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AGGREGATION OF LOG-LINEAR RISKS

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Abstract

In this paper we work in the framework of a k-dimensional vector of log-linear risks. Under weak conditions on the marginal tails and the dependence structure of a vector of positive risks we derive the asymptotic tail behaviour of the aggregated risk and present an application concerning log-normal risks with stochastic volatility.

Keywords: Risk aggregation; log-linear model; subexponential distribution; Gumbel max-domain of attraction.

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1. Introduction

The recent contribution [17] discusses important aspects of linear models of heavy-tailed risks related to risk diversification. Following the aforementioned paper, a good starting point for explaining linear models is the stochastic representation of multivariate normal risks in terms of regression models. For the purpose of this introduction, we confine ourselves for the moment to two random variables (rvs). Specifically, if X_1 and X_2 are jointly normal with mean 0, variance 1 (i.e., with standard

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normal distribution function (df)) and correlation $\rho \in [0,1)$, we assume

$$X_i = \sqrt{\rho} W_0 + W_{i,\rho}, \qquad W_{i,\rho} = \sqrt{1-\rho} W_i, \quad i = 1, 2, \tag{1}$$

where W_0, W_1, W_2 are independent $N(0,1)$ rvs. Motivated by (1), we refer to the rvs $Z_1 = e^{X_1}, Z_2 = e^{X_2}$ as *log-linear risks*. Clearly, Z_1, Z_2, are risks with some positive dependence structure, which is a common feature of many financial and insurance risks. In numerous finance, insurance and risk management applications a prevailing model for aggregating dependent risks is the log-normal one with positive dependence; see [1, 2, 12, 13, 15, 16, 18].

The importance of this paradigm lies in the fact that on the log-scale a linear relationship such as (1) is assumed. Numerous applications based on the log-normal assumption explain the behaviour of aggregated and maximum risk. Despite the tractability and the wide applicability of log-normal based models, the asymptotic tail behaviour of the aggregated risk $S_2 = e^{X_1} + e^{X_2}$ has been unknown for a long time; in [3] it is first shown that the principle of a single big-jump applies for log-normal rvs defined by (1), i.e.,

$$\mathbb{P}\{S_2 > u\} \quad \sim \quad \mathbb{P}\Big\{\max(e^{X_1}, e^{X_2}) > u\Big\} \sim 2\mathbb{P}\{X_1 > \ln u\}, \quad u \to \infty.$$

We use the standard notation $a(u) \sim b(u)$ meaning $\lim_{u\to\infty} a(u)/b(u) = 1$ for non-negative functions $a(\cdot)$ and $b(\cdot)$.

In this paper, instead of making specific distributional assumptions on the W_i's we shall impose only weak conditions on the marginal tails and on the joint dependence structure. For instance, for the simple setup of bivariate log-linear risks, the main finding of this contribution is that under such assumptions, the asymptotic tail behaviour of the aggregated risk S_2 is still determined by the tail asymptotics of Z_1 and Z_2. In the special case when W_i, $i = 1, 2$, satisfy

$$\lim_{x\to\infty} \frac{\mathbb{P}\{W_i > x\}}{\mathbb{P}\{W > x\}} = c_i \in (0, \infty), \tag{2}$$

where W is an $N(0,1)$ rv, the tail asymptotics of S_2 are determined by those of W.

The organisation of the paper is as follows: Section 2 consists of some preliminary results. In Section 3 we present our main result. Section 4 discusses a log-normal model with stochastic volatility. The proofs of the results are relegated to Section 5.

2. Preliminaries

In this section we briefly discuss some classes of univariate dfs characterised by their tail behaviour. Additionally we present two results on the tail asymptotics of products and sums of independent rvs.

A well-known class of univariate dfs with numerous applications to risk aggregation is that of *subexponential dfs*, see [7, 11]. A rv U and its df F with support on $(0,\infty)$ are called subexponential if

$$\lim_{u\to\infty} \mathbb{P}\left\{U+U^*>u\right\} \sim 2\mathbb{P}\left\{U>u\right\}, \quad u\to\infty,$$

where U^* is an independent copy of U. It is well-known (e.g., [10]) that a subexponential df F is *long-tailed*, i.e., there exists some positive measurable function $a(\cdot)$, such that $\lim_{u\to\infty} a(u) = \lim_{u\to\infty} u/a(u) = \infty$ and further

$$\mathbb{P}\left\{U>u+a(u)\right\} \sim \mathbb{P}\left\{U>u\right\}, \quad u\to\infty.$$

Canonical examples of such dfs are the Pareto and the log-normal. A further interesting example is the Weibull df with tail df $\overline{F}(x) = 1-F(x) = \exp(-x^\beta), \beta\in(0,1)$, for which we can choose

$$a(u) = o(u^{1-\beta}), \quad u\to\infty.$$

An important property of log-normal and Weibull rvs is that their dfs are in the Gumbel max-domain of attraction (MDA) with some positive scaling function $b(\cdot)$ meaning that

$$\lim_{u\to\infty} \frac{\mathbb{P}\left\{U>u+tb(u)\right\}}{\mathbb{P}\left\{U>u\right\}} = \exp(-t)\,, \quad t>0\,. \tag{3}$$

Condition (3) is equivalent to the fact that the rv U has df in the Gumbel MDA; see [7, 21]. We shall abbreviate the limit relation (3) as $U\in GMDA(b)$. Random variables satisfying (3) have rapidly varying tail df $\overline{F}$, i.e., for any $\lambda>1$,

$$\lim_{u\to\infty} \frac{\overline{F}(\lambda u)}{\overline{F}(u)} = 0\,. \tag{4}$$

The next lemma is crucial for the derivation of the asymptotic tail behaviour of the product of two independent non-negative rvs.

Lemma 1. *Let X, Y, Y^* be non-negative rvs with infinite right endpoint and such that X is independent of (Y, Y^*). If the tail df $\overline{F}$ of X satisfies* (4) *for some $\lambda_0 > 1$ and further $\lim_{u\to\infty} \mathbb{P}\{Y > u\}/\mathbb{P}\{Y^* > u\} = c \in (0,\infty)$, then we have that*

$$\mathbb{P}\{XY > u\} \sim c\,\mathbb{P}\{XY^* > u\}, \quad u \to \infty. \tag{5}$$

Remark 1. a) The class of dfs F with infinite right endpoint and such that (4) holds for some $\lambda_0 > 1$ strictly contains the class of dfs with a rapidly varying tail. For $\overline{F}$ rapidly varying, (5) was proved in [23], Lemma A.5.
b) We notice that condition (4) on $X = e^W$ holds if and only if the rv W has an infinite right endpoint and for some $\eta > 0$,

$$\lim_{u\to\infty} \frac{\mathbb{P}\{W > u + \eta\}}{\mathbb{P}\{W > u\}} = 0\,. \tag{6}$$

Lemma 1 implies the following result for convolutions.

Corollary 1. *Let $W_1, \ldots, W_k$ be independent rvs with infinite right endpoints. Suppose that condition* (6) *holds for each W_i with a suitable constant $\eta_i > 0$, $i = 1, \ldots, k$. Further, if the independent rvs $V_1, \ldots, V_k$ satisfy $\lim_{u\to\infty} \mathbb{P}\{W_i > u\}/\mathbb{P}\{V_i > u\} = p_i \in (0,\infty)$, $i \le k$, then for any positive constants $\theta_1, \ldots, \theta_k$ we have*

$$\mathbb{P}\left\{\sum_{i=1}^k \theta_i W_i > u\right\} \sim \left(\prod_{i=1}^k p_i\right) \mathbb{P}\left\{\sum_{i=1}^k \theta_i V_i > u\right\}, \quad u \to \infty. \tag{7}$$

Remark 2. If $W_1, \ldots, W_k$ are independent rvs satisfying $\mathbb{P}\{W_i > u\} \sim p_i u^{\alpha_i} e^{-u^2/2}$ as $u \to \infty$ for some $\alpha_i \in \mathbb{R}, p_i \in (0,\infty), i \le k$, then $W_i \in GMDA(b)$ with $b(u) = 1/u$, see e.g., [7], p. 155. Consequently, (7) holds for any $\theta_1, \ldots, \theta_k$ positive and $V_1, \ldots, V_k$ independent rvs such that each V_i has a density f_i satisfying $f_i(u) \sim u^{\alpha_i+1} e^{-u^2/2}$ as $u \to \infty$. By Theorem 1.1 in [22] we obtain

$$\mathbb{P}\left\{\sum_{i=1}^k \theta_i V_i > u\right\} \sim (\sqrt{2\pi})^{k-1} \sigma^{1-2k-2\alpha} \prod_{i=1}^k \theta_i^{\alpha_i+1} u^{\alpha+k-1} e^{-u^2/(2\sigma^2)} \tag{8}$$

as $u \to \infty$, where $\alpha = \sum_{i=1}^k \alpha_i, \sigma^2 = \sum_{i=1}^k \theta_i^2$. The asymptotic expansion (8) is shown in [19], Lemma 8.6; see also [9], Theorem 2.2.

3. Main Result

Motivated by (1), we introduce a k-dimensional *log-linear model* of positive risks. For this reason, let $W_i, i = 0, \dots, k$, be independent rvs. Writing for non-negative constants $\rho \in [0, 1)$, $W_{i,\rho} = \sqrt{1-\rho} W_i$, we introduce the linearly dependent rvs $X_i = \sqrt{\rho_0} W_0 + W_{i,\rho_i}$, for constants $\rho_0 > 0$ and $\rho_i \in [0, 1)$ and define the log-linear model for positive constants θ_i as follows

$$Z_i = \theta_i e^{X_i}, \quad i = 1, \dots, k. \tag{9}$$

In the credit risk literature, the model (9) with iid standard normal $W_0, \dots, W_k$ is usually referred to as the one-factor (or Vasicek) model and forms the mathematical basis underlying the CreditMetrics^{TM}/KMV approach; see for instance [6], Section 2.5.

In Theorem 1 below, an explicit expansion for the tail of the aggregated risk is derived by assuming subexponentiality of certain factors, which in particular implies that the aggregated risk $S_k = \sum_{i=1}^k Z_i$ is tail equivalent to the maximum risk $M_k = \max_{1 \le i \le k} Z_i$. As in the log-normal case, investigated by Asmussen and Rojas-Nandayapa in [3], the principle of a single big-jump applies in our framework; see [11] for an insightful explanation of this phenomenon. In fact, the asymptotic tail behaviour of the aggregated risk is controlled by the base risk W_0 and the index set

$$J = \{1 \le j \le k : \rho_j = \varrho\}, \qquad \text{where } \varrho = \min_{1 \le i \le k} \rho_i.$$

Theorem 1. *Consider the log-linear model $Z_1, \dots, Z_k$ defined by* (9)*. In addition, we assume the following conditions.*

1. *The rvs W_i satisfy the tail equivalence condition* (2) *for a rv W and positive constants $c_i, i = 1, \dots, k$.*
2. *The rv W satisfies* (6) *for some $\eta > 0$ and $e^{\sqrt{1-\varrho} W}$ is subexponential.*
3. *The rv W_0 satisfies* (6) *for some $\eta = \eta_0 > 0$.*

Then the following relation holds as $u \to \infty$,

$$\begin{aligned} \mathbb{P}\{S_k > u\} &\sim \mathbb{P}\{M_k > u\} \\ &\sim \sum_{i \in J} c_i \, \mathbb{P}\Big\{\sqrt{\rho_0} W_0 + \sqrt{1-\varrho} W^* > \ln(u/\theta_i)\Big\}, \end{aligned}$$

where W_0 and W^* are independent and W^* is an independent copy of W. Moreover, if $\widetilde{\theta} = \max_{i\in J} \theta_i$ and

$$0 < \eta \le (1-\varrho)^{-1/2} \min_{i\in J:\theta_i<\widetilde{\theta}} \ln(\widetilde{\theta}/\theta_i), \tag{10}$$

we have

$$\begin{aligned}\mathbb{P}\{S_k > u\} &\sim \mathbb{P}\{M_k > u\}\\ &\sim \sum_{i\in J:\theta_i=\widetilde{\theta}} c_i\, \mathbb{P}\Big\{\sqrt{\rho_0}W_0 + \sqrt{1-\varrho}W^* > \ln(u/\widetilde{\theta})\Big\}.\end{aligned}$$

In this context, we interpret $\min_{i\in J:\theta_i<\widetilde{\theta}} \ln(\tilde{\theta}/\theta_i) = \infty$ if $\theta_i = \widetilde{\theta}$ for all $i \in J$.

Theorem 1 can be formulated to cover also differences of log-linear risks by allowing some θ_i's to be negative. Under the assumptions of the aforementioned theorem, if $\widetilde{\theta} > 0$, then any i such that $\theta_i < 0$ does not belong to the index set J.

Theorem 1 does in general not follow from the results in [12] since we do not impose conditions on the hazard rate function. In this context, we also mention the recent contribution [13] which investigates the asymptotic tail behaviour of the differences of log-normal risks.

Example 1. If $W \in GMDA(b)$ with scaling function $b(u) = 1/u$, then $e^{\sqrt{1-\varrho}W} \in GMDA(b^*)$ with $b^*(u) = (1-\varrho)u/\ln u$ and $e^{\sqrt{1-\varrho}W}$ is subexponential by virtue of the Goldie-Resnick condition; see e.g., [7] p. 149. In particular, if V is an $N(0,1)$ rv with density φ, then $V \in GMDA(b)$ with $b(u) = 1/u$. Hence for W_0 and W with tail behaviour proportional to that of the standard normal rv V, i.e., for positive ν, ν_0,

$$\mathbb{P}\{W_0 > u\} \sim \nu_0\, \mathbb{P}\{V > u\}, \quad \mathbb{P}\{W > u\} \sim \nu\, \mathbb{P}\{V > u\} \quad u\to\infty, \tag{11}$$

the conditions of Theorem 1 are satisfied. In view of Corollary 1, for an independent copy V^* of V and with $\sigma = \sqrt{1+\rho_0-\varrho}$,

$$\begin{aligned}\mathbb{P}\Big\{\sqrt{\rho_0}W_0 + \sqrt{1-\varrho}W^* > u\Big\} &\sim \nu_0\nu\mathbb{P}\Big\{\sqrt{\rho_0}V + \sqrt{1-\varrho}V^* > u\Big\}\\ &= \nu_0\nu\mathbb{P}\{\sigma V > u\} \sim \nu_0\nu\frac{\sigma}{u}\varphi\Big(\frac{u}{\sigma}\Big), \quad u\to\infty.\end{aligned}$$

Thus we derived the following result under the assumptions of Theorem 1 and the additional conditions (10) and (11):

$$\mathbb{P}\{S_k > u\} \sim \mathbb{P}\{M_k > u\} \sim \nu_0\,\nu \sum_{j\in J} c_j\, \frac{\sigma}{\ln(u/\widetilde{\theta})}\, \varphi\Big(\frac{\ln(u/\widetilde{\theta})}{\sigma}\Big), \quad u\to\infty.$$

We mention in passing that, under the assumptions above, each rv $e^{W_{i,\rho_i}}$ has df in the Gumbel MDA, hence the approach suggested in [18] is applicable.

4. Log-Normal Risks with Stochastic Volatility

Next we discuss the log-normal model with stochastic volatility. We consider the log-linear model of the previous section, where we assume that the rvs W_i are independent conditionally mean-zero normal rvs with stochastic volatility $I_i > 0$, $i = 0, \dots, k$. This means we have the representation

$$W_i = I_i Y_i\,, \quad i = 0, \dots, k\,,$$

where $Y_i, I_i,\ i = 0, \dots, k$, are independent rvs and (Y_i) is an iid $N(0,1)$ sequence. We note that there is a close relationship of our model with normal variance mixture models; see e.g., [4].

In a practical setting, the I_i's can be understood as random deflators. Therefore we assume that the I_i's are supported on $(0, 1]$ with an atom at 1, i.e., for every $i \le k$ there exists $c_i = \mathbb{P}\{I_i = 1\} > 0$. The asymptotic tail behaviour of the W_i's in this model is very close to that of the Y_i's; see Lemma 2 in Section 6.

4.1. Maximum and Aggregated Risk

The log-normal model with stochastic volatility, defined for given constants $\theta_1 \ge \dots \ge \theta_k > 0$ by

$$Z_i = \theta_i \exp(\sqrt{\rho_0} I_0 Y_0 + \sqrt{1-\rho_i} I_i Y_i), \quad i = 1, \dots, k,$$

is of special interest since it allows for the incorporation of random deflation effects. The positive weights θ_i correspond to a deterministic trend $\ln \theta_i$ in the log-linear relationship for Z_i.

In view of Lemma 2 in the Appendix we have

$$\mathbb{P}\{W_i > u\} \sim c_i \frac{\varphi(u)}{u}, \quad u \to \infty\,, \quad i = 0, \dots, k\,,$$

where φ is the density of an $N(0,1)$ rv. Applying Example 1 and assuming for simplicity that $\varrho = \rho_1$ and $\theta_i = 1$ for all $i \in J$, we obtain for fixed $k \ge 1$,

$$\mathbb{P}\{S_k > u\} \sim \mathbb{P}\{M_k > u\} \sim c_0 \sum_{j \in J} c_j \frac{\sigma}{\ln u} \varphi\Big(\frac{\ln u}{\sigma}\Big), \quad u \to \infty, \tag{12}$$

where $\sigma = \sqrt{1+\rho_0-\rho_1}$.

4.2. Asymptotic Behaviour of VAR and CTE

Since the Y_i's have continuous dfs, the Z_i's have continuous dfs as well. Hence by definition the conditional tail expectation (also referred to as Expected Shortfall) of S_k is given by

$$\mathrm{CTE}_q(S_k) = \mathbb{E}\{S_k | S_k > \mathrm{VaR}_q(S_k)\},$$

where $q \in (0,1)$ is a predefined confidence level and $\mathrm{VaR}_q(X)$ is the Value-at-Risk at level q for the risk X, i.e., $\mathrm{VaR}_q(X) = \inf\{s \in \mathbb{R} : \mathbb{P}\{X \le s\} \ge q\}$.

Relation (12) implies

$$\mathbb{E}\{(S_k - u) \mid S_k > u\} \sim b(u) \sim \sigma^2 u / \ln u\,, \quad u \to \infty$$

see e.g., [7]. Since $v_n = \mathrm{VaR}_{1-1/n}(S_k) \to \infty$ as $n \to \infty$, we obtain

$$\begin{aligned} \mathrm{CTE}_{1-1/n}(S_k) &= \mathbb{E}\{(S_k - v_n)|S_k > v_n\} + v_n \\ &= v_n\Big(\frac{\sigma^2}{\ln v_n}(1+o(1)) + 1\Big) \\ &\sim \mathrm{VaR}_{1-1/n}(S_k)\,, \quad n \to \infty\,. \end{aligned} \tag{13}$$

In practice, the level $q = 1 - 1/n$ is fixed with n typically large leading to confidence levels 0.95, 0.99, 0.995, 0.999 and even 0.9997. For instance, the capital charge for credit and operational risk is calculated with a confidence level of $q = 0.999$ and a holding period (horizon) of one year. For the calculation of economic capital, one typically takes $q = 0.9997$. Although CTE_q is more conservative than VaR_q, (13) implies that their asymptotic behaviour (for q close to 1) is similar. The conclusion is that, in terms of the asymptotic behaviour of VaR and CTE this model is similar to the log-normal model.

Remark 3. These properties of VaR and CTE very much link up with the recent discussion around the regulatory document [5]. On p. 41 of the latter document, Question 8 reads as: "What are the likely constraints with moving from Value-at-Risk (VaR) to Expected Shortfall (ES = CTE), including any challenges in delivering robust backtesting and how might these be best overcome?" The "moving from" has to be interpreted as "using ES as an alternative risk measure to VaR for setting capital adequacy standards". An important aspect of this discussion concerns the understanding of portfolio structures where it does not matter much (hence Section 4.2), and more importantly those for which significant differences do exist. For a discussion on the latter, see for instance [20]. For results on risk measure estimation and model uncertainty (mainly at the level of VaR), see [8].

5. Proofs

PROOF OF LEMMA 1 Let F and G denote the dfs of X and Y, respectively. Since X and Y are independent, for $0 < \delta < \gamma < \infty$ given constants and any $u > 0$ we have that

$$\begin{aligned}\mathbb{P}\{XY > u\} &= \int_0^\delta \mathbb{P}\{X > u/y\}\, dG(y) + \int_\delta^\gamma \mathbb{P}\{X > u/y\}\, dG(y)\\ &\quad + \int_\gamma^\infty \mathbb{P}\{X > u/y\}\, dG(y).\end{aligned}$$

Choosing γ such that $\gamma/\delta > \lambda_0$, condition (4) for $\lambda = \lambda_0$ implies as $u \to \infty$ that

$$\frac{\int_0^\delta \mathbb{P}\{X > u/y\}\, dG(y)}{\int_\gamma^\infty \mathbb{P}\{X > u/y\}\, dG(y)} \le \frac{\mathbb{P}\{X > u/\delta\}}{\mathbb{P}\{X > u/\gamma\}\mathbb{P}\{Y > \gamma\}} \to 0.$$

Since we also have

$$\mathbb{P}\{XY > u\} \ge \int_\delta^\infty \mathbb{P}\{X > u/y\}\, dG(y)$$

for any $u > 0$, we conclude that the following relation holds for any fixed $\delta > 0$,

$$\mathbb{P}\{XY > u\} \sim \int_\delta^\infty \mathbb{P}\{X > u/y\}\, dG(y), \quad u \to \infty.$$

Next, applying integration by parts, we have that

$$\begin{aligned}\int_\delta^\infty \mathbb{P}\{X > u/y\}\, dG(y) &= \overline{F}(u/\delta)\mathbb{P}\{Y > \delta\} + \int_0^{u/\delta} \mathbb{P}\{Y > u/y\}\, dF(y)\\ &= (1+o(1))\int_0^{u/\delta} \mathbb{P}\{Y^* > u/y\}\Big(\frac{\mathbb{P}\{Y > u/y\}}{\mathbb{P}\{Y^* > u/y\}} - c\Big)dF(y)\\ &\quad +(1+o(1))c\int_0^{u/\delta} \mathbb{P}\{Y^* > u/y\}\, dF(y).\end{aligned}$$

The first term on the right hand side is of smaller order than the second one because $\lim_{u\to\infty} \mathbb{P}\{Y > u\}/\mathbb{P}\{Y^* > u\} = c$ and one can choose $\delta > 0$ arbitrarily large. Moreover, another integration by parts and the first part of the proof show that the second term has the asymptotic order

$$c\int_0^{u/\delta} \mathbb{P}\left\{Y^* > u/y\right\} dF(y) \sim c\mathbb{P}\left\{XY^* > u\right\},$$

for any fixed $\delta > 0$. This finishes the proof. $\square$

Proof of Corollary 1 In what follows, we assume without loss of generality that all $\theta_i = 1$. We can make this assumption because W_i satisfies (6) for $\eta = \eta_i$ if and only if $\theta_i W_i$ does it for $\eta = \eta_i/\theta_i$. We prove the result by induction on k. For $k = 1$, the result is just a consequence of the condition $\lim_{u\to\infty} \mathbb{P}\left\{W_1 > u\right\}/\mathbb{P}\left\{V_1 > u\right\} = p_1$. Let (V_i') be a copy of (V_i) which is independent of (W_i). For $k = 2$, take $X = e^{W_2}$, $Y = e^{W_1}$ and $Y^* = e^{V_1'}$. In view of (6) for W_2 the assumptions of Lemma 1 are satisfied. Therefore

$$\mathbb{P}\left\{W_1 + W_2 > u\right\} = \mathbb{P}\left\{XY > e^u\right\} \sim p_1\mathbb{P}\left\{XY^* > e^u\right\} = \mathbb{P}\left\{V_1' + W_2 > u\right\}.$$

Next choose $X = e^{V_1'}$, $Y = e^{W_2}$ and $Y^* = e^{V_2'}$ and apply Lemma 1 to obtain (7) for $k = 2$. Notice that we also used the fact that (6) holds for V_1'.

Now assume that (7) holds for $k = n \geq 2$. In view of the proof above we may also assume that we proved

$$\prod_{i=2}^n p_i\mathbb{P}\left\{(V_2' + \cdots + V_n') + W_{n+1} > u\right\} \sim \mathbb{P}\left\{(W_2 + \cdots + W_n) + W_{n+1} > u\right\}. \tag{14}$$

Take $X = e^{W_{n+1}}$, $Y = e^{W_1+\cdots+W_n}$ and $Y^* = e^{V_1'+\cdots+V_n'}$. By the induction hypothesis and (6) for W_{n+1} the assumptions of Lemma 1 are satisfied. Therefore

$$\begin{aligned}
&\mathbb{P}\left\{\big(W_1 + \cdots + W_n\big) + W_{n+1} > u\right\}\\
&\quad = \mathbb{P}\left\{XY > e^u\right\}\\
&\quad \sim \prod_{i=1}^n p_i\mathbb{P}\left\{XY^* > e^u\right\}\\
&\quad = \prod_{i=1}^n p_i\mathbb{P}\left\{\big(V_1' + \cdots + V_n'\big) + W_{n+1} > u\right\}. \qquad (15)
\end{aligned}$$

Now we choose $X = e^{V_1'}$, $Y = e^{(V_2'+\cdots+V_n')+W_{n+1}}$ and $Y^* = e^{(W_2+\cdots+W_n)+W_{n+1}}$ and again apply Lemma 1:

$$\begin{aligned}
&\prod_{i=1}^n p_i\,\mathbb{P}\left\{\big(V_1' + (V_2' + \cdots + V_n'\big) + W_{n+1} > u\right\}\\
&\quad \sim\ p_1\,\mathbb{P}\left\{\big(V_1' + (W_2 + \cdots + W_n\big) + W_{n+1} > u\right\} \sim \mathbb{P}\left\{\sum_{i=1}^{n+1} W_i > u\right\},
\end{aligned}$$

also taking into account (14). Together with (15) this proves (7) for $k = n+1$. $\square$

PROOF OF THEOREM 1 First assume that the index $i \le k$ is such that $\rho_i > \varrho = \min_{1\le j\le k} \rho_j$. Then for sufficiently large u,

$$\begin{aligned}
\mathbb{P}\left\{\theta_i e^{W_{i,\rho_i}} > u\right\} &\le \mathbb{P}\left\{\widetilde{\theta} e^{W_{i,\rho_i}} > u\right\}\\
&= \mathbb{P}\left\{W_i > \ln(u/\widetilde{\theta})(1-\varrho_i)^{-0.5}\right\}\\
&\sim c_i\,\mathbb{P}\left\{W > \ln(u/\widetilde{\theta})(1-\varrho)^{-0.5} + \left((1-\rho_i)^{-0.5} - (1-\varrho)^{0.5}\right)\ln(u/\widetilde{\theta})\right\}\\
&\le c_i\,\mathbb{P}\left\{W > \ln(u/\widetilde{\theta})(1-\varrho)^{-0.5} + \eta\right\}\\
&= o\big(\mathbb{P}\left\{\widetilde{\theta} e^{\sqrt{1-\varrho}W} > u\right\}\big), \quad u \to \infty. \qquad (16)
\end{aligned}$$

In the last step we used (6) for any choice of $\eta > 0$.

Next consider an index $i \in J$. A similar argument as above shows that

$$\begin{aligned}
\mathbb{P}\left\{\theta_i e^{W_{i,\rho_i}} > u\right\} &= \mathbb{P}\left\{W_i > \ln(u/\theta_i)(1-\varrho)^{-0.5}\right\}\\
&\sim c_i\,\mathbb{P}\left\{W > \ln(u/\theta_i)(1-\varrho)^{-0.5}\right\}, \quad u \to \infty,
\end{aligned}$$

and under the additional assumption (10) a similar argument proves (16) if $\theta_i < \widetilde{\theta}$. By subexponentiality of $e^{\sqrt{1-\varrho}W}$, applying Corollary 3.19 in [11], we get that

$$\begin{aligned}
\mathbb{P}\Big\{\sum_{i=1}^k \theta_i e^{W_{i,\rho_i}} > u\Big\} &\sim \sum_{i=1}^k \mathbb{P}\Big\{\theta_i e^{W_{i,\rho_i}} > u\Big\}\\
&\sim \sum_{i\in J} c_i\,\mathbb{P}\Big\{\theta_i e^{\sqrt{1-\varrho}W} > u\Big\}, \quad u \to \infty. \qquad (17)
\end{aligned}$$

Moreover, under the additional condition (10), the right hand side is equivalent to

$$\sum_{i\in J:\theta_i=\widetilde{\theta}} c_i\,\mathbb{P}\Big\{\widetilde{\theta} e^{\sqrt{1-\varrho}W} > u\Big\}.$$

By assumption, W_0 satisfies (6) for some $\eta = \eta_0 > 0$, hence $X = e^{\sqrt{\rho_0}W_0}$ satisfies (4) for some $\lambda_0 > 1$. Writing $Y = \sum_{i=1}^k \theta_i e^{W_{i,\rho_i}}$ and interpreting (17) as the right tail of a rv $Y^* = \Theta e^{\sqrt{1-\varrho}W^*}$, where W^* is a copy of W independent of W_0, and Θ is a rv independent of W^*, we obtain from Lemma 1 that

$$\mathbb{P}\{XY > u\} \sim \mathbb{P}\{XY^* > u\} = \sum_{i\in J} c_i\,\mathbb{P}\left\{\theta_i e^{\sqrt{\rho_0}W_0 + \sqrt{1-\varrho}W^*} > u\right\}, \quad u \to \infty.$$

In view of Bonferroni's inequalities, we also have

$$\mathbb{P}\Big\{\max_{1\le i\le k} \theta_i e^{W_{i,\rho_i}} > u\Big\} \sim \sum_{i=1}^k \mathbb{P}\Big\{\theta_i e^{W_{i,\rho_i}} > u\Big\} \sim \sum_{i\in J} c_i\,\mathbb{P}\Big\{\theta_i e^{\sqrt{1-\varrho}W} > u\Big\}, \quad u \to \infty.$$

Again applying Lemma 1 for $Y = \max_{1\le i\le k} \theta_i e^{W_{i,\rho_i}}$ and $Y^* = \Theta e^{\sqrt{1-\varrho}W^*}$, we obtain

$$\mathbb{P}\left\{ e^{\sqrt{\rho_0}W_0} \max_{1\le i\le k} \theta_i e^{W_{i,\rho_i}} > u \right\} \sim \sum_{i\in J} c_i\, \mathbb{P}\left\{ \theta_i e^{\sqrt{\rho_0}W_0+\sqrt{1-\varrho}W^*} > u \right\}, \quad u\to\infty .$$

This concludes the proof. □

6. Appendix

Below we give the tail asymptotics of deflated risks assuming that the deflator is bounded and the risk has a rapidly varying tail. For the case that the deflator is bounded and has a regularly varying tail at the right endpoint of the df, see e.g., [14].

Lemma 2. *Consider independent rvs I, W such that I is supported on $(0,1]$ and the tail of W is rapidly varying. Then*

$$\lim_{u\to\infty} \frac{\mathbb{P}\{IW > u\}}{\mathbb{P}\{W>u\}} = \mathbb{P}\{I=1\} . \tag{18}$$

Proof of Lemma 2 For any $u>0, z\in(0,1)$,

$$\mathbb{P}\{IW>u\} = \mathbb{P}\{IW>u, I\in(0,z]\} + \mathbb{P}\{IW>u, I\in(z,1)\} + \mathbb{P}\{IW>u, I=1\} .$$

We observe that $\mathbb{P}\{IW>u, I=1\} = \mathbb{P}\{W>u\}\,\mathbb{P}\{I=1\}$,

$$\lim_{z\uparrow 1}\limsup_{u\to\infty} \frac{\mathbb{P}\{IW>u, I\in(z,1)\}}{\mathbb{P}\{W>u\}} \le \lim_{z\uparrow 1} \mathbb{P}\{I\in(z,1)\} = 0 ,$$

and by rapid variation,

$$\limsup_{u\to\infty} \frac{\mathbb{P}\{IW>u, I\in(0,z]\}}{\mathbb{P}\{W>u\}} \le \lim_{u\to\infty} \frac{\mathbb{P}\left\{W>uz^{-1}\right\}}{\mathbb{P}\{W>u\}} = 0 .$$

This proves the lemma. □

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References

[1] Arbenz, P., Embrechts, P. and Puccetti, G. (2011). The AEP algorithm for the fast computation of the distribution of the sum of dependent random variables. *Bernoulli* **17,** 562–591.

[2] Asmussen, S., Blanchet, J., Juneja, S. and Rojas-Nandayapa, L. (2011). Efficient simulation of tail probabilities of sums of correlated lognormals. *Ann. Oper. Res.* **189,** 5–23.

[3] Asmussen, S. and Rojas-Nandayapa, L. (2008). Asymptotics of sums of lognormal random variables with Gaussian copula. *Statist. Probab. Lett.* **78,** 2709–2714.

[4] Barndorff-Nielsen, O., Kent, J. and Sørensen, M. (1982). Normal variance-mean mixtures and z distributions. *Internat. Statist. Rev.* **50,** 145–159.

[5] Basel. *Consultative document May 2102: Fundamental Review of the Trading Book.* Bank for International Settlements, Basel.

[6] Bluhm, C., Overbeck, L. and Wagner, C. (2002). *An Introduction to Credit Risk Modeling.* BocaRaton, Chapman&Hall/CRC.

[7] Embrechts, P., Klüppelberg, C. and Mikosch, T. (1997). *Modelling Extremal Events for Insurance and Finance.* Springer-Verlag, Berlin.

[8] Embrechts, P., Puccetti, G. and Rüschendorf, L. (2013). Model uncertainty and VaR aggregation. *J. Bank. Finance* **37,** 2750–2764.

[9] Farkas, J. and Hashorva, E. (2013). Tail approximation for reinsurance portfolios of Gaussian-like risks. *Scand. Actuar. J.* DOI 10.1080/03461238.2013.825639, in press.

[10] Foss, S., Korshunov, D. and Zachary, S. (2009). Convolutions of long-tailed and subexponential distributions. *J. Appl. Probab.* 756–767.

[11] Foss, S., Korshunov, D. and Zachary, S. (2013). *An Introduction to Heavy-tailed and Subexponential Distributions.* Springer-Verlag, 2nd Edition, New York.

[12] Foss, S. and Richards, A. (2010). On sums of conditionally independent subexponential random variables. *Math. Oper. Res.* **35,** 102–119.

[13] Gulisashvili, A. and Tankov, P. (2013). Tail behavior of sums and differences of log-normal random variables. *arXiv preprint arXiv:1309.3057*.

[14] Hashorva, E. (2012). Exact tail asymptotics in bivariate scale mixture models. *Extremes* **15,** 109–128.

[15] Hashorva, E. (2013). Exact tail asymptotics of aggregated parametrised risk. *J. Math. Anal. Appl.* **400,** 187–199.

[16] Kortschak, D. and Hashorva, E. (2013). Efficient simulation of tail probabilities for sums of log-elliptical risks. *J. Comp. Appl. Math.* **247,** 53–67.

[17] Mainik, G. and Embrechts, P. (2013). Diversification in heavy-tailed portfolios: properties and pitfalls. *Ann. Actuar. Science* **7,** 26–45.

[18] Mitra, A. and Resnick, S. I. (2009). Aggregation of rapidly varying risks and asymptotic independence. *Adv. in Appl. Probab.* **41,** 797–828.

[19] Piterbarg, V. I. (1996). *Asymptotic Methods in the Theory of Gaussian Processes and Fields* vol. 148 of *Translations of Mathematical Monographs.* American Mathematical Society, Providence, RI.

[20] Puccetti, G. and Rüschendorf, L. (2013). Sharp bounds for sums of dependent risks. *J. Appl. Probab.* **50,** 42–53.

[21] Resnick, S. I. (1987). *Extreme Values, Regular Variation, and Point Processes.* Springer-Verlag, New York.

[22] Rootzén, H. (1987). A ratio limit theorem for the tails of weighted sums. *Ann. Probab.* **15,** 728–747.

[23] Tang, Q. and Tsitsiashvili, G. (2004). Finite- and infinite-time ruin probabilities in the presence of stochastic returns on investments. *Adv. in Appl. Probab.* **36,** 1278–1299.