV}(\mu,\alpha)$ and for some $0 < \delta' < \alpha$,

$$\int_{\mathbb{R}^d} \big(\|\mathbf{y}\|^{\alpha-\delta'} \vee \|\mathbf{y}\|^{\alpha+\delta'}\big)\, \rho(d\mathbf{y}) < \infty\,, \tag{2.2}$$

and for each $j = 1,\dots,d$,

$$\lim_{b\downarrow 0} \limsup_{s\to\infty} \frac{\int_{0<|y_j|\le b} \rho\big(\{\mathbf{x} : |x_j| > s/|y_j|\}\big)\, \nu(d\mathbf{y})}{(\nu \circledast \rho)\big(\{\mathbf{x} : \|\mathbf{x}\| > s\}\big)} = 0\,. \tag{2.3}$$

Then the family of measures on $\mathbb{R}^d_0$ given by

$$\mu_s(\cdot) = \frac{\nu(s\cdot)}{(\nu \circledast \rho)\big(\{\mathbf{x} : \|\mathbf{x}\| > s\}\big)}, \quad s \ge 1\,, \tag{2.4}$$

is relatively compact in the vague topology on $\overline{\mathbb{R}}^d_0$. Further, any limiting (as $s \to \infty$) point μ_* does not charge the set of infinite points and satisfies the equation

$$\mu_* \circledast \rho = \mu\,. \tag{2.5}$$

Proof. By (2.1), we can choose $\theta > 0$ such that $\rho\big(\{\mathbf{x} : |x_j| \ge \theta\}\big) \ge \delta > 0$ for every $j = 1,\dots,d$, and a sufficiently small δ. For every j and $s > 0$,

$$\rho\big(\{\mathbf{x} : |x_j| \ge \theta\}\big)\nu\big(\{\mathbf{x} : |x_j| > s/\theta\}\big) \le (\nu \circledast \rho)\big(\{\mathbf{x} : \|\mathbf{x}\| > s\}\big)\,.$$

Therefore,

$$\begin{aligned}
\nu\big(\{\mathbf{x} : \|\mathbf{x}\| > s\}\big) &\le \sum_{j=1}^d \nu\big(\{\mathbf{x} : |x_j| > s/d\}\big) \\
&\le (\nu \circledast \rho)\big(\{\mathbf{x} : \|\mathbf{x}\| > \theta s/d\}\big) \sum_{j=1}^d \frac{1}{\rho\big(\{\mathbf{x} : |x_j| \ge \theta\}\big)} \\
&\le \frac{d}{\delta} (\nu \circledast \rho)\big(\{\mathbf{x} : \|\mathbf{x}\| > \theta s/d\}\big)\,.
\end{aligned}$$

By B_τ, we denote the closed ball of radius $\tau > 0$ centered at the origin. Then we have as $s \to \infty$,

$$\mu_s\big(B_\tau^c\big) \le \frac{d}{\delta} \frac{(\nu \circledast \rho)\big(\{\mathbf{x} : \|\mathbf{x}\| > \theta\tau s/d\}\big)}{(\nu \circledast \rho)\big(\{\mathbf{x} : \|\mathbf{x}\| > s\}\big)} \to \frac{d}{\delta}\Big(\frac{\theta\tau}{d}\Big)^{-\alpha} \mu(B_1) < \infty\,.$$

Hence (μ_s) is relatively compact (see Proposition 3.16 in Resnick (1987)). Let $s_k \to \infty$ be a sequence such that $\mu_{s_k} \stackrel{v}{\to} \mu_*$ in $\overline{\mathbb{R}}_0^d$ for some limiting point μ_*. Then μ_* does not charge the set of infinite points. For $\mathbf{a} \in \mathbb{R}^d$ denote

$$D_{\mathbf{a}} = \Big\{ \mathbf{y} \in \mathbb{R}_0^d : \mu_* \Big(\big\{ \mathbf{z} : \, z_j = a_j / y_j \ \text{ for some } j = 1, \ldots, d \big\} \Big) = 0 \Big\}.$$

The argument after (2.22) in Jacobsen et al. (2008) shows that there are at most countable sets $A_1, \ldots, A_d$ of real numbers such that

$$\rho\big(D_{\mathbf{a}}\big) = 0, \quad \mathbf{a} \in \prod_{j=1}^d A_j^c. \tag{2.6}$$

Consider $\mathbf{a}$ such that

$$a_1 > 0, \quad a_j \geq 0, \quad j = 2, \ldots, d, \quad a_j \notin A_j, \quad j = 1, \ldots, d\,. \tag{2.7}$$

The set $C_d(\mathbf{a}) = \prod_{j=1}^d [a_j, \infty)$ is bounded away from the origin and a continuity set for the tail measure μ of $\nu \circledast \rho$. Therefore

$$\begin{aligned} \mu(C_d(\mathbf{a})) &= \lim_{k \to \infty} \frac{(\nu \circledast \rho)\Big(s_k C_d(\mathbf{a}) \Big)}{(\nu \circledast \rho)\big(\{ \mathbf{x} : \, \|\mathbf{x}\| > s_k \} \big)} \\ &= \lim_{k \to \infty} \sum_{I_+ \subseteq \{1, \ldots, d\}} \int_{D(I_+)} f_{k, I_+}(\mathbf{z}) \, \rho(d\mathbf{z})\,, \end{aligned} \tag{2.8}$$

where for $I_+ \subseteq \{1, \ldots, d\}$,

$$D(I_+) = \big\{ \mathbf{z} : \, z_j \geq 0 \ \text{ for } j \in I_+ \text{ and } \ z_j < 0 \ \text{ for } j \notin I_+ \big\}\,,$$

interpreting $[0/0, \infty) = \mathbb{R}$ and writing for $k \geq 1$ and v such that $v_j \geq 0$ for $j \in I_+$,

$$f_{k, I_+}(v) \;=\; \mu_{s_k} \Big(\prod_{j \in I_+} \big[a_j / v_j, \infty \big) \times \prod_{j \notin I_+} \big(-\infty, -a_j / |v_j| \big] \Big)\,.$$

Choosing $\varepsilon > 0$ so small that $c := \rho\big(\{ \mathbf{z} : \, |z_1| \geq \varepsilon \} \big) > 0$, and proceeding similarly to the beginning of the proof, we get for $I_+ \subseteq \{1, \ldots, d\}$ and $\mathbf{z} = \mathbf{1} = (1, \ldots, 1)$,

$$\begin{aligned} f_{k, I_+}(\mathbf{1}) &\leq \mu_{s_k} \big(\{ \mathbf{y} : \, |y_1| > a_1 \} \big) \\ &\leq c^{-1} \frac{(\nu \circledast \rho)\big(\{ \mathbf{x} : \, |x_1| \geq a_1 \varepsilon s_k \} \big)}{(\nu \circledast \rho)\big(\{ \mathbf{x} : \, \|\mathbf{x}\| > s_k \} \big)}\,. \end{aligned}$$

Therefore, on $D(I_+) \cap \{ \mathbf{z} : \, |z_1| \leq M s_k \}$, $M > 0$,

$$\begin{aligned} f_{k, I_+}(\mathbf{z}) &\leq c^{-1} \frac{(\nu \circledast \rho)\big(\{ \mathbf{x} : \, |x_1| \geq a_1 \varepsilon s_k / z_1 \} \big)}{(\nu \circledast \rho)\big(\{ \mathbf{x} : \, \|\mathbf{x}\| > s_k \} \big)} \\ &\leq C(a_1, \varepsilon, M) \big(|z_1|^{\alpha - \delta'} \vee |z_1|^{\alpha + \delta'} \big)\,. \end{aligned}$$

Here $C(a_1,\varepsilon,M)$ is a finite positive constant, and in the last step we used the Potter bounds; see Proposition 0.8 in Resnick (1987). Recalling that (2.6) holds for our choice of $\mathbf{a}$, using (2.2) and the dominated convergence theorem, we conclude that for every $M>0$, as $k\to\infty$,

$$\int_{D(I_+)\cap\{\mathbf{z}:\,|z_1|\le Ms_k\}} f_{k,I_+}(\mathbf{z})\,\rho(d\mathbf{z}) \to \int_{D(I_+)} \mu_*\left(\prod_{j\in I_+}\big[a_j/z_j,\infty\big)\times\prod_{j\notin I_+}\big(-\infty,-a_j/|z_j|\big]\right)\rho(d\mathbf{z})\,.$$

Furthermore,

$$\begin{aligned}\int_{D(I_+)\cap\{\mathbf{z}:\,|z_1|>Ms_k\}} & f_{k,I_+}(\mathbf{z})\,\rho(d\mathbf{z})\\ \le\;& \frac{\rho\big(\{\mathbf{z}:\,|z_1|>Ms_k\}\big)\nu\big(\{\mathbf{y}:\,|y_1|>a_1/M\}\big)}{(\nu\circledast\rho)\big(\{\mathbf{x}:\,\|\mathbf{x}\|>s_k\}\big)}\\ &+\frac{\int_{0<|y_1|\le a_1/M}\rho\big(\{\mathbf{z}:\,|z_1|>s_ka_1/|y_1|\}\big)\,\nu(d\mathbf{y})}{(\nu\circledast\rho)\big(\{\mathbf{x}:\,\|\mathbf{x}\|>s_k\}\big)} := A_k+B_k\,.\end{aligned}$$

Since

$$\rho\big(\{\mathbf{z}:\,|z_1|>Ms_k\}\big)\le(Ms_k)^{-(\alpha+\delta)}\int_{\mathbb{R}^d}|z_1|^{\alpha+\delta}\,\rho(d\mathbf{z})\,,$$

it follows from (2.2) that $A_k\to0$ as $k\to\infty$, once again for each $M>0$, and by (2.3), $\lim_{M\to\infty}\limsup_{k\to\infty}B_k=0$. Thus we proved that for any $\mathbf{a}$ satisfying (2.7) and $I_+\subseteq\{1,\ldots,d\}$, as $k\to\infty$,

$$\int_{D(I_+)} f_{k,I_+}(\mathbf{z})\,\rho(d\mathbf{z}) \to \int_{D(I_+)} \mu_*\left(\prod_{j\in I_+}\big[a_j/z_j,\infty\big)\times\prod_{j\notin I_+}\big(-\infty,-a_j/|z_j|\big]\right)\rho(d\mathbf{z})\,.$$

Then, in view of (2.8),

$$\begin{aligned}&\mu(C_d(\mathbf{a}))\\ =\;&\sum_{I_+\subseteq\{1,\ldots,d\}}\int_{D(I_+)}\mu_*\left(\prod_{j\in I_+}\big[a_j/z_j,\infty\big)\times\prod_{j\notin I_+}\big(-\infty,-a_j/|z_j|\big]\right)\rho(d\mathbf{z})\\ =\;&(\mu_*\circledast\rho)(C_d(\mathbf{a}))\,.\end{aligned}\tag{2.9}$$

Using the continuity of measures from above, we can now extend (2.9) to any $\mathbf{a}$ satisfying $a_1>0$, $a_j\ge0, j=2,\ldots,d$. This means that the measures μ and $\mu_*\circledast\rho$ coincide on the set $\{\mathbf{x}:\,x_1>0,\,x_j\ge0, j=2,\ldots,d\}$.

Of course, this argument can be repeated while distinguishing any coordinate $k=1,\ldots,d$, so that we see that the measures μ and $\mu_*\circledast\rho$ coincide on each of the d sets

$\{\mathbf{x} : x_k > 0, x_j \geq 0, j = 1, \ldots, d\}$, $k = 1, \ldots, d$. Since the union of these sets is the first quadrant $[0, \infty)^d \backslash \{\mathbf{0}\}$ we conclude that these two measures coincide on this set. An identical argument can be used for all other quadrants of $\overline{\mathbb{R}}_0^d$. Thus (2.5) holds and the proof of the theorem is complete.

There is only an apparently small step remaining between the conclusion of Theorem 1 and the statement that ν is regularly varying with index α. This step consists of showing that (with ρ and μ fixed) equation (2.5) has a unique solution μ_*. Indeed, if this could be established, then all subsequential limits as $s \to \infty$ of the family (μ_s) in (2.4) would be equal. In turn, (μ_s) would converge vaguely and ν would be regularly varying.

Unfortunately, this step is not so small and it turns out that, in some cases, (2.5) has multiple solutions; see the following discussion and, in particular, Remark 2. Therefore, our next step aims at establishing conditions under which the solution to (2.5) is, indeed, unique. We start by reducing the problem to a slightly different form. Uniqueness of the solution to (2.5) would follow if the measure ρ had the following property: within a relevant class of σ-finite measures ν_1, ν_2,

$$\text{if} \;\; \nu_1 \circledast \rho = \nu_2 \circledast \rho \;\; \text{then} \;\; \nu_1 = \nu_2 \,. \tag{2.10}$$

This property can be viewed as the *cancellation property of the measure ρ with respect to the operation $\circledast$.*

A similar situation was considered in Jacobsen et al. (2008), in which the case $d = 1$ was treated. There it was assumed that all measures are supported on the positive half-line $(0, \infty)$. In particular, all regularly varying measures supported on $(0, \infty)$ have tail measures proportional to one another. It is natural in this situation to study the cancellation property if one of the measures ν_1, ν_2 is such a canonical measure. Correspondingly, one defines a measure ν^α on $(0, \infty)$, $\alpha \in \mathbb{R}$, with a power density given by

$$\nu^\alpha(dx) = \alpha \, x^{-(\alpha+1)} \, dx \,. \tag{2.11}$$

Actually, Jacobsen et al. (2008) allow at this point for any real value of α. In the present paper, we will look only at positive α, even though the statement of Theorem 2 below can be extended to the more general case.

The paper Jacobsen et al. (2008) addresses the question which measures ρ have the following cancellation property:

$$\nu \circledast \rho = \nu^\alpha \circledast \rho \qquad \text{implies } \nu = \nu^\alpha,$$

and it was shown that a measure ρ satisfying

$$\int_0^\infty y^{\alpha-\delta} \vee y^{\alpha+\delta} \, \rho(dy) < \infty \,, \quad \text{for some } \delta > 0,$$

has this cancellation property if and only if

$$\int_0^\infty y^{\alpha+i\theta} \, \rho(dy) \neq 0 \quad \text{for all } \; \theta \in \mathbb{R} \,.$$

In order to understand the more general cancellation property (2.10), we start by replacing the single equation by a system of linear equations that include only measures concentrated on the positive quadrant of $\mathbb{R}^d$.

For $d \geq 1$, consider the set $Q_d = \{-1,1\}^d$ equipped with the coordinate-wise (binary) multiplication. Let $\alpha_1,\ldots,\alpha_d$ be positive numbers, $\big(\rho_v,\, v\in Q_d\big)$ be σ-finite measures on $(0,\infty)^d$, and $\big(\nu_v^{(i)},\, v\in Q_d\big)$, $i=1,2$, be two collections of σ-finite measures on $[0,\infty)^d$. We assume that for a certain non-empty subset K of $\{1,\ldots,d\}$

$$\int_{(0,\infty)^d} x_j^{\alpha_j}\,\rho_v(d\mathbf{x})<\infty \quad \text{for each } v\in Q_d \text{ and } j\in K, \tag{2.12}$$

and for $i=1,2$,

$$\sup_{s>0} s^{\alpha_j}\nu_v^{(i)}\big(\{\mathbf{x}:\, x_j>s\}\big)<\infty \quad \text{for each } v\in Q_d \text{ and } j\in K. \tag{2.13}$$

We now assume that these measures satisfy the following system of 2^d linear equations.

$$\sum_{w\in Q_d}\nu_w^{(1)}\circledast\rho_{vw}=\sum_{w\in Q_d}\nu_w^{(2)}\circledast\rho_{vw} \quad \text{for each } v\in Q_d. \tag{2.14}$$

The following result characterizes those measures $\big(\rho_v,\, v\in Q_d\big)$ which can be "cancelled" in this system of equations.

Theorem 2. *Let $\big(\rho_v,\, v\in Q_d\big)$ be σ-finite measures on $(0,\infty)^d$ and $\big(\nu_v^{(i)},\, v\in Q_d\big)$, $i=1,2$, be σ-finite measures on $[0,\infty)^d$. Assume that for some non-empty set $K\subseteq\{1,\ldots,d\}$,*

$$\nu_v^{(i)}\big(\{\mathbf{x}:\, x_k=0 \ \ \text{for each } k\in K\}\big)=0\,,\quad i=1,2,\quad v\in Q_d\,, \tag{2.15}$$

and that (2.12) and (2.13) hold for this set K. Suppose that for each $j\in K$, $m_1,\ldots,m_d\in\{0,1\}$ and $\theta_1,\ldots,\theta_d\in\mathbb{R}$,

$$\sum_{v\in Q_d}\prod_{k=1}^d v_k^{m_k}\int_{(0,\infty)^d} x_j^{\alpha_j}\prod_{k=1}^d x_k^{i\theta_k}\rho_v(d\mathbf{x})\neq 0 \tag{2.16}$$

with the usual notation $v=(v_1,\ldots,v_d)\in Q_d$ and $\mathbf{x}=(x_1,\ldots,x_d)\in[0,\infty)^d$. If these measures satisfy the system of equations (2.14), then

$$\nu_v^{(1)}=\nu_v^{(2)} \quad \text{for each } v\in Q_d. \tag{2.17}$$

Remark 1. In applications to regular variation the measures $\big(\nu_v^{(i)},\, v\in Q_d\big)$, $i=1,2$, will appear as (restrictions to the different quadrants of) certain vague limits ν in $\overline{\mathbb{R}}_0^d$, hence will automatically put no mass at the origin. Hence the set $K=\{1,\ldots,d\}$ will always satisfy (2.15). This is the maximal possible choice of K which requires the largest possible set of conditions in (2.16). The smaller the set K can be chosen, the fewer conditions one needs to check. If, for example, ν is absolutely continuous, then $K=\{1\}$ and (2.15) gives 2^d conditions.

Before proving Theorem 2, we consider some special cases. We start by considering the scalar case, $d=1$. In this case, the system of equations (2.14) becomes

$$\nu_1^{(1)}\circledast\rho_1+\nu_{-1}^{(1)}\circledast\rho_{-1}=\nu_1^{(2)}\circledast\rho_1+\nu_{-1}^{(2)}\circledast\rho_{-1}\,, \tag{2.18}$$

$$\nu_1^{(1)} \circledast \rho_{-1} + \nu_{-1}^{(1)} \circledast \rho_1 = \nu_1^{(2)} \circledast \rho_{-1} + \nu_{-1}^{(2)} \circledast \rho_1 \,.$$

The only choice is $K = \{1\}$ and the conditions (2.16) for the cancellation property become

$$\begin{cases} \int_0^\infty x^{\alpha_1 + i\theta} \rho_1(dx) + \int_0^\infty x^{\alpha_1 + i\theta} \rho_{-1}(dx) \neq 0\,, \\ \int_0^\infty x^{\alpha_1 + i\theta} \rho_1(dx) - \int_0^\infty x^{\alpha_1 + i\theta} \rho_{-1}(dx) \neq 0\,, \end{cases} \quad, \quad \theta \in \mathbb{R}\,. \tag{2.19}$$

In dimension one the measure ν^α, $\alpha > 0$, given in (2.11), is particularly important when studying regular variation. Suppose that $\nu_i^{(2)} = c_i \nu^\alpha$, $i = \pm 1$, where $c_1\,, c_{-1}$ are nonnegative constants. If we choose $\alpha_1 = \alpha$, then the assumption (2.13) automatically holds for the measures $\nu_1^{(2)}$ and $\nu_{-1}^{(2)}$. Assuming that the measures ρ_1, ρ_{-1} satisfy (2.12) and $\|\rho_i\|_\alpha^\alpha = \int_0^\infty x^\alpha \rho_i(dx)$, $\;i = \pm 1$, the system (2.18) takes the form

$$\nu_1^{(1)} \circledast \rho_i + \nu_{-1}^{(1)} \circledast \rho_{-i} = \big(c_1 \|\rho_i\|_\alpha^\alpha + c_{-1} \|\rho_{-i}\|_\alpha^\alpha\big)\, \nu^\alpha, \quad i = \pm 1\,. \tag{2.20}$$

Notice that the two equations (2.20) already imply that (2.13) holds for the measures $\nu_1^{(1)}$ and $\nu_{-1}^{(1)}$ as well. We therefore obtain the following corollary of Theorem 2.

Corollary 1. *Let* $\alpha_1 = \alpha > 0$, *and* ρ_1, ρ_{-1} *be* σ*-finite measures on* $(0, \infty)$ *satisfying* (2.12). *If the* σ*-finite measures on* $[0, \infty)$, $\nu_1^{(1)}$ *and* $\nu_{-1}^{(1)}$, *satisfy the system of equations* (2.20), *and if the cancellation conditions* (2.19) *are satisfied, then* $\nu_i^{(1)} = c_i \nu^\alpha$, $i = \pm 1$.

Remark 2. Assume that all conditions of Corollary 1 but (2.19) are satisfied. For example, if the first condition in (2.19) is not satisfied for some $\theta = \theta_0 \in \mathbb{R}$, then the measures

$$\nu_i^{(1)}(dx) = [c_i + a\, \cos(\theta_0 \log x) + b \sin(\theta_0 \log x)]\, \nu^\alpha(dx)\,, \quad i = \pm 1\,,$$

for a, b such that $0 \leq a^2 + b^2 \leq 1$ solve the system of equations (2.20). Similarly, if the second condition in (2.19) fails for some $\theta = \theta_0 \in \mathbb{R}$, then the measures

$$\nu_i^{(1)}(dx) = \big[c_i + (-1)^i \big(a\, \cos(\theta_0 \log x) + b\, \sin(\theta_0 \log x)\big)\big]\, \nu^\alpha(dx)\,, \quad i = \pm 1\,,$$

with the same choice of a, b as above satisfy (2.20).

Another useful special case of Theorem 2 corresponds to the situation, where only one of the measures $\big(\rho_v,\, v \in Q_d\big)$ is non-null; as we will see in the sequel this case naturally arises in inverse problems for regular variation. We assume without loss of generality that the non-null measure corresponds to the unity in Q_d, $v = (1, \ldots, 1)$. For simplicity denoting this measure by ρ, we see that the system of equations (2.14) decouples, and becomes

$$\nu_v^{(1)} \circledast \rho = \nu_v^{(2)} \circledast \rho \;\text{ for each } v \in Q_d.$$

However, the decoupled system of equations does not provide us with any additional insight over a single equation, so the right thing to do is to drop the subscript and consider the equation

$$\nu^{(1)} \circledast \rho = \nu^{(2)} \circledast \rho \tag{2.21}$$

for two σ-finite measures $\nu^{(1)}$ and $\nu^{(2)}$. If we interpret (2.12), (2.13) and (2.15) by disregarding the subscripts, we obtain another corollary of Theorem 2.

Corollary 2. *Let $\alpha_1, \ldots, \alpha_d$ be positive numbers, ρ a σ-finite measure on $(0,\infty)^d$ and $\nu^{(1)}$, $\nu^{(2)}$ σ-finite measures on $[0,\infty)^d$. Suppose that the non-empty set $K \subseteq \{1,\ldots,d\}$ satisfies* (2.15) *and* (2.12) *and* (2.13) *hold.*

If the equation (2.21) *is fulfilled, and*

$$\int_{(0,\infty)^d} x_j^{\alpha_j} \prod_{k=1}^d x_k^{i\theta_k} \rho(d\mathbf{x}) \neq 0 \tag{2.22}$$

for each $j \in K$, and $\theta_1, \ldots, \theta_d \in \mathbb{R}$, then $\nu^{(1)} = \nu^{(2)}$.

In the case $d = 1$, the conclusion of Corollary 2 is the same as the direct part of Theorem 2.1 in Jacobsen et al. (2008).

Proof of Theorem 2. The general idea of the proof is similar to the proof of Theorem 2.1 in Jacobsen et al. (2008). Fix $j \in K$ and define

$$h_j^{(v,i)}(\mathbf{y}) = y_j^{\alpha_j} \nu_v^{(i)}\big(\{\mathbf{z} :\, 0 \le z_k \le y_k,\, k \neq j,\, z_j > y_j\}\big), \quad v \in Q_d,\, i = 1,2$$

for $\mathbf{y} = (y_1, \ldots, y_d)$ with all $y_k > 0$. It follows from (2.13) that all these functions are bounded on their domain. The equations (2.14) then tell us that

$$\begin{aligned}\sum_{w\in Q_d} \int_{[0,\infty)^d} & h_j^{(w,1)}\big(x_1/z_1, \ldots, x_d/z_d\big)\, \rho_{vw}(d\mathbf{z}) \\ = & \sum_{w\in Q_d} \int_{[0,\infty)^d} h_j^{(w,2)}\big(x_1/z_1, \ldots, x_d/z_d\big)\, \rho_{vw}(d\mathbf{z})\end{aligned}$$

for each $v \in Q_d$, $x_k > 0$, $k = 1, \ldots, d$. Next, we define functions

$$g_j^{(v,i)}(\mathbf{y}) = h_j^{(v,i)}\big(e^{y_1}, \ldots, e^{y_d}\big), \quad v \in Q_d,\, i = 1,2,$$

for $\mathbf{y} \in \mathbb{R}^d$, and finite measures on $\mathbb{R}^d$ by

$$\mu_j^{(v)}(d\mathbf{x}) = \big(e^{\alpha_j x_j} \rho_v\big) \circ T_{\log}^{-1}(d\mathbf{x})\,,$$

where $T_{\log}(\mathbf{y}) = (\log y_1, \ldots, \log y_d)$, $\mathbf{y} \in (0,\infty)^d$. We can now write

$$\sum_{w\in Q_d} \int_{\mathbb{R}^d} g_j^{(w,1)}(\mathbf{z}-\mathbf{y})\, \mu_j^{(vw)}(d\mathbf{y}) = \sum_{w\in Q_d} \int_{\mathbb{R}^d} g_j^{(w,2)}(\mathbf{z}-\mathbf{y})\, \mu_j^{(vw)}(d\mathbf{y})$$

for each $v \in Q_d$, $\mathbf{z} \in \mathbb{R}^d$. Therefore, the bounded functions

$$g_j^{(v)}(\mathbf{y}) = g_j^{(v,1)}(\mathbf{y}) - g_j^{(v,2)}(\mathbf{y}),\ \mathbf{y} \in \mathbb{R}^d, \quad v \in Q_d,$$

satisfy

$$\sum_{w\in Q_d} \int_{\mathbb{R}^d} g_j^{(w)}(\mathbf{z}-\mathbf{y})\, \mu_j^{(vw)}(d\mathbf{y}) = 0 \tag{2.23}$$

for each $v \in Q_d$, $\mathbf{z} \in \mathbb{R}^d$.

For fixed $m_k \in \{0,1\}$, $k=1,\ldots,d$, and $j \in K$, we define a signed bounded measure on $\mathbb{R}^d$ by

$$\mu_j = \sum_{v\in Q_d} \prod_{k=1}^d v_k^{m_k} \mu_j^{(v)}$$

and a bounded function on $\mathbb{R}^d$ by

$$g_j = \sum_{v\in Q_d} \prod_{k=1}^d v_k^{m_k} g_j^{(v)} .$$

Then the system of equations (2.23) implies

$$\int_{\mathbb{R}^d} g_j(\mathbf{z}-\mathbf{y})\,\mu_j(d\mathbf{y}) = 0\,, \quad \mathbf{z}\in\mathbb{R}^d\,, \tag{2.24}$$

and we want to show that $g_j = 0$ everywhere.

Notice now that the right-hand side of (2.16) is exactly the Fourier transform of μ_j at the point $\mathbf{s} = (\theta_1,\ldots,\theta_d)$. Let φ be the standard normal density in $\mathbb{R}^d$. Then, in the standard notation for the additive convolution, we have $\varphi * \mu_j \in L^1(\mathbb{R}^d)$, and the equation (2.24) tells us that $g_j * \big(\varphi * \mu_j\big) \equiv 0$. Let the symbol $\widehat{\ }$ denote the distributional Fourier transform of a function or a signed measure. By Theorem 9.3 in Rudin (1973) we have that, in the distributional sense,

$$\mathrm{supp}(\widehat{g}_j) \subseteq \{\mathbf{s}\in\mathbb{R}^d : \widehat{\varphi}(\mathbf{s})\widehat{\mu}_j(\mathbf{s}) = 0\} = \{\mathbf{s}\in\mathbb{R}^d : \widehat{\mu}_j(\mathbf{s}) = 0\} = \emptyset\,,$$

where the last equation is just the condition (2.16). Therefore, we conclude that the support of the Fourier transform $\widehat{g}_j$ is empty, hence $g_j = 0$ almost everywhere. Since the function g_j is coordinate-wise right-continuous, we see that $g_j = 0$ everywhere.

The $2^d \times 2^d$ matrix A with the entries

$$a_{m_1,\ldots,m_d,v_1,\ldots,v_d} = \prod_{k=1}^d v_k^{m_k}, \quad m_j\in\{0,1\},\ v_j\in\{-1,1\}, \quad j=1,\ldots,d\,,$$

is non-degenerate (in fact, $|\det A| = 2^{d2^{d-1}}$). Therefore, it follows from the definition of the function g_j that for each $v \in Q_d$, $g_j^{(v)} \equiv 0$, hence $g_j^{(v,1)} \equiv g_j^{(v,2)}$. We conclude that

$$\begin{aligned} &\nu_v^{(1)}\big(\{\mathbf{z} :\, 0\le z_k\le y_k,\, k\ne j,\, z_j > y_j\}\big) \\ &\quad = \ \nu_v^{(2)}\big(\{\mathbf{z} :\, 0\le z_k\le y_k,\, k\ne j,\, z_j > y_j\}\big) \end{aligned}$$

for each $v \in Q_d$, $\mathbf{y} \in (0,\infty)^d$ and $j \in K$. This means that, for each $v \in Q_d$, the measures $\nu_v^{(1)}$ and $\nu_v^{(2)}$ coincide on the set $\{y_j > 0\}$ for each $j \in K$. By the definition of the set K we obtain (2.17) and, hence, complete the proof.

The conditions for the cancellation property in (2.16) and its special cases above, are somewhat implicit. On the other hand, in the case of one dimension and a single equation, the presence of a sufficiently large atom in the measure ρ already guarantees the cancellation property; see Corollary 2.2 in Jacobsen et al. (2008). A similar phenomenon, described in the following statement, occurs in general.

Corollary 3. *Let $\big(\rho_v,\, v \in Q_d\big)$ be σ-finite measures on $(0,\infty)^d$, and let $\big(\nu_v^{(i)},\, v \in Q_d\big)$, $i = 1,2$, be σ-finite measures on $[0,\infty)^d$. Suppose that K is a nonempty set satisfying* (2.15). *Assume, further, that* (2.12) *and* (2.13) *hold for this set K.*

Suppose that these measures satisfy the system of equations (2.14). *If for every $j \in K$ there is $v^{(j)} \in Q_d$ and an atom $\mathbf{x}^{(j)} = \big(x_1^{(j)},\dots,x_d^{(j)}\big)$ of $\rho_{v^{(j)}}$ with mass $w^{(j)}$ so large that*

$$w^{(j)}\big(x_j^{(j)}\big)^{\alpha_j} > \int_{\mathbf{x}\neq\mathbf{x}^{(j)}} x_j^{\alpha_j}\,\rho_{v^{(j)}}(d\mathbf{x}) + \sum_{v\neq v^{(j)}} \int_{(0,\infty)^d} x_j^{\alpha_j}\,\rho_v(d\mathbf{x})\,,$$

then the conclusion (2.17) *holds.*

Proof. An application of the triangle inequality shows that the assumptions of the corollary, in fact, imply (2.16). Indeed, let $j \in K$. We have, for any $m_1,\dots,m_d \in \{0,1\}$ and $\theta_1,\dots,\theta_d \in \mathbb{R}$,

$$\begin{aligned}&\left|\sum_{v\in Q_d}\prod_{k=1}^d v_k^{m_k}\int_{(0,\infty)^d} x_j^{\alpha_j}\prod_{k=1}^d x_k^{i\theta_k}\,\rho_v(d\mathbf{x})\right|\\ &\quad\geq\; w^{(j)}\big(x_j^{(j)}\big)^{\alpha_j} - \int_{\mathbf{x}\neq\mathbf{x}^{(j)}} x_j^{\alpha_j}\,\rho_{v^{(j)}}(d\mathbf{x}) - \sum_{v\neq v^{(j)}}\int_{(0,\infty)^d} x_j^{\alpha_j}\,\rho_v(d\mathbf{x}) > 0\end{aligned}$$

by the assumption, so none of the expressions in (2.16) can vanish.

We now put together Theorems 1 and 2 and obtain an inverse regular variation result for multiplicative convolutions. It is a multivariate extension of Theorem 2.3 in Jacobsen et al. (2008).

Theorem 3. *Let $\alpha > 0$ and ρ, ν be σ-finite measures on $\mathbb{R}^d$ such that*

$$\rho\big(\{\mathbf{x}:\, x_i = 0\}\big) = 0 \quad \textit{for every } i = 1,\dots,d,$$

and $(\nu \circledast \rho) \in \mathrm{RV}(\alpha,\mu)$. Assume (2.2), (2.3) *and*

$$\int_{\mathbb{R}^d} |x_j|^\alpha \prod_{k=1}^d |x_k|^{i\theta_k} \prod_{k=1}^d \big(\mathrm{sign}(x_k)\big)^{m_k}\,\rho(d\mathbf{x}) \neq 0 \tag{2.25}$$

for each $j = 1,\dots,d$, $m_1,\dots,m_d \in \{0,1\}$ and $\theta_1,\dots,\theta_d \in \mathbb{R}$. Then the measure ν is regularly varying with index α. Moreover, the measures (μ_s) in (2.4) *converge vaguely as $s \to \infty$, in $\overline{\mathbb{R}}_0^d$, to a measure μ_* satisfying* (2.5).

Proof. Because of the statement of Theorem 1, we only need to prove that any two subsequential vague limits $\nu^{(1)}$ and $\nu^{(2)}$ in that theorem coincide. Note that $\nu^{(1)}$ and $\nu^{(2)}$ are two solutions to the equation (2.5), so in order to prove that $\nu^{(1)} = \nu^{(2)}$ we translate our problem to the cancellation property situation of Theorem 2. For $v \in Q_d$ denote

$$\mathcal{Q}_v = \big\{\mathbf{x}:\, x_j v_j \geq 0 \;\text{ for each }\; j = 1,\dots,d\big\}\,,$$

and define

$$\rho_v(\cdot) = \rho\big(\{\mathbf{x} \in \mathcal{Q}_v :\, (|x_1|,\dots,|x_d|) \in \cdot\}\big)\,.$$

Similarly, we define two collections of σ-finite measures on $[0,\infty)^d$, $\big(\nu_v^{(i)},\, v \in Q_d\big)$, $i = 1,2$, by restricting the measures $\nu^{(1)}$ and $\nu^{(2)}$ to the appropriate quadrants. By assumption, $\nu^{(1)} \circledast \rho = \nu^{(2)} \circledast \rho$. Writing up this equality of measures on $\mathbb{R}^d$ for each quadrant of $\mathbb{R}^d$, we immediately see that the measures $(\rho_v,\, v \in Q_d)$ and $\big(\nu_v^{(i)},\, v \in Q_d\big)$, $i = 1,2$, satisfy the system of equations (2.14).

We let $\alpha_j = \alpha$ for $j = 1,\dots,d$ and $K = \{1,\dots,d\}$. Then (2.15) holds since the measures $\nu^{(1)}$ and $\nu^{(2)}$ are vague limits in $\overline{\mathbb{R}}_0^d$ and, hence, place no mass at the origin in $\mathbb{R}^d$. The assumption (2.12) follows from (2.2). The assumption (2.13) follows from the fact that both $\nu^{(1)}$ and $\nu^{(2)}$ satisfy (2.5) and the scaling property of the tail measure μ. Finally, the condition (2.16) follows from (2.25) and elementary manipulation of the sums and integrals. Therefore, Theorem 2 applies, and $\nu_v^{(1)} = \nu_v^{(2)}$ for each $v \in Q_d$. This means that $\nu^{(1)} = \nu^{(2)}$.

Remark 3. If the tail measure μ of $\nu \circledast \rho$ satisfies

$$\mu\big(\{\mathbf{x} :\, x_k = 0 \ \text{ for each } k \in K\}\big) = 0 \tag{2.26}$$

for some non-empty set $K \subseteq \{1,\dots,d\}$, then every measure μ_* satisfying (2.5) has the same property:

$$\mu_*\big(\{\mathbf{x} :\, x_k = 0 \ \text{ for each } k \in K\}\big) = 0\,.$$

Therefore the measures $\big(\nu_v^{(i)},\, v \in Q_d\big)$, $i = 1,2$, defined in the proof of Theorem 3 satisfy (2.15), and we can apply Theorem 2 with this smaller set K. In other words, if (2.26) holds, then the condition (2.25) in Theorem 3 has to be checked only for $j \in K$.

We can extend Theorem 3 to the situation where the measure ρ puts a positive mass on the axes. The next corollary follows from the theorem by splitting the space $\mathbb{R}^d$ into subspaces of different dimensions, by setting some of the coordinates equal to zero. We omit details.

Corollary 4. *Let $\alpha > 0$ and ρ, ν be σ-finite measures on $\mathbb{R}^d$ such that* (2.2) *and* (2.3) *hold and $\nu \circledast \rho \in \mathrm{RV}(\alpha,\mu)$. Assume that for every $I_0 \subset \{1,\dots,d\}$ such that*

$$\rho\Big(\big\{\mathbf{x} \in \mathbb{R}^d :\, x_i = 0 \ \textit{ for all } i \in I_0\big\}\Big) > 0$$

we have for every I such that $I_0 \cup I = \{1,\dots,d\}$,

$$\int_{\mathbb{R}^d} |x_j|^\alpha \prod_{k\in I} |x_k|^{i\theta_k} \prod_{k\in I} \big(\mathrm{sign}(x_k)\big)^{m_k}\, \rho(d\mathbf{x}) \neq 0 \tag{2.27}$$

for each $j \in I$, $m_k \in \{0,1\}$ and $\theta_k \in \mathbb{R}$, $k \in I$. Then the conclusions of Theorem 3 hold.

3. The inverse problem for weighted sums

In this section we revisit the weighted sums of iid random vectors introduced in Example 1. We consider the special case of diagonal coefficient matrices. Our goal is to apply the generalized cancellation theory of the previous section to investigate under what conditions on the coefficient matrices regular variation of the weighted sum implies regular variation of the underlying iid random vectors.

Let $(\mathbf{Z}^{(i)})$ be an iid sequence of $\mathbb{R}^d$-valued random column vectors with a generic element $\mathbf{Z}$ and $(\mathbf{d}^{(i)})$ be deterministic vectors in $\mathbb{R}^d$. The ith coefficient matrix Ψ_i is a diagonal matrix with $\mathbf{d}^{(i)}$ on the main diagonal: $\Psi_i = \mathrm{diag}(\mathbf{d}^{(i)})$. The following theorem is the main result of this section. The corresponding result for $d = 1$ and positive weights ψ_i was proved in Jacobsen et al. (2008), Theorem 3.3.

Theorem 4. *Assume that the series $\mathbf{X} = \sum_{i=1}^\infty \Psi_i \mathbf{Z}^{(i)}$ converges a.s., $\mathbf{X} \in \mathrm{RV}(\alpha, \mu_\mathbf{X})$ and for some $0 < \delta' < \alpha$,*

$$\sum_{i=1}^\infty \|\mathbf{d}^{(i)}\|^{\alpha-\delta'} < \infty\,. \tag{3.1}$$

Suppose also that all non-zero vectors $(\mathbf{d}^{(i)})$ have non-vanishing coordinates. If for all $j = 1, \dots, d$, for $m_1, \dots, m_d \in \{0, 1\}$ and $\theta_1, \dots, \theta_d \in \mathbb{R}$,

$$\sum_{l=1}^\infty \Big[|d_j^{(l)}|^\alpha \prod_{k=1}^d |d_k^{(l)}|^{i\theta_k} \prod_{k=1}^d \big(\mathrm{sign}(d_k^{(l)})\big)^{m_k}\Big] \neq 0\,, \tag{3.2}$$

then $\mathbf{Z}$ is regularly varying with index α and (1.1) *holds.*

Remark 4. Of course, if some of the non-zero vectors $(\mathbf{d}^{(i)})$ have vanishing coordinates, we can use Corollary 4 instead of Theorem 3, and obtain regular variation of the vector $\mathbf{Z}$ under a more extensive set of conditions than (3.2).

We start the proof with the following lemma.

Lemma 1. *Assume the conditions of Theorem 4 but the vectors $\mathbf{d}_i$, $i = 1, 2, \dots,$ may also contain zero components. Then, for any Borel set $A \subset \mathbb{R}^d$ bounded away from the origin and such that A is a $\mu_\mathbf{X}$-continuity set,*

$$P(s^{-1}\mathbf{X} \in A) \sim \sum_{i=1}^\infty P\big(s^{-1}\Psi_i \mathbf{Z} \in A\big)\,, \quad s \to \infty\,. \tag{3.3}$$

Proof. For every $j = 1, \dots, d$, we may assume that there is $i(j) = 1, 2, \dots$ such that $d_j^{(i(j))} \neq 0$ for, if this is not the case, we can simply delete the jth coordinate. Denote

$$\mathbf{Y}^{(j)} = \mathbf{X} - \Psi_{i(j)}\, \mathbf{Z}^{(i(j))}$$

and choose $M_j > 0$ such that $P(\|\mathbf{Y}^{(j)}\| \le M_j) > 0$, $j = 1, \dots, d$. We have for $s > 0$ and $j = 1, \dots, d$,

$$P(\|\mathbf{X}\| > s) \ge P(\|\mathbf{Y}^{(j)}\| \le M_j) P\big(|d_j^{(i(j))}||Z_j| > s + M_j\big)\,,$$

and the regular variation of $\mathbf{X}$ implies that there is $C_j > 0$ such that

$$P(|Z_j| > s) \le C_j P(\|\mathbf{X}\| > s)\,, \quad s > 0\,,$$

and therefore there is $C > 0$ such that

$$P(\|\mathbf{Z}\| > s) \le C\, P(\|\mathbf{X}\| > s) \quad \text{for all } s > 0\,. \tag{3.4}$$

We write $\mathbf{X}_q = \sum_{i=1}^q \Psi_i \mathbf{Z}^{(i)}$ and $\mathbf{X}^q = \mathbf{X} - \mathbf{X}_q$ for $q \geq 1$. In the usual notation,

$$A^\epsilon = \{\mathbf{y} \in \mathbb{R}_0^d : d(\mathbf{y}, A) \leq \epsilon\}, \quad A_\epsilon = \{\mathbf{y} \in A : d(\mathbf{y}, A^c) > \epsilon\}\,,$$

we have

$$\begin{aligned} & P(s^{-1}\mathbf{X}_q \in A_\epsilon)\, P(\|\mathbf{X}^q\|) \leq \epsilon s) \\ \leq \quad & P(s^{-1}\mathbf{X} \in A) \leq P(s^{-1}\mathbf{X}_q \in A^\epsilon) + P(\|\mathbf{X}^q\| > \epsilon s)\,. \end{aligned} \tag{3.5}$$

Proceeding as in the Appendix of Mikosch and Samorodnitsky (2000) and using (3.4), we obtain

$$\lim_{q\to\infty} \limsup_{s\to\infty} \frac{P(\|\mathbf{X}^q\| > s)}{P(\|\mathbf{X}\| > s)} = 0\,.$$

Therefore and by virtue of (3.5) it suffices to prove the lemma for $\mathbf{X}_q$ instead of $\mathbf{X}$. In what follows, we assume $q < \infty$ and suppress the dependence of $\mathbf{X}$ on q in the notation.

Let $M = \max_{i=1,\ldots,q,\, j=1,\ldots,d} |d_j^{(i)}|$. For $\epsilon > 0$ we have

$$\begin{aligned} & P(s^{-1}\mathbf{X} \in A_\epsilon) \\ \leq \quad & \sum_{j=1}^q P(s^{-1}\Psi_j \mathbf{Z} \in A) + \frac{q\,(q-1)}{2} \left(P\left(\|\mathbf{Z}\| > \frac{s\,\epsilon}{(q-1)M}\right)\right)^2\,. \end{aligned}$$

Hence, by (3.4) and regular variation of $\mathbf{X}$,

$$\mu_{\mathbf{X}}(A_\epsilon) \leq \liminf_{s\to\infty} \frac{P(s^{-1}\mathbf{X} \in A_\epsilon)}{P(\|\mathbf{X}\| > s)} \leq \liminf_{s\to\infty} \frac{\sum_{j=1}^q P(s^{-1}\Psi_j \mathbf{Z} \in A)}{P(\|\mathbf{X}\| > s)}\,.$$

Letting $\epsilon \downarrow 0$ and using that A is a $\mu_{\mathbf{X}}$-continuity set, we have

$$\mu_{\mathbf{X}}(A) \leq \liminf_{s\to\infty} \frac{\sum_{j=1}^q P(s^{-1}\Psi_j \mathbf{Z} \in A)}{P(\|\mathbf{X}\| > s)}\,,$$

and (3.3) will follow once we show that

$$\mu_{\mathbf{X}}(A) \geq \limsup_{s\to\infty} \frac{\sum_{j=1}^q P(s^{-1}\Psi_j\, \mathbf{Z} \in A)}{P(\|\mathbf{X}\| > s)}\,. \tag{3.6}$$

Let $\delta := \inf\{\|\mathbf{x}\| : \mathbf{x} \in A\} > 0$. For $0 < \epsilon < \delta$ write

$$\begin{aligned} P(s^{-1}\mathbf{X} \in A^\epsilon) \quad & \geq \quad P\left(\bigcup_{i=1}^q \left\{ s^{-1}\Psi_i \mathbf{Z}^{(i)} \in A\,, \left\| \sum_{1\leq l\neq i\leq q} \Psi_l\, \mathbf{Z}^{(l)} \right\| \leq s\epsilon \right\}\right) \\ & \geq \quad \sum_{i=1}^q P\left(s^{-1}\Psi_i \mathbf{Z}^{(i)} \in A\,, \left\| \sum_{1\leq l\neq i\leq q} \Psi_l\, \mathbf{Z}^{(l)} \right\| \leq s\epsilon\right) \\ & \qquad -\frac{q(q-1)}{2}\,(P(\|\mathbf{Z}\| \geq s\delta/M))^2 \end{aligned}$$

$$\geq \sum_{i=1}^{q} P\Big(s^{-1}\Psi_i \mathbf{Z}^{(i)} \in A\Big) - \frac{q(q-1)}{2}\left(P(\|\mathbf{Z}\| \geq s\delta/M)\right)^2$$
$$-q(q-1)P(\|\mathbf{Z}\| \geq s\delta/M)P(\|\mathbf{Z}\| \geq s\epsilon/((q-1)M))\,.$$

Thus by regular variation of $\mathbf{X}$ and (3.4),

$$\mu_{\mathbf{X}}(A^\epsilon) \geq \limsup_{s\to\infty} \frac{P(s^{-1}\mathbf{X} \in A^\epsilon)}{P(\|\mathbf{X}\| > s)} \geq \limsup_{s\to\infty} \frac{\sum_{i=1}^{q} P\left(s^{-1}\Psi_j \mathbf{Z} \in A\right)}{P(\|\mathbf{X}\| > s)}\,.$$

Letting $\epsilon \downarrow 0$ and using the $\mu_{\mathbf{X}}$-continuity of A, we obtain the desired relation (3.6).

Proof of Theorem 4. It follows from Lemma 1 that the measure

$$\mu(\cdot) = \sum_{i=1}^{\infty} P\big(\Psi_i \mathbf{Z} \in \cdot\big) \quad \text{on } \mathbb{R}^d$$

is regularly varying with index α. Note that $\mu = \nu \circledast \rho$, where ν is the law of $\mathbf{Z}$ (a probability measure), and

$$\rho = \sum_{i=1}^{\infty} \delta_{\mathbf{d}^{(i)}}\,,$$

with the usual notation $\delta_{\mathbf{a}}$ standing for the unit mass at the point $\mathbf{a} \in \mathbb{R}^d$. Note that the conditions of Theorem 3 are satisfied; in particular (2.3) holds because the measure ρ has bounded support. Therefore, the conclusion of Theorem 4 follows.

Example 3. Consider the vector AR(1) difference equation $\mathbf{X}_t = \Psi\mathbf{X}_{t-1} + \mathbf{Z}_t$, $i \in \mathbb{Z}$, for an iid $\mathbb{R}^d$-valued sequence $(\mathbf{Z}_t)$ and a matrix $\Psi = \mathrm{diag}(\mathbf{d})$ for some deterministic vector $\mathbf{d} \in \mathbb{D}^d$ with nonvanishing coordinates. A unique stationary causal solution to the AR(1) equation exists if and only if $\max_{i=1,\ldots,d} |d_i| < 1$ and $\mathbf{Z}_1$ has some finite logarithmic moment. A generic element $\mathbf{X}$ of the solution satisfies the relation $\mathbf{X} \stackrel{d}{=} \sum_{j=0}^{\infty} \Psi^j \mathbf{Z}_j$. Assume that $\mathbf{X}$ is regularly varying with index $\alpha > 0$. Then (3.1) is trivially satisfied and (3.2) reads as follows: for every $j = 1,\ldots,d$, any $m_i \in \{0,1\}, \theta_i \in \mathbb{R}$, $i = 1,\ldots,d$,

$$|d_j|^\alpha \prod_{k=1}^{d} |d_k|^{i\theta_k} \prod_{k=1}^{d} (\mathrm{sign}(d_k))^{m_k}\Big(1 - |d_j|^\alpha \prod_{k=1}^{d} |d_k|^{i\theta_k} \prod_{k=1}^{d} (\mathrm{sign}(d_k))^{m_k}\Big)^{-1} \neq 0\,,$$

This condition is always satisfied. Hence any $\mathbf{Z}_t$ is regularly varying with index $\alpha > 0$.

A special case of the setup of this section is a sum with scalar weights, of the type $\mathbf{X} = \sum_{i=1}^{\infty} \psi_i \mathbf{Z}^{(i)}$, where (ψ_i) is a sequence of scalars. Applying Theorem 4 with $d_j^{(i)} = \psi_i$, $j = 1,\ldots,d$ for $i = 1,2,\ldots$, we obtain the following corollary.

Corollary 5. *Let $\alpha > 0$, and suppose that for some $0 < \delta < \alpha$,*

$$\sum_{i=1}^{\infty} |\psi_i|^{\alpha-\delta} < \infty\,. \tag{3.7}$$

Assume that the series $\mathbf{X} = \sum_{i=1}^{\infty} \psi_i \mathbf{Z}^{(i)}$ *converges a.s. and* $\mathbf{X}$ *is regularly varying with index* α*. If*

$$\sum_{j=1}^{\infty} |\psi_j|^{\alpha+i\theta} \neq 0\,, \quad \theta \in \mathbb{R}\,, \quad \textit{and} \tag{3.8}$$

$$\sum_{j:\psi_j>0} \psi_j^{\alpha+i\theta} \neq \sum_{j:\psi_j<0} |\psi_j|^{\alpha+i\theta}\,, \quad \theta \in \mathbb{R}\,, \tag{3.9}$$

then $\mathbf{Z} \in \mathrm{RV}(\alpha, \mu_{\mathbf{Z}})$ *and the tail measure* $\mu_{\mathbf{Z}}$ *satisfies*

$$\frac{P(s^{-1}\mathbf{X} \in \cdot)}{P(|\mathbf{Z}| > s)} \xrightarrow{v} \psi_+ \, \mu_{\mathbf{Z}}(\cdot) + \psi_- \, \mu_{\mathbf{Z}}(-\cdot)\,, \quad s \to \infty\,,$$

where

$$\psi_+ = \sum_{j:\psi_j>0} \psi_j^{\alpha} \quad \textit{and} \quad \psi_- = \sum_{j:\psi_j<0} |\psi_j|^{\alpha}\,. \tag{3.10}$$

Remark 5. Corollary 5 has a natural converse statement. Specifically, if either (3.8) or (3.9) fail to hold for some real θ, then there is a random vector $\mathbf{Z}$ that is not regularly varying but $\mathbf{X} = \sum_{i=1}^{\infty} \psi_i \mathbf{Z}^{(i)}$ is regularly varying. Indeed, suppose, for example, that (3.8) fails for some real θ_0. We use a construction similar to that in Jacobsen et al. (2008). Choose real numbers a, b satisfying $0 < a^2 + b^2 \leq 1$, and define a measure on $(0, \infty)$ by

$$\nu_0(dx) = [1 + a \, \cos(\theta_0 \, \log x) + b \, \sin(\theta_0 \, \log x)] \, \nu^{\alpha}(dx)\,, \tag{3.11}$$

where ν^{α} is given in (2.11). Choose $r > 0$ large enough so that $\nu_0(r, \infty) \leq 1$, define a probability law on $(0, \infty)$ by

$$\mu_0(B) = \nu_0\big(B \cap (r, \infty)\big) + \big[1 - \nu_0(r, \infty)\big] \mathbf{1}_B(1) \quad \text{for any Borel set } B,$$

and a probability law on $\mathbb{R}$ by

$$\mu_*(\cdot) = \frac{1}{2}\mu_0(\cdot) + \frac{1}{2}\mu_0(-\cdot).$$

Obviously, μ_* is not a regularly varying probability measure. Therefore, neither is the random vector $\mathbf{Z} = (Z, 0, \ldots, 0)$ regularly varying, where Z has distribution μ_*.

Since the vector $\mathbf{Z}$ is symmetric, the series $\mathbf{X} = \sum_{i=1}^{\infty} \psi_i \mathbf{Z}^{(i)}$ converges a.s. under the assumption (3.7); see Lemma A.3 in Mikosch and Samorodnitsky (2000), and the argument in Jacobsen et al. (2008) shows that $\mathbf{X}$ is regularly varying with index α.

On the other hand, suppose that (3.9) fails for some real θ_0. Define ν_0 as in (3.11), and define another measure on $(0, \infty)$ by

$$\nu_1(dx) = [1 - a \, \cos(\theta_0 \, \log x) - b \, \sin(\theta_0 \, \log x)] \, \nu^{\alpha}(dx)\,.$$

Convert ν_0 into a probability measure μ_0 as above, and similarly convert ν_1 into a probability measure μ_1. Define a probability measure on $\mathbb{R}$ by

$$\mu_*(\cdot) = \frac{1}{2}\mu_0(\cdot) + \frac{1}{2}\mu_1(-\cdot).$$

Once again, let $\mathbf{Z} = (Z_1, 0, \ldots, 0)$, where $Z_1 \sim \mu_*$. Then $\mathbf{Z}$ is not regularly varying, and neither is the vector

$$\tilde{\mathbf{Z}} = \begin{cases} \mathbf{Z}\,, & \text{if } 0 < \alpha \le 1\,, \\ \mathbf{Z} - E(\mathbf{Z})\,, & \text{if } \alpha > 1\,. \end{cases}$$

As before, the series $\mathbf{X} = \sum_{i=1}^{\infty} \psi_i \tilde{\mathbf{Z}}^{(i)}$ converges a.s. under the assumption (3.7), and $\mathbf{X}$ is regularly varying with index α.

We proceed with several examples of the situation described in Corollary 5. We say that the coefficients $\psi_1, \psi_2, \ldots$, are *α-regular variation determining* if regular variation of $\mathbf{X} = \sum_{i=1}^{\infty} \psi_i \mathbf{Z}^{(i)}$ implies regular variation of $\mathbf{Z}$. In other words, both conditions (3.8) and (3.9) must be satisfied.

Example 4. Let $q < \infty$ and assume that $\psi_i = 1$, $i = 1, \ldots, q$, $\psi_i = 0$ for $i > q$. By Corollary 5 these coefficients are α-regular variation determining and $P(s^{-1}\mathbf{X} \in \cdot) \sim qP(s^{-1}\mathbf{Z} \in \cdot)$ as $s \to \infty$. For $d = 1$ (only in this case the notion of subexponentiality is properly defined) this property is in agreement with the *convolution root property* of subexponential distributions; see Embrechts et al. (1979); cf. Proposition A3.18 in al. Embrechts et al. (1997). Indeed, if X is a positive random variable then regular variation of Z implies subexponentiality.

Example 5. Again, let $q < \infty$ and $\psi_j = 0$ for $j > q$. If, say, $|\psi_1|^\alpha > \sum_{j=2}^{q} |\psi_j|^\alpha$, then both conditions (3.8) and (3.9) are satisfied and therefore the coefficients are α-regular variation determining. This is, of course, the same phenomenon as in Corollary 3. In the special case, $q = 2$, if $\psi_1 \neq -\psi_2$, then the coefficients are α-regular variation determining. On the other hand, If $\psi_1 = -\psi_2$, then condition (3.9) fails, and the coefficients are not α-regular variation determining. This means that regular variation of $\mathbf{X} = \mathbf{Z}_1 - \mathbf{Z}_2$ does not necessarily imply regular variation of $\mathbf{Z}$.

4. The inverse problem for products

We now apply the generalized cancellation theory to Example 2 above. We concentrate on the case of multiplication by a random diagonal matrix. Specifically, let $\mathbf{A} = \mathrm{diag}(A_1, \ldots, A_d)$ for some random variables (A_i), $i = 1, \ldots, d$. The following theorem is an easy application of Theorem 3.

Theorem 5. *Assume that $P(A_j = 0) = 0$ for $j = 1, \ldots, d$. Let $\mathbf{A} = \mathrm{diag}(A_1, \ldots, A_d)$. Let $\mathbf{Z}$ be a d-dimensional random vector independent of $\mathbf{A}$, such that $\mathbf{X} = \mathbf{A}\mathbf{Z}$ is regularly varying with index $\alpha > 0$. If $E\|\mathbf{A}\|^{\alpha+\delta} < \infty$ for some $\delta > 0$ and*

$$E\left(|A_j|^\alpha \prod_{k=1}^{d} \left(|A_k|^{i\theta_k} (\mathrm{sign}(A_k))^{m_k} \right) \right) \neq 0 \tag{4.1}$$

for each $j = 1, \ldots, d$, $m_1, \ldots, m_d \in \{0, 1\}$ and $\theta_1, \ldots, \theta_d \in \mathbb{R}$, then $\mathbf{Z}$ is regularly varying with index $\alpha > 0$. Moreover, (1.3) *holds.*

A special case is multiplication of a random vector by an independent scalar random variable, corresponding to $A_1 = \ldots = A_d = A$ for some random variable A. The following corollary restates Theorem 5 in this special case.

Corollary 6. *Let A be a random variable independent of a d-dimensional random vector $\mathbf{Z}$ such that $\mathbf{X} = A\mathbf{Z}$ is regularly varying with index $\alpha > 0$. If $E|A|^{\alpha+\delta} < \infty$ for some $\delta > 0$ and*

$$E|A|^{\alpha+i\theta} \neq 0\,, \quad \theta \in \mathbb{R}\,, \tag{4.2}$$

$$EA_+^{\alpha+i\theta} \neq EA_-^{\alpha+i\theta}\,, \quad \theta \in \mathbb{R}\,, \tag{4.3}$$

then $\mathbf{Z}$ is regularly varying with index $\alpha > 0$. Moreover, the tail measure $\mu_{\mathbf{Z}}$ of $\mathbf{Z}$ satisfies

$$\frac{P(s^{-1}\mathbf{X} \in \cdot)}{P(|\mathbf{Z}| > s)} \stackrel{v}{\rightarrow} EA_+^{\alpha}\,\mu_{\mathbf{Z}}(\cdot) + EA_-^{\alpha}\,\mu_{\mathbf{Z}}(-\cdot)\,, \quad s \to \infty\,,$$

where $A_+ = \max(A, 0)$, $A_- = \max(-A, 0)$.

Using terminology similar to that of the previous section, we say that a random variable A is *α-regular variation determining* if regular variation with index α of $\mathbf{X} = A\mathbf{Z}$ implies regular variation of $\mathbf{Z}$. Corollary 6 shows that if A satisfies both conditions (4.2) and (4.3), then A is α-regular variation determining. On the other hand, a construction similar to that in Remark 5 shows that, if one of the conditions (4.2) and (4.3) fails, then one can construct an example of a random vector $\mathbf{Z}$ that is not regularly varying but $\mathbf{X} = A\mathbf{Z}$ is regularly varying with index α. Therefore, conditions (4.2) and (4.3) are necessary and sufficient for A being α-regular variation determining.

Jacobsen et al. (2008), Theorem 4.2, proved this result for positive A. They gave various examples of distributions on $(0, \infty)$ which are α-regular variation determining, including the gamma, log-normal, Pareto distributions, the distribution of the powers of the absolute value of a symmetric normal random variable, of the absolute values of a Cauchy random variable (for $\alpha < 1$) and any positive random variable whose log-transform is infinitely divisible. The condition in (4.3) rules out a whole class of important distributions: no member of the class of symmetric distributions is α-regular variation determining. Even non-symmetric distributions with $EA_+^{\alpha} = EA_-^{\alpha}$ are not α-regular variation determining. For a further example, consider a uniform random variable $A \sim U(a, b)$ for $a < b$. If $a = -b$, then A cannot be α-regular variation determining since it has a symmetric distribution. On the other hand, an elementary calculation shows that in all other cases both conditions (4.2) and (4.3) hold. Therefore, the only non-α-regular variation determining uniform random variables are the symmetric ones.

In financial time series analysis, models for returns are often of the form $X_t = A_t Z_t$, where (A_t) is some volatility sequence and (Z_t) is an iid multiplicative noise sequence such that A_t and Z_t are independent for every t and (X_t) constitutes a strictly stationary sequence. In most parts of the literature it is assumed that the volatility A_t is non-negative. It is often assumed that X_t is heavy-tailed, e.g. regularly varying with some index $\alpha > 0$; see Davis and Mikosch (2009b,a). Notice that A_t and Z_t are not observable; it depends on the model to which of the variables A_t or Z_t one assigns regular variation. For example, in the case of a GARCH process (X_t), (A_t) is regularly varying with index $\alpha > 0$ and the iid noise (Z_t) has lighter tails and is symmetric. On the other hand, if one only assumes that X_t is regularly varying with index α and $E|Z|^{\alpha+\delta} < \infty$ for some $\delta > 0$ and Z is symmetric, one cannot conclude that A_t is regularly varying.

5. Non-diagonal matrices

The (direct) statements of Examples 1 and 2 of Section 1 deal with transformations of regularly varying random vectors involving matrices that do not have to be diagonal matrices. On the other hand, all the converse statements of Sections 3 and 4 deal only with diagonal matrices. Generally, we do not know how to solve inverse problems involving non-diagonal matrices. This section describes one of the very few "non-diagonal" situations where we can prove a converse statement. We restrict ourselves to the case of finite weighted sums and square matrices.

Theorem 6. *Let* $\mathbf{X} = \sum_{j=1}^{q} \Psi_j \mathbf{Z}_j$*, where* $\mathbf{Z}_j$*,* $j = 1, \ldots, q$*, are iid* $\mathbb{R}^d$*-valued random vectors and* Ψ_j*,* $j = 1, \ldots, q$*, deterministic* $(d \times d)$*-matrices. Assume that* $\mathbf{X} \in \mathrm{RV}(\alpha, \mu_{\mathbf{X}})$ *for some* $\alpha > 0$*. If all the matrices* Ψ_j*,* $j = 1, \ldots, q$ *are invertible, and*

$$(\gamma(\Psi_1))^\alpha > \sum_{j=2}^{q} \|\Psi_j\|^\alpha \,, \tag{5.1}$$

where $\gamma(\Psi_1) = \min_{\mathbf{z} \in \mathbb{S}^{d-1}} |\Psi_1 \mathbf{z}|$ *and* $\|\Psi_j\|$ *is the operator norm of* Ψ_j*,* $j = 1, \ldots, q$*, then* $\mathbf{Z} \in \mathrm{RV}(\alpha, \mu_{\mathbf{Z}})$ *and* $\mu_{\mathbf{Z}}$ *satisfies* (1.1).

Proof. An argument similar to that in Lemma 1 shows that under the assumptions of the theorem a finite version of (3.3) holds: for any Borel set $A \subset \mathbb{R}^d$ bounded away from the origin such that A is a continuity set with respect to the tail measure $\mu_{\mathbf{X}}$,

$$P(s^{-1}\mathbf{X} \in A) \sim \sum_{i=1}^{q} P\big(s^{-1}\Psi_i \mathbf{Z} \in A\big) \,, \quad s \to \infty \,.$$

This allows us to proceed as in Theorem 1 to see that the family of measures

$$\left(\frac{P(s^{-1}\mathbf{Z} \in \cdot)}{P(|\mathbf{X}| > s)} \right)_{s \geq 1} \tag{5.2}$$

is vaguely tight in $\overline{\mathbb{R}}_0^m$, and any vague subsequent limit μ_* of this family satisfies

$$\mu_{\mathbf{X}} = \sum_{j=1}^{q} \mu_* \circ \Psi_j^{-1} \,. \tag{5.3}$$

Let $T_j = \Psi_j^{-1}\Psi_1$, $j = 2, \ldots, q$. Then by (5.3), for any measurable set $B \subset \mathbb{R}^d$ bounded away from zero,

$$\mu_*(B) = \mu_{\mathbf{X}}(\Psi_1 B) - \sum_{j=2}^{q} \mu_*(T_j B) \,. \tag{5.4}$$

Replacing B with $T_j B$ for $j = 2, \ldots, q$ and iterating (5.4), we obtain for $n = 1, 2, \ldots$,

$$\begin{aligned} \mu_*(B) &= \mu_{\mathbf{X}}(\Psi_1 B) - \sum_{j=2}^{q} \mu_{\mathbf{X}}(\Psi_1 T_j B) + \sum_{j_1=2}^{q} \sum_{j_2=2}^{q} \mu_*(T_{j_2} T_{j_1} B) \\ &= \cdots \end{aligned}$$

$$= \sum_{k=0}^{n}(-1)^k \sum_{j_1=2}^{q}\cdots\sum_{j_k=2}^{q}\mu_{\mathbf{X}}(\Psi_1 T_{j_k}\cdots T_{j_1}B) \qquad (5.5)$$
$$+(-1)^{n+1}\sum_{j_1=2}^{q}\cdots\sum_{j_{n+1}=2}^{q}\mu_*(T_{j_{n+1}}\cdots T_{j_1}B)\,.$$

Clearly, for every $n \ge 1$ and $j_1,\dots,j_{n+1} = 2,\dots,q$,

$$\inf_{\mathbf{z}\in T_{j_{n+1}}\cdots T_{j_1}B}|\mathbf{z}| \ge \inf_{\mathbf{z}\in B}|\mathbf{z}|\,(\gamma(\Psi_1))^{n+1}\prod_{k=1}^{n+1}\|\Psi_{j_k}\|^{-1}\,. \qquad (5.6)$$

Furthermore, it follows from (5.3) that, for some $c > 0$,

$$\mu_*(\{\mathbf{z}\in\mathbb{R}^d : |\mathbf{z}| > s\}) \le c\, s^{-\alpha}\,, \quad s > 0\,.$$

Therefore we conclude by (5.6) and (5.1) that

$$\sum_{j_1=2}^{q}\cdots\sum_{j_{n+1}=2}^{q}\mu_*(T_{j_{n+1}}\cdots T_{j_1}B)$$
$$\le c\,\big(\inf_{\mathbf{z}\in B}|\mathbf{z}|\big)^{-\alpha}\sum_{j_1=2}^{q}\cdots\sum_{j_{n+1}=2}^{q}\left((\gamma(\Psi_1))^{n+1}\prod_{k=1}^{n+1}\|\Psi_{j_k}\|^{-1}\right)^{-\alpha}$$
$$= c\,\big(\inf_{\mathbf{z}\in B}|\mathbf{z}|\big)^{-\alpha}(\gamma(\Psi_1))^{-\alpha\,(n+1)}\left(\sum_{j=2}^{q}\|\Psi_j\|^{\alpha}\right)^{n+1} \to 0\,.$$

Thus by virtue of (5.5),

$$\mu_*(B) = \lim_{n\to\infty}\sum_{k=0}^{n}(-1)^k\sum_{j_1=2}^{q}\cdots\sum_{j_k=2}^{q}\mu_{\mathbf{X}}(T_{j_k}\cdots T_{j_1}B)\,.$$

This means that μ_* is uniquely determined by the measure $\mu_{\mathbf{X}}$. Hence all the subsequential vague limits of (5.2) coincide. Therefore, $\mathbf{Z}\in \mathrm{RV}(\alpha,\mu_{\mathbf{Z}})$ and (1.1) holds.

Remark 6. Note that in the special case of diagonal matrices (Ψ_j) with identical elements on the diagonals, the conditions in Theorem 6 coincide with those in Example 5 above.

Remark 7. The conditions in Theorem 6 can be slightly weakened by assuming, instead of (5.1), that

$$(\gamma(A\Psi_1))^{\alpha} > \sum_{j=2}^{q}\|A\Psi_j\|^{\alpha} \qquad (5.7)$$

for some invertible matrix A. Indeed, regular variation of $\mathbf{X}$ implies regular variation of the vector $A\mathbf{X}$, and regular variation of the $A\mathbf{Z}$ implies regular variation of $\mathbf{Z}$. It is not difficult to construct examples where (5.7) holds but (5.1) fails.

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References

B. Basrak, R. Davis and T. Mikosch (2002a): A characterization of multivariate regular variation. *Annals of Applied Probability* 12:908–920.

B. Basrak, R. Davis and T. Mikosch (2002b): Regular variation of GARCH processes. *Stochastic Processes and Their Applications* 99:95–115.

R. Davis and T. Mikosch (2009a): Extreme value theory for GARCH processes. In *Handbook of Financial Time Series*, T. Andersen, R. Davis, J.-P. Kreiss and T. Mikosch, editors. Springer, Berlin, pp. 355–364.

R. Davis and T. Mikosch (2009b): Extremes of stochastic volatility models. In *Handbook of Financial Time Series*, T. Andersen, R. Davis, J.-P. Kreiss and T. Mikosch, editors. Springer, Berlin, pp. 355–364.

P. Embrechts, C. Goldie and N. Veraverbeke (1979): Subexponentiality and infinite divisibility. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* 49:335–347.

P. Embrechts, C. Klüppelberg and T. Mikosch (1997): *Modelling Extremal Events for Insurance and Finance*. Springer-Verlag, Berlin.

H. Hult and G. Samorodnitsky (2008): Tail probabilities for infinite series of regularly varying random vectors. *Bernoulli* 14:838–864.

H. Hult and G. Samorodnitsky (2010): Large deviations for point processes based on stationary sequecies with heavy tails. *Advances in Applied Probability* 42:1–40.

M. Jacobsen, T. Mikosch, J. Rosiński and G. Samorodnitsky (2008): Inverse problems for regular variation of linear filters, a cancellation property for σ-finite measures, and identification of stable laws. *Annals of Applied Probability* 19:210–242.

O. Kallenberg (1983): *Random Measures*. Akademie–Verlag, Berlin, 3rd edition.

T. Mikosch and G. Samorodnitsky (2000): The supremum of a negative drift random walk with dependent heavy–tailed steps. *Annals of Applied Probability* 10:1025–1064.

S. Resnick (1987): *Extreme Values, Regular Variation and Point Processes*. Springer-Verlag, New York.

S. Resnick and E. Willekens (1991): Moving averages with random coefficients and random coefficient autoregressive models. *Comm. Statist. Stochastic Models* 7:511–525.

W. Rudin (1973): *Functional Analysis*. McGraw Hill, New York.