Stable limits for sums of dependent infinite variance random variables

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Abstract   The aim of this paper is to provide conditions which ensure that the affinely
transformed partial sums of a strictly stationary process converge in distribution to
an infinite variance stable distribution. Conditions for this convergence to hold are
known in the literature. However, most of these results are qualitative in the sense
that the parameters of the limit distribution are expressed in terms of some limiting
point process. In this paper we will be able to determine the parameters of the limit-
ing stable distribution in terms of some tail characteristics of the underlying stationary
sequence. We will apply our results to some standard time series models, including the
GARCH(1, 1) process and its squares, the stochastic volatility models and solutions
to stochastic recurrence equations.

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1 Introduction

Whereas there exists a vast amount of papers and books on the limit theory for sums
$S_n = X_1 + \cdots + X_n$ of finite variance strictly stationary sequences (X_t), less attention
has been given to the case of sums of infinite variance stationary sequences. Follow-
ing classical work (for example, Gnedenko and Kolmogorov [27], Feller [26], Petrov
[46]), we know that an iid sequence (X_t) satisfies the limit relation

$$a_n^{-1}(S_n - b_n) \xrightarrow{d} Y_\alpha, \tag{1.1}$$

for suitable constants $a_n > 0, b_n \in \mathbb{R}$ and an infinite variance α-stable random variable
Y_α if and only if the random variable $X = X_1$ has a distribution with regularly varying
tails with index $-\alpha \in (-2, 0)$, i.e., there exist constants $p, q \geq 0$ with $p + q = 1$
and a slowly varying function L such that

$$\mathbb{P}(X > x) \sim p \, \frac{L(x)}{x^\alpha} \quad \text{and} \quad \mathbb{P}(X \leq q - x) \sim q \, \frac{L(x)}{x^\alpha}, \quad x \to \infty. \tag{1.2}$$

This relation is often referred to as *tail balance condition.* It will be convenient to refer
to X and its distribution as *regularly varying with index α.*

The limit relation (1.1) is a benchmark result for weakly dependent stationary
sequences with regularly varying marginal distribution. However, in the presence of
dependence, conditions for the convergence of the partial sums towards a stable limit
are in general difficult to obtain, unless some special structure is assumed. Early on,
α-stable limit theory has been established for the partial sums of linear processes (X_t)
with iid regularly varying noise with index $\alpha \in (0, 2)$. Then the linear process (X_t)
has regularly varying marginals, each partial sum $S_d, d \geq 1$, is regularly varying with
index α and (S_n) satisfies (1.1) for suitable (a_n) and (b_n). These results, the corre-
sponding limit theory for the partial sums S_n and the sample autocovariance function
of linear processes were proved in a series of papers by Davis and Resnick [18–20].
They exploited the relations between regular variation and the weak convergence of
the point processes $N_n = \sum_{t=1}^n \varepsilon_{a_n^{-1} X_t}$, where ε_x denotes Dirac measure at x. Starting
from the convergence $N_n \xrightarrow{d} N$, they used a continuous mapping argument acting on
the points of the processes N_n and N in conjunction with the series representation of
infinite variance stable random variables. Their proofs heavily depend on the linear
dependence structure. A different, not point process oriented, approach was chosen
by Phillips and Solo [47] who decomposed the partial sums of the linear process
into an iid sum part and a negligible remainder term. Then the limit theory for the
partial sums follows from the one for iid sequences with regularly varying marginal

distribution. The first result on stable limits for stationary processes more general than linear models, assuming suitable conditions for non-Gaussian limits, was proved by Davis [11]. Davis's ideas were further developed for mixing sequences by Denker and Jakubowski [22] and Jakubowski and Kobus [34]. The latter paper provides a formula for the stable limit for sums of stationary sequences which are m-dependent and admit local clusters of big values. A paper by Dabrowski and Jakubowski [10] opened yet another direction of studies: stable limits for associated sequences.

Results for special non-linear time series models, exploiting the structure of the model, were proved later on. Davis and Resnick [21] and Basrak et al. [4] studied the sample autocovariances of bilinear processes with heavy-tailed and light-tailed noise, respectively. Mikosch and Straumann [44] proved limit results for sums of stationary martingale differences of the form $X_t = G_t Z_t$, where (Z_t) is an iid sequence with regularly varying Z_t's with index $\alpha \in (0, 2)$, (G_t) is adapted to the filtration generated by $(Z_s)_{s \le t}$ and $\mathbb{E}|G_t|^{\alpha+\delta} < \infty$ for some $\delta > 0$. Stable limit theory for the sample autocovariances of solutions to stochastic recurrence equations, GARCH processes and stochastic volatility models was considered in Davis and Mikosch [13,14], Mikosch and Stărică [43], Basrak et al. [5]; see the survey papers Davis and Mikosch [15–17].

The last mentioned results are again based on the weak convergence of the point processes $N_n = \sum_{t=1}^n \varepsilon_{a_n^{-1} X_t}$ in combination with continuous mapping arguments. The results make heavy use of the fact that any α-stable random variable, $\alpha \in (0, 2)$, has a series representation, involving the points of a Poisson process. A general asymptotic theory for partial sums of strictly stationary processes, exploiting the ideas of point process convergence mentioned above, was given in Davis and Hsing [12]. The conditions in Davis and Hsing [12] are relatively straightforward to verify for various concrete models. However, the α-stable limits are expressed as infinite series of the points of a Poisson process. This fact makes it difficult to identify the parameters of the α-stable distributions: these parameters are functions of the distribution of the limiting point process.

Jakubowski [31,33] followed an alternative approach based on classical blocking and mixing techniques for partial sums of weakly dependent random variables. A basic idea of these papers consists of approximating the distribution of the sum $a_n^{-1} S_n$ by the sum of the iid block sums $(a_n^{-1} S_{mi})_{i=1,\dots,k_n}$ such that $k_n = [n/m] \to \infty$ and $S_{mi} \stackrel{d}{=} S_m$. Then one can use the full power of classical summation theory for row sums of iid triangular arrays. It is also possible to keep under control clustering of big values and calculate the parameters of the α-stable limit in terms of quantities depending on the finite-dimensional distributions of the underlying stationary process. Thus the direct method is in some respects advantageous over the point process approach.

At a first glance, the conditions and results in Jakubowski [31,33] and Davis and Hsing [12] look rather different. Therefore we shortly discuss these conditions in Sect. 2 and argue that they are actually rather close. Our main result (Theorem 1) is given in Sect. 3. Using an argument going back to Jakubowski [31,33], we provide an α-stable limit theorem for the partial sums of weakly dependent infinite variance stationary sequences. The proof only depends on the characteristic functions of the converging partial sums. The result and its proof are new and give insight into the

dependence structure of a heavy-tailed stationary sequence. In Sect. 3.2 we discuss the conditions of Theorem 1 in detail. In particular, we show that our result is easily applicable for strongly mixing sequences. In Sect. 4 we explicitly calculate the parameters of the α-stable limits of the partial sums of the GARCH(1, 1) process and its squares, solutions to stochastic recurrence equations, the stochastic volatility model and symmetric α-stable processes.

2 A discussion of the conditions in α-stable limit theorems

2.1 Regular variation conditions

We explained in Sect. 1 that regular variation of X with index $\alpha \in (0, 2)$ in the sense of (1.2) is necessary and sufficient for the limit relation (1.1) with an α-stable limit Y_α for an iid sequence (X_t). The necessity of regular variation of X with index $\alpha \in (0, 2)$ in the case of dependent X_i's is difficult to establish and, in general, incorrect; see Remark 2. It is, however, natural to assume such a condition as long as one takes the conditions for an iid sequence as a benchmark result.

Davis and Hsing [12] assume the stronger condition that the strictly stationary sequence (X_t) is *regularly varying with index* $\alpha \in (0, 2)$. This means that the finite-dimensional distributions of (X_t) have a jointly regularly varying distribution in the following sense. For every $d \geq 1$, there exists a non-null Radon measure μ_d on the Borel σ-field of $\overline{\mathbb{R}}^d \setminus \{\mathbf{0}\}$ (this means that μ_d is finite on sets bounded away from zero), $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, such that

$$n \, \mathbb{P}\left(a_n^{-1}(X_1, \ldots, X_d) \in \cdot\right) \xrightarrow{v} \mu_d(\cdot), \tag{2.1}$$

where $\xrightarrow{v}$ denotes vague convergence (see Kallenberg [37], Resnick [48]) and (a_n) satisfies

$$n \, \mathbb{P}(|X| > a_n) \sim 1. \tag{2.2}$$

The limiting measure has the property $\mu_d(xA) = x^{-\alpha}\mu_d(A)$, $t > 0$, for Borel sets A. We refer to α as the *index of regular variation* of (X_t) and its finite-dimensional distributions. Note that Theorem 3 in [32] provides conditions under which regular variation of the one-dimensional marginals implies *joint* regular variation (2.1).

Jakubowski [31,33] does not directly assume regular variation of X. However, his condition U1 requires that the normalizing sequence (a_n) in (1.1) is regularly varying with index $1/\alpha$. In [33] he also requires the conditions $\mathbf{T}_+(d)$ and $\mathbf{T}_-(d)$, $d \geq 1$, i.e., the existence of the limits

$$\lim_{n\to\infty} n \, \mathbb{P}(S_d > a_n) = b_+(d) \quad \text{and} \quad \lim_{n\to\infty} n \, \mathbb{P}(S_d \leq -a_n) = b_-(d), \quad d \geq 1. \tag{2.3}$$

If $b_+(d) + b_-(d) > 0$, the regular variation of (a_n) with index $1/\alpha$ is equivalent to regular variation of S_d with index α; see Bingham et al. [7]. Condition U2 in [33] restricts the class of all regularly varying distributions to a subclass. The proof of Theorem 1 below shows that this condition can be avoided.

Remark 1 Condition (2.3) is automatically satisfied for regularly varying (X_t), where

$$b_\pm(d) = \mu_d(\{\mathbf{x} \in \overline{\mathbb{R}}^d : \pm(x_1 + \cdots + x_d) > 1\}). \qquad (2.4)$$

Since μ_d is non-null for every $d \geq 1$ and $\mu_d(tA) = t^{-\alpha}\mu(A)$, $t > 0$, we have $b_+(d) + b_-(d) > 0$, $d \geq 1$. Since (a_n) is regularly varying with index $1/\alpha$ it then follows that S_d is regularly varying with index α for every $d \geq 1$. Since (a_n) satisfies relation (2.2) it then follows that $b_+(1) = p$ and $b_-(1) = q$ with p and q defined in Eq. (1.2). In particular $p + q = 1$. The coefficients $b_+(d)$ and $b_-(d)$ for $d > 1$ can be considered as a measure of extremal dependence in the sequence (X_t). The two benchmarks are the iid case, $b_+(d) = p\,d$ and $b_-(d) = q\,d$ and the case $X_i = X$ for all i, $b_+(d) = p\,d^\alpha$ and $b_-(d) = q\,d^\alpha$.

Regular variation of a stationary sequence (X_t) is a well accepted concept in applied probability theory. One of the reasons for this fact is that some of the important time series models (ARMA with regularly varying noise, GARCH, solutions to stochastic recurrence equations, stochastic volatility models with regularly varying noise) have this property. Basrak and Segers [6] give some enlightening results about the structure of regularly varying sequences. In what follows, we will always assume:

Condition (**RV**): The strictly stationary sequence (X_t) is regularly varying with index $\alpha \in (0, 2)$ in the sense of condition (2.1) with non-null Radon measures μ_d, $d \geq 1$, and (a_n) chosen in (2.2).

2.2 Mixing conditions

Assuming condition (**RV**), Davis and Hsing [12] require the mixing condition $\mathcal{A}(a_n)$ defined in the following way. Consider the point process $N_n = \sum_{t=1}^n \varepsilon_{X_t/a_n}$ and assume that there exists a sequence $m = m_n \to \infty$ such that $k_n = [n/m_n] \to \infty$, where $[x]$ denotes the integer part of x. The condition $\mathcal{A}(a_n)$ requires that

$$\mathbb{E}e^{-\int f\,dN_n} - \left(\mathbb{E}e^{-\int f\,dN_m}\right)^{k_n} \to 0, \qquad (2.5)$$

where f belongs to a sufficiently rich class of non-negative measurable functions on $\mathbb{R}$ such that the convergence of the Laplace functional $\mathbb{E}e^{-\int f\,dN_n}$ for all f from this class ensures weak convergence of (N_n). Relation (2.5) ensures that N_n can be approximated in law by a sum of k_n iid copies of N_m, hence the weak limits of (N_n) must be infinitely divisible point processes.

The condition $\mathcal{A}(a_n)$ is difficult to be checked directly, but it follows from standard mixing conditions such as strong mixing with a suitable rate. For future use, recall

that the stationary sequence (X_t) is *strongly mixing with rate function* (α_h) if

$$\sup_{A \in \sigma(\dots, X_{-1}, X_0), B \in \sigma(X_h, X_{h+1}, \dots)} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| = \alpha_h \to 0, \quad h \to \infty.$$

Jakubowski [31] showed that (1.1) with $b_n = 0$ and regularly varying (a_n) implies the condition

$$\max_{1 \le k, l \le n, k+l \le n} \left| \mathbb{E}\mathrm{e}^{ixa_n^{-1}S_{k+l}} - \mathbb{E}\mathrm{e}^{ixa_n^{-1}S_k}\mathbb{E}\mathrm{e}^{ixa_n^{-1}S_l} \right| \to 0, \quad n \to \infty, \quad x \in \mathbb{R} \tag{2.6}$$

which is satisfied for strongly mixing (X_t). We also refer to the discussion in Sections 4–6 of [33] for alternative ways of verifying (2.6). Under assumptions on the distribution of X more restrictive than regular variation it is shown that (1.1) implies the existence of a sequence $l_n \to \infty$ such that for any $k_n = o(l_n)$ the following relation holds

$$\left(\mathbb{E}\mathrm{e}^{i x k_n^{-1/\alpha}(a_n^{-1}S_n)} \right)^{k_n} - \mathbb{E}\mathrm{e}^{i x a_n^{-1}S_n} \to 0, \quad x \in \mathbb{R}. \tag{2.7}$$

It is similar to condition (2.5) at the level of partial sums.

We will assume a similar mixing condition in terms of the characteristic functions of the partial sums of (X_t). Write

$$\varphi_{nj}(x) = \mathbb{E}\mathrm{e}^{ixa_n^{-1}S_j}, \quad j = 1, 2, \dots, \quad \varphi_n = \varphi_{nn}, \quad x \in \mathbb{R}.$$

*Condition (**MX**).* Assume that there exist $m = m_n \to \infty$ such that $k_n = [n/m] \to 0$ and

$$\left| \varphi_n(x) - (\varphi_{nm}(x))^{k_n} \right| \to 0, \quad n \to \infty, \quad x \in \mathbb{R}. \tag{2.8}$$

This condition is satisfied for a strongly mixing sequence provided the rate function (α_h) decays sufficiently fast; see Sect. 3.2.4. But (2.8) is satisfied for classes of stationary processes much wider than strongly mixing ones. Condition (**MX**) is analogous to $\mathcal{A}(a_n)$. The latter condition is formulated in terms of the Laplace functionals of the underlying point processes. It is motivated by applications in extreme value theory, where the weak convergence of the point processes is crucial for proving limit results of the maxima and order statistics of the samples $X_1, \dots, X_n$. Condition (**MX**) implies that the partial sum processes $(a_n^{-1}S_n)$ and $(a_n^{-1}\sum_{i=1}^{k_n} S_{mi})$ have the same weak limits, where $S_{mi}, i = 1, \dots, k_n$, are iid copies of S_m. This observation opens the door to classical limit theory for partial sums based on triangular arrays of independent random variables. Since we are dealing with the limit theory for the partial sum process $(a_n^{-1}S_n)$ condition (**MX**) is more natural than $\mathcal{A}(a_n)$ which is only indirectly (via a non-trivial continuous mapping argument acting on converging point processes) responsible for the convergence of the normalized partial sum process $(a_n^{-1}S_n)$.

2.3 Anti-clustering conditions

Assuming condition (**RV**), Davis and Hsing [12] require the anti-clustering condition

$$\lim_{d\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\max_{d\le|i|\le m_n} |X_i| > x\,a_n \mid |X_0| > x\,a_n\right) = 0, \quad x > 0,$$

(2.9)

where, as before, $m = m_n \to \infty$ is the block size used in the definition of the mixing condition $\mathcal{A}(a_n)$. It follows from recent work by Basrak and Segers [6] that the index set $\{i : d \le |i| \le m_n\}$ can be replaced by $\{i : d \le i \le m_n\}$, reducing the efforts for verifying (2.9). With this modification, a sufficient condition for (2.9) is then given by

$$\lim_{d\to\infty} \limsup_{n\to\infty} n \sum_{i=d}^{m_n} \mathbb{P}\left(|X_i| > x\,a_n, |X_0| > x\,a_n\right) = 0, \quad x > 0.$$

(2.10)

Relation (2.10) is close to the anti-clustering condition $D'(x\,a_n)$ used in extreme value theory; see Leadbetter et al. [40], Leadbetter and Rootzén [39] and Embrechts et al. [24, Chapter 5].

An alternative anti-clustering condition is (38) in Jakubowski [33]:

$$\lim_{d\to\infty} \limsup_{x\to\infty} \limsup_{n\to\infty} x^\alpha \sum_{h=d}^{n-1} (n-h)\,\mathbb{P}(|X_0| > x\,a_n, |X_h| > x\,a_n) = 0.$$

(2.11)

Assuming regular variation of X and defining (a_n) as in (2.2), we see that (2.11) is implied by the condition

$$\lim_{d\to\infty} \limsup_{x\to\infty} \limsup_{n\to\infty} n \sum_{h=d}^{n-1} \mathbb{P}(|X_h| > x\,a_n, |X_0| > x\,a_n) = 0,$$

which is close to condition (2.10).

For our results we will need an anti-clustering condition as well. It is hidden in assumption (**AC**) in Theorem 1; see the discussion in Sect. 3.2.3.

2.4 Vanishing small values conditions

Davis, Hsing, and Jakubowski prove convergence of the normalized partial sums by showing that the limiting distribution is infinitely divisible with a Lévy triplet corresponding to an α-stable distribution. In particular, they need conditions to ensure that the sum of the small values (summands) in the sum $a_n^{-1} S_n$ does not contribute to the limit. Such a condition for a dependent sequence (X_t) is often easily established for $\alpha \in (0, 1)$, whereas the case $\alpha \in [1, 2)$ requires some extra work.

Davis and Hsing [12] assume the condition (3.2):

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \mathbb{P}\left(\left|\sum_{t=1}^{n} X_t I_{\{|X_t| \leq \epsilon a_n\}} - n\,\mathbb{E}XI_{\{|X| \leq \epsilon a_n\}}\right| > x\,a_n\right) = 0, \quad x > 0,$$

(2.12)

for $\alpha \in (0, 2)$. For an iid sequence (X_t) the relation

$$\text{median}(a_n^{-1} S_n) - a_n^{-1} n \mathbb{E}[XI_{\{|X| \leq \epsilon a_n\}}] \to 0, \quad \epsilon > 0,$$

holds. Therefore $a_n^{-1} n \mathbb{E}XI_{\{|X| \leq \epsilon a_n\}}$ are the natural centering constants for $a_n^{-1} S_n$ in stable limit theory. In the case of dependent (X_t), the choice of the latter centering constants is less straightforward; it is dictated by truncation of the points in the underlying weakly converging point processes.

The analogous condition (35) in Jakubowski [33] reads as follows: for each $x > 0$

$$\lim_{\epsilon \downarrow 0} \limsup_{l \to \infty} \limsup_{n \to \infty} l^{\alpha}\, \mathbb{P}\left(\left|\sum_{t=1}^{n}[X_t\, I_{\{|X_t| \leq \epsilon\, l\, a_n\}} - \mathbb{E}XI_{\{|X| \leq \epsilon\, l\, a_n\}}]\right| > x\, l\, a_n\right) = 0.$$

(2.13)

As shown in [33], this condition is automatically satisfied for $\alpha \in (0, 1)$.

In our approach the anti-clustering condition **(AC)** (see below) is imposed on the sum of "small" and "moderate" values and we do not need to verify conditions such as (2.12) and (2.13).

3 Main result

In this section we formulate and prove our main result. Recall the regular variation condition (**RV**) and the mixing condition (**MX**) from Sects. 2.1 and 2.2. We will use the following notation for any random variable Y:

$$\overline{Y} = (Y \wedge 2) \vee (-2).$$

Notice that $|\overline{Y}| = |Y| \wedge 2$ is subadditive.

Theorem 1 *Assume that (X_t) is a strictly stationary process satisfying the following conditions.*

1. *The regular variation condition (**RV**) holds for some $\alpha \in (0, 2)$.*
2. *The mixing condition (**MX**) holds.*
3. *The anti-clustering condition*

$$\lim_{d \to \infty} \limsup_{n \to \infty} \frac{n}{m} \sum_{j=d+1}^{m} \mathbb{E}\left|\overline{x\, a_n^{-1}(S_j - S_d)}\, \overline{x\, a_n^{-1} X_1}\right| = 0, \quad x \in \mathbb{R}, \quad \textbf{(AC)}$$

*holds, where $m = m_n$ is the same as in (**MX**).*

4. *The limits*

$$\lim_{d \to \infty} (b_+(d) - b_+(d-1)) = c_+ \text{ and } \lim_{d \to \infty} (b_-(d) - b_-(d-1)) = c_- ,$$

(**TB**)

exist. Here $b_+(d)$, $b_-(d)$ *are the tail balance parameters given in* (2.3).

5. *For* $\alpha > 1$ *assume* $\mathbb{E}(X_1) = 0$ *and for* $\alpha = 1$,

$$\lim_{d \to \infty} \limsup_{n \to \infty} n \, |\mathbb{E}(\sin(a_n^{-1} S_d))| = 0.$$

(**CT**)

Then c_+ *and* c_- *are non-negative and* $(a_n^{-1} S_n)$ *converges in distribution to an* α-*stable random variable* (*possibly zero*) *with characteristic function* $\psi_\alpha(x) = \exp(-|x|^\alpha \chi_\alpha(x, c_+, c_-))$, *where for* $\alpha \neq 1$ *the function* $\chi_\alpha(x, c_+, c_-)$, $x \in \mathbb{R}$, *is given by the formula*

$$\frac{\Gamma(2-\alpha)}{1-\alpha} \left((c_+ + c_-) \cos(\pi\alpha/2) - i \, \text{sign}(x)(c_+ - c_-) \sin(\pi \alpha/2) \right),$$

while for $\alpha = 1$ *one has*

$$\chi_1(x, c_+, c_-) = 0.5 \, \pi (c_+ + c_-) + i \, \text{sign}(x) \, (c_+ - c_-) \log |x|, \quad x \in \mathbb{R}.$$

We discuss the conditions of Theorem 1 in Sect. 3.2. In particular, we compare them with the conditions in Jakubowski [31,33] and Davis and Hsing [12]. If the sequence (X_n) is m_0-dependent for some integer $m_0 \geq 1$, i.e., the σ-fields $\sigma(\dots, X_{-1}, X_0)$ and $\sigma(X_{m_0+1}, X_{m_0+2}, \dots)$ are independent, the conditions (**MX**), (**AC**) and (**TB**) of Theorem 1 are automatic; see Sect. 4.1. The surprising fact that c_+ and c_- are non-negative is explained at the end of Sect. 3.2.2.

Remark 2 Although Theorem 1 covers a wide range of strictly stationary sequences (see in particular Sect. 4) condition (**RV**) limits the applications to infinite variance (in particular unbounded) random variables X_n. The referee of this paper pointed out the surprising fact that there exist strictly stationary Markov chains (Y_n), suitable bounded functions f and a sequence (a_n) with $a_n = n^{1/\alpha} \ell(n)$ for some slowly varying function ℓ such that the sequence of the normalized partial sums $(a_n^{-1} S_n)$ of the sequence $(X_n) = (f(Y_n))$ converges in distribution to an infinite variance stable random variable. Then (**RV**) is obviously violated. Such an example is contained in Gouëzel [29, Theorem 1.3].

Other examples of stable limits for sums of bounded stationary random variables (of different nature—non-Markov and involving long-range dependence) are given in [54, Theorems 2.1 (ii) and 2.2 (ii)].

Remark 3 It might be instructive to realize that in limit theorems for weakly dependent sequences properties of finite dimensional distributions can be as bad as possible. For example it is very easy to build a 1-dependent sequence having no moment of any

order and such that its (centered and normalized) partial sums still converge to a stable law of order $\alpha \in (0, 2]$. Let

$$X_n = Y_n + \varepsilon_n - \varepsilon_{n-1},$$

where (Y_n) is an iid sequence of α-stable random variables and (ε_n) is an iid sequence without any moments (that is $E(|\varepsilon_0|^a) = +\infty$ for any positive a) and the two sequences are independent. Then the (centered and normalized) partial sums of (X_n)) have the same limit behavior (in distribution) as that of (Y_n).

3.1 Proof of Theorem 1

For any strictly stationary sequence $(X_t)_{t\in\mathbb{Z}}$ it will be convenient to write

$$S_0 = 0, \quad S_n = X_1 + \cdots + X_n, \quad S_{-n} = X_{-n} + \cdots + X_{-1}, \quad n \geq 1.$$

Let S_{mi}, $i = 1, 2, \ldots$, be iid copies of S_m. In view of (2.8) the theorem is proved if we can show that $(a_n^{-1} \sum_{i=1}^{k_n} S_{mi})$ has an α-stable limit with characteristic function ψ_α. For such a triangular array, it is implied by the relation

$$k_n(\varphi_{nm}(x) - 1) \to \log \psi_\alpha(x), \quad x \in \mathbb{R}. \tag{3.1}$$

Indeed, notice first that the triangular array $(a_n^{-1} S_{mi})_{i=1,\ldots,k_n}$ of iid random variables satisfies the infinite smallness condition. Then apply Lemma 3.5 in Petrov [46] saying that for all $x \in \mathbb{R}$ and sufficiently large n,

$$\log \varphi_{nm}(x) = \varphi_{nm}(x) - 1 + \theta_{nm}(\varphi_{nm}(x) - 1)^2,$$

where $|\theta_{nm}| \leq 1$. Thus

$$|k_n(\varphi_{nm}(x) - 1) - k_n \log \varphi_{nm}(x)| \leq k_n |\varphi_{nm}(x) - 1|^2 \leq c\, k_n^{-1} \to 0.$$

Here and in what follows, c denotes any positive constants. Our next goal is to find a suitable approximation to the left-hand side in (3.1).

Lemma 1 *Under* **(RV)** *and* **(AC)** *the following relation holds*:

$$\lim_{d\to\infty} \limsup_{n\to\infty} \left|k_n(\varphi_{nm}(x) - 1) - n\,(\varphi_{nd}(x) - \varphi_{n,d-1}(x))\right| = 0, \quad x \in \mathbb{R}. \tag{3.2}$$

Moreover, if (X_n) is m_0-dependent for some integer $m_0 \geq 1$, then

$$\lim_{n\to\infty} \left|k_n(\varphi_{nm}(x) - 1) - n\,(\varphi_{nd}(x) - \varphi_{n,d-1}(x))\right| = 0, \quad x \in \mathbb{R}, \quad d > m_0. \tag{3.3}$$

The proof is given at the end of this section. By virtue of (**RV**) and (2.3), S_d is regularly varying with index $\alpha \in (0, 2)$; see Remark 1. Therefore it belongs to the domain of attraction of an α-stable law. Theorem 3 in Section XVII.5 of Feller [26] yields that for every $d \geq 1$ there exists an α-stable random variable $Z_\alpha(d)$ such that

$$a_n^{-1} \sum_{i=1}^{n} (S_{di} - e_{nd}) \xrightarrow{d} Z_\alpha(d) \quad \text{where } e_{nd} = \begin{cases} 0 & \alpha \neq 1, \\ \mathbb{E}(\sin(S_d/a_n)) & \alpha = 1. \end{cases} \quad (3.4)$$

The limiting variable $Z_\alpha(d)$ has the characteristic function

$$\widetilde{\psi}_{\alpha,d}(x) = \exp(-|x|^\alpha \chi_\alpha(x, b_+(d), b_-(d))), \quad x \in \mathbb{R}.$$

Applying Theorem 1 in Section XVII.5 of Feller [26], we find the equivalent relation

$$n \, (\varphi_{nd}(x) \mathrm{e}^{-i e_{nd} x} - 1) \to \log \widetilde{\psi}_{\alpha,d}(x), \quad x \in \mathbb{R},$$

and exploiting condition (**TB**), for $x \in \mathbb{R}$,

$$\begin{aligned} n \, (\varphi_{nd}(x) \mathrm{e}^{-i e_{nd} x} &- \varphi_{n,d-1}(x) \mathrm{e}^{-i e_{n,d-1} x}) \\ &\to \log \widetilde{\psi}_{\alpha,d}(x) - \log \widetilde{\psi}_{\alpha,d-1}(x) \quad \text{as } n \to \infty \\ &\to \log \psi_\alpha(x) \quad \text{as } d \to \infty. \end{aligned} \quad (3.5)$$

For $\alpha \neq 1$ we have $e_{nd} = 0$. Therefore (3.2) implies

$$k_n \, (\varphi_{nm}(x) - 1) \to \log \psi_\alpha(x), \quad x \in \mathbb{R}.$$

This finishes the proof in this case. For $\alpha = 1$ we use the same arguments but we have to take into account that e_{nd} does not necessarily vanish. However, we have

$$|\varphi_{nd}(x) - \varphi_{nd}(x) \mathrm{e}^{-i e_{nd} x}| \leq |1 - \mathrm{e}^{-i e_{nd} x}| \leq t \, |e_{nd}|,$$

and using (**CT**), we obtain

$$\lim_{d \to \infty} \limsup_{n \to \infty} n \, \left| (\varphi_{nd}(x) - \varphi_{n,d-1}(x)) - (\varphi_{nd}(x) \mathrm{e}^{-i e_{nd} x} - \varphi_{n,d-1}(x) \mathrm{e}^{-i e_{n,d-1} x}) \right| = 0.$$

This proves the theorem.   □

Proof of Lemma 1 Consider the following telescoping sum for any n, $m \leq n$ and $d < m$:

$$\varphi_{nm}(x) - 1 = \varphi_{nd}(x) - 1 + \sum_{j=1}^{m-d} (\varphi_{n,d+j}(x) - \varphi_{n,d-1+j}(x)).$$

By stationarity of (X_t) we also have

$$
\begin{aligned}
&m\left(\varphi_{nd}(x) - \varphi_{n,d-1}(x)\right) \\
&= d\left(\varphi_{nd}(x) - \varphi_{n,d-1}(x)\right) + \sum_{j=1}^{m-d}\left[\mathbb{E}e^{ixa_n^{-1}(S_{-d-j}-S_{-j})} - \mathbb{E}e^{ixa_n^{-1}(S_{-d-j+1}-S_{-j})}\right].
\end{aligned}
$$

Taking the difference between the previous two identities, we obtain

$$
\begin{aligned}
&(\varphi_{nm}(x) - 1) - m\left(\varphi_{nd}(x) - \varphi_{n,d-1}(x)\right) \\
&\quad = -(d-1)\left(\varphi_{n,d}(x) - 1\right) + d\left(\varphi_{n,d-1}(x) - 1\right) \\
&\qquad + \sum_{j=1}^{m-d}\left[\varphi_{n,d+j}(x) - \mathbb{E}e^{ixa_n^{-1}(S_{-d-j}-S_{-j})} - \varphi_{n,d-1+j}(x) + \mathbb{E}e^{ixa_n^{-1}(S_{-d-j+1}-S_{-j})}\right].
\end{aligned}
$$

By stationarity, for any $k \geq 1$, $\varphi_{nk}(x) = \mathbb{E}e^{ixa_n^{-1}S_{-k}}$. Therefore any summand in the latter sum can be written in the following form

$$
\begin{aligned}
&\mathbb{E}\left(e^{ixa_n^{-1}S_{-d-j}} - e^{ixa_n^{-1}(S_{-d-j}-S_{-j})} - e^{ixa_n^{-1}S_{-d-j+1}} + e^{ixa_n^{-1}(S_{-d-j+1}-S_{-j})}\right) \\
&\quad = \mathbb{E}\left(e^{ixa_n^{-1}S_{-d-j}}\left(1 - e^{-ixa_n^{-1}S_{-j}}\right)\left(1 - e^{-ixa_n^{-1}X_{-d-j}}\right)\right).
\end{aligned}
$$

Using the fact that $x \to \exp(ix)$ is a 1-Lipschitz function bounded by 1, the absolute value of the expression on the right-hand side is bounded by

$$
\mathbb{E}\left((|xa_n^{-1}S_{-j}| \wedge 2)(|xa_n^{-1}X_{-d-j}| \wedge 2)\right).
$$

Collecting the above identities and bounds, we finally arrive at the inequality

$$
\begin{aligned}
&|k_n\left(\varphi_{nm}(x) - 1\right) - n\left(\varphi_{nd}(x) - \varphi_{n,d-1}(x)\right)| \\
&\quad \leq k_n(d-1)|\varphi_{nd}(x) - 1| + k_n d|\varphi_{n,d-1}(x) - 1| \\
&\qquad + k_n\sum_{j=d+1}^{m}\mathbb{E}\left|\overline{xa_n^{-1}(S_j - S_d)}\,\overline{xa_n^{-1}X_1}\right|.
\end{aligned}
$$

The last term on the right-hand side converges to zero in view of assumption (**AC**) when first $n \to \infty$ and then $d \to \infty$. In the m_0-dependent case, the last term on the right-hand side converges to zero whenever $d > m_0$ and $n \to \infty$. To prove that the first two terms also converge to zero, let us notice that, under (**RV**), $n(\varphi_{nd}(x) - 1) \to \chi_\alpha(x, b_+(d), b_-(d))$ and so

$$
\lim_{n\to\infty} k_n(d-1)(\varphi_{nd}(x) - 1) = \lim_{n\to\infty} m_n^{-1}(d-1)n(\varphi_{nd}(x) - 1) = 0.
$$

This proves the lemma. $\qquad\qquad\square$

Remark 4 Balan and Louhichi [2] have taken a similar approach to prove limit theorems for triangular arrays of stationary sequences with infinitely divisible limits. Their paper combines ideas from Jakubowski [33], in particular condition (**TB**), and the point process approach in Davis and Hsing [12]. They work under a mixing condition close to $\mathcal{A}(a_n)$. One of their key results (Theorem 2.6) is the analog of Lemma 1 above. It is formulated in terms of the Laplace functionals of point processes instead of the characteristic functions of the partial sums. Then they sum the points in the converging point processes and in the limiting point process to get an infinitely divisible limit. The sum of the points of the limiting process represent an infinitely divisible random variable by virtue of the Lévy-Itô representation. As in Davis and Hsing [12] the method of proof is indirect, i.e., one does not directly deal with the partial sums, and therefore the results are less explicit.

3.2 A discussion of the conditions of Theorem 1

3.2.1 Condition 5

It is a natural centering condition for the normalized partial sums in the cases $\alpha = 1$ and $\alpha \in (1, 2)$. In the latter case, $\mathbb{E}|X| < \infty$, and therefore $\mathbb{E}X = 0$ can be assumed without loss of generality. As usual in stable limit theory, the case $\alpha = 1$ is special and therefore we need condition (**CT**). It is satisfied if S_d is symmetric for every d.

*3.2.2 Condition (**TB**)*

If $c_+ + c_- = 0$ the limiting stable random variable is zero. For example, assume $X_n = Y_n - Y_{n-1}$ for an iid regularly varying sequence (Y_n) with index $\alpha \in (0, 2)$. Then $S_d = Y_d - Y_0$ is symmetric and regularly varying with index α. By the definition of (a_n), $b_+(d) = b_-(d) = 0.5$, hence $c_+ = c_- = 0$. Of course, $a_n^{-1} S_n \xrightarrow{P} 0$.

In the context of Theorem 3.1 in Jakubowski [33] (although the conditions of that result are more restrictive as regards the tail of X) it is shown that (**TB**) is *necessary* for convergence of $(a_n^{-1} S_n)$ towards a stable limit. Condition (**TB**) can be verified for various standard time series models; see Sect. 4. The meaning of this condition is manifested in Lemma 1. It provides the link between the regular variation of the random variables S_d for every $d \geq 1$ (this is a property of the finite-dimensional distributions of the partial sum process (S_d)) and the Lévy measure ν_α of the α-stable limit. Indeed, notice that (**TB**) implies that, for every $x > 0$, with $b_+(0) = 0$,

$$
\begin{aligned}
c_+ x^{-\alpha} &= \lim_{d \to \infty} \frac{b_+(d)}{d} x^{-\alpha} \\
&= \lim_{d \to \infty} \frac{1}{d} \sum_{i=1}^{d} (b_+(i) - b_+(i-1)) x^{-\alpha} \\
&= \lim_{d \to \infty} \frac{1}{d} \sum_{i=1}^{d} \lim_{n \to \infty} n \left(\mathbb{P}(S_i > x \, a_n) - \mathbb{P}(S_{i-1} > x \, a_n) \right),
\end{aligned}
$$

and a similar relation applies to $c_- x^{-\alpha}$. Then

$$v_\alpha(x, \infty) = c_+ x^{-\alpha} \quad \text{and} \quad v_\alpha(-\infty, -x) = c_- x^{-\alpha}, \quad x > 0,$$

determine the Lévy measure v_α of the α-stable limit distribution with the characteristic function ψ_α given in Theorem 1. In particular, Lemma 1 implies that as $n \to \infty$

$$
\begin{aligned}
k_n \, \mathbb{P}(S_m > x \, a_n) &\to v_\alpha(x, \infty), \\
k_n \, \mathbb{P}(S_m \le -x \, a_n) &\to v_\alpha(-\infty, -x],
\end{aligned}
\quad x > 0. \tag{3.6}
$$

The latter relation opens the door to the limit theory for partial sums of triangular arrays of iid copies $(S_{mi})_{i=1,\dots,k_n}$, of S_m. Notice that the relations (3.6) are of large deviations type in the sense of [33]. We refer to [35] for their multi-dimensional counterparts.

Let us notice that although one cannot ensure that $b_\pm(d) \ge b_\pm(d-1) \ge 0$ for sufficiently large d, the constants c_+, c_- are non-negative. It is immediate from the observation that

$$
\begin{aligned}
c_\pm &= \lim_{d \to \infty} (b_\pm(d) - b_\pm(d-1)) \\
&= \lim_{d \to \infty} \frac{1}{d} \sum_{i=1}^d (b_\pm(i) - b_\pm(i-1)) = \lim_{d \to \infty} \frac{b_\pm(d)}{d} \ge 0.
\end{aligned}
$$

Remark 5 Recall the two benchmark examples of Remark 1. If (X_t) is an iid sequence regularly varying with index $\alpha > 0$ the limits $c_+ = p$ and $c_- = q$ always exist and conditions (**MX**), (**AC**) are automatically satisfied. Then, under (**CT**), we recover the classical limit results for partial sums with α-stable limit. On the other hand, if $X_i = X$ for all i, then $c_+ = c_- = 0$ if $0 < \alpha < 1$, $c_+ = p$ and $c_- = q$ if $\alpha = 1$ and c_+ and c_- are not defined otherwise. This observation is in agreement with the fact that $a_n^{-1} S_n = n^{1-1/\alpha} \ell(n) X$ for some slowly varying function ℓ.

3.2.3 Sufficient conditions for (**AC**)

Condition (**AC**) is close to the anti-clustering conditions in [12,33] discussed in Sect. 2.3. In what follows, we give some sufficient conditions for (**AC**). These conditions are often simple to verify.

Lemma 2 *Assume the conditions of Theorem 1 and that (X_t) is strongly mixing with rate function (α_h). Moreover, assume that there exists a sequence $r_n \to \infty$ such that $r_n/m_n \to 0$, $n\alpha_{r_n} \to 0$ and one of the following three conditions is satisfied.*

$$
\begin{aligned}
\lim_{d \to \infty} \limsup_{n \to \infty} n &\left[\sum_{i=d+1}^{r_n} \mathbb{P}(|X_i| > a_n, |X_1| > a_n) \right. \\
&\left. + \mathbb{P}\left(\left| \sum_{i=d+1}^{r_n} X_i I_{\{|X_i| \le a_n\}} \right| > a_n, |X_1| > a_n \right) \right] = 0. \tag{3.7}
\end{aligned}
$$

or

$$\lim_{d \to \infty} \limsup_{n \to \infty} n \, \mathbb{P} \left(\max_{i=d+1,\dots,r_n} |X_i| > a_n/r_n, |X_1| > a_n \right) = 0. \tag{3.8}$$

or

$$\lim_{d \to \infty} \limsup_{n \to \infty} n \, \mathbb{P} \left(|S_{r_n} - S_d| > a_n, |X_1| > a_n \right) = 0. \tag{3.9}$$

Then (**AC**) *holds.*

Proof Let us recall that the function $y \mapsto |\bar{y}|$ is subadditive. We decompose the sum in (**AC**) as follows.

$$k_n \left(\sum_{j=d+1}^{r_n} + \sum_{j=r_n+1}^{m} \right) \mathbb{E} \left| \overline{x \, a_n^{-1}(S_j - S_d)} \, \overline{x \, a_n^{-1} X_1} \right| = J_1(n) + J_2(n).$$

We will deal with the two terms $J_1(n)$ and $J_2(n)$ in different ways. For the sake of simplicity we assume $x = 1$.

We start by bounding $J_2(n)$.

$$J_2(n) \le k_n \sum_{j=r_n+1}^{m} \left[\mathbb{E} \left| \overline{a_n^{-1}(S_j - S_{r_n})} \, \overline{a_n^{-1} X_1} \right| + \left| \overline{a_n^{-1}(S_{r_n} - S_d)} \right| \right]$$

$$= k_n \sum_{j=r_n+1}^{m} \mathbb{E} \left| \overline{a_n^{-1}(S_j - S_{r_n})} \, \overline{a_n^{-1} X_1} \right| + n \, \mathbb{E} \left| \overline{a_n^{-1}(S_{r_n} - S_d)} \, \overline{a_n^{-1} X_1} \right|$$

$$= J_{21}(n) + J_{22}(n).$$

We bound a typical summand in $J_{21}(n)$, using the strong mixing property

$$\mathbb{E} \left| \overline{a_n^{-1}(S_j - S_{r_n})} \, \overline{a_n^{-1} X_1} \right|$$

$$= \text{cov} \left(|\overline{a_n^{-1}(S_j - S_{r_n})}|, |\overline{a_n^{-1} X_1}| \right) + \mathbb{E} \left| \overline{a_n^{-1} S_{j-r_n+1}} \right| \mathbb{E} \left| \overline{a_n^{-1} X_1} \right|$$

$$\le c \, \alpha_{r_n} + \mathbb{E} \left| \overline{a_n^{-1} S_{j-r_n+1}} \right| \mathbb{E} \left| \overline{a_n^{-1} X_1} \right|.$$

Moreover, we have for $j > r_n$

$$\left| \overline{a_n^{-1}(S_{j-r_n+1})} \right| \le \sum_{i=1}^{j-r_n+1} \left| \overline{a_n^{-1} X_i} \right|,$$

hence

$$\mathbb{E} \left| \overline{a_n^{-1} S_{j-r_n+1}} \right| \mathbb{E} \left| \overline{a_n^{-1} X_1} \right| \le j \left(\mathbb{E} \left| \overline{a_n^{-1} X_1} \right| \right)^2.$$

Thus we arrive at the bound

$$J_{21}(n) \leq c\, n\alpha_{r_n} + c\, n\, m \left(\mathbb{E} \left| \overline{a_n^{-1} X_1} \right| \right)^2.$$

Observe that

$$\mathbb{E} \overline{|a_n^{-1} X_1|} = \mathbb{E} \left(a_n^{-1} |X_1| I_{\{a_n^{-1}|X_1|\leq 2\}} \wedge 2 I_{\{a_n^{-1}|X_1|>2\}} \right)$$
$$\leq 2\mathbb{P}(|X_1| > 2\, a_n).$$

Therefore, by definition of (a_n) and since $n\alpha_{r_n} \to 0$ by assumption,

$$J_{21}(n) = O(n\alpha_{r_n}) + O(m/n) = o(1).$$

We also have

$$J_{22}(n) \leq c\, n\, \mathbb{P}(|S_{r_n} - S_d| > 2\, a_n, |X_1| > 2\, a_n)$$
$$\leq c\, n \sum_{i=d+1}^{r_n} \mathbb{P}(|X_i| > a_n, |X_1| > 2\, a_n)$$
$$+ c\, n\, \mathbb{P}\left(\left| \sum_{i=d+1}^{r_n} X_i I_{\{|X_i|\leq a_n\}} \right| > 2\, a_n, |X_1| > 2\, a_n \right).$$

and

$$J_{22}(n) \leq c\, \mathbb{P}\left(\max_{i=d+1,\ldots,r_n} |X_i| > 2\, a_n/r_n \mid |X_1| > 2\, a_n \right).$$

Thus, under any of the assumptions (3.7)–(3.9), $\lim_{d\to\infty} \limsup_{n\to\infty} J_{22}(n) = 0$. Finally,

$$J_1(n) \leq c\, k_n \sum_{j=d+1}^{r_n} \mathbb{P}(|S_j - S_d| > 2a_n, |X_1| > 2a_n)$$
$$\leq \frac{r_n}{m}\, n\, \mathbb{P}(|X_1| > 2a_n) = o(1).$$

Collecting the bounds above we proved that (**AC**) holds.

3.2.4 Condition (**MX**)

As we have already discussed in Sects. 2.1 and 2.2, the condition (**MX**) is a natural one in the context of stable limit theory for dependent stationary sequences. Modifications of these conditions appear in Davis and Hsing [12], Jakubowski [31,33]. We also discuss the existence of sequencees $m = m_n \to \infty$ and $r = r_n \to \infty$ such that $r/m \to 0$ and $m/n \to 0$ to be used in Lemma 2 in the context of strong mixing.

Lemma 3 *Assume that (X_t) is strongly mixing with rate function (α_h). In addition, assume that there exists a sequence $\epsilon_n \to 0$ satisfying*

$$n\, \alpha_{[\epsilon_n(a_n^2/n \wedge n)]} \to 0. \tag{3.10}$$

Then **(MX)** *holds for some $m = m_n \to \infty$ with $k_n = [n/m] \to \infty$. Moreover, writing $r_n = [\epsilon_n(a_n \wedge n)]$, then $r_n/m_n \to 0$ for this choice of (m_n) and*

$$n\, \alpha_{r_n} \to 0. \tag{3.11}$$

If the tail index $\alpha \leq 1$, then (3.11) turns into $n\alpha_{[\epsilon_n n]} \to 0$ which is more restrictive than (3.10). On the other hand, if the tail index α is close to 2, (3.10) is not implied by polynomial decay of the coefficients α_h. Then a subexponential decay condition of the type $\alpha_n \leq C \exp(-cn^b)$ for some $C, c, b > 0$ implies (3.10), and then (3.11) follows.

Proof We start by showing that (2.8) holds for a suitable sequence (m_n). Let $\varphi_{nm\delta}$ be the characteristic function of $a_n^{-1} \sum_{i=1}^{k_n} U_{m-\delta,i}$ for some $\delta = \delta_n$ and

$$U_{ji} = \sum_{k=(i-1)j+1}^{ij} X_k$$

a block sum of size j. Using that characteristic functions are Lipschitz functions bounded by 1 and writing $q = k_n m$, for $x \in \mathbb{R}$,

$$\begin{aligned}
|\varphi_q(x) - \varphi_{nm\delta}(x)| &\leq \mathbb{E}\left(\left|\frac{x}{a_n}\sum_{j=1}^{k_n} U_{\delta j}\right| \wedge 2\right) \\
&\leq \mathbb{E}\left(\frac{x\delta n}{ma_n}|X_1| \wedge 2\right) \\
&\leq \int_0^2 \mathbb{P}(|X_1| > ma_n/(\delta n)s)\,ds.
\end{aligned}$$

The right-hand side approaches zero if $m_n a_n/(\delta_n n) \to \infty$ as $n \to \infty$. Under this condition, the same arguments yield

$$|(\varphi_{nm}(x))^{k_n} - (\varphi_{n,m-\delta}(x))^{k_n}| \to 0 \quad \text{and} \quad |\varphi_q(x) - \varphi_n(x)| \to 0,$$

as soon as $a_n/m \to \infty$. Next we use a standard mixing argument to bound

$$\begin{aligned}
&|\varphi_{nm\delta}(x) - (\varphi_{n,m-\delta}(x))^{[m/n]}| \\
&\leq |\varphi_{nm\delta}(x) - \varphi_{m-\delta}(x)\varphi_{n-m,m\delta}(x)| \\
&\quad + |\varphi_{m-\delta}(x)\varphi_{n-m,m\delta}(x) - (\varphi_{n,m-\delta}(x))^{[m/n]}|.
\end{aligned}$$

The first term on the right-hand side is the covariance of bounded Lipschitz functions of $S_{m-\delta}$ and $S_n - S_m$. Hence it is bounded by α_δ. Iterative use of this argument, recursively on distinct blocks, shows that the right-hand side is of the order $(n/m)\alpha_\delta$. Thus we proved that (2.8) is satisfied if

$$n/m\alpha_\delta \to 0, \quad ma_n/(\delta n) \to \infty \quad \text{and} \quad a_n/m \to \infty. \tag{3.12}$$

Choose $m_n = [\sqrt{\epsilon_n}(a_n \wedge n)]$, $\delta = [m^2/n]$ and assume (3.10). Then (3.12) holds, (2.8) is satisfied and $m/n \sim \sqrt{\delta/n} \to 0$.

Finally, if (3.10) is satisfied choose $r_n = [\epsilon_n(a_n \wedge n)]$. Then $n\alpha_{r_n} \to 0$ and $r_n/m \to 0$ are automatic.

4 Examples

4.1 m_0-Dependent sequences

Consider a strictly stationary sequence (X_n) satisfying condition (**RV**) and which is m_0-dependent for some integer $m_0 \geq 1$. In this case, $\alpha_h = 0$ for $h > m_0$. Then, by virtue of Lemma 3 condition (**MX**) is satisfied for any choice of sequences (m_n) such that $m_n \to \infty$ and $m_n = o(n)$. Moreover, (**AC**) follows from Lemma 2 for any (r_n) such that $r_n \to \infty$ and $r_n = o(m_n)$. We verify the validity of condition (3.8). Then for (r_n) growing sufficiently slowly,

$$n\,\mathbb{P}\left(\max_{i=d+1,\dots,r_n} |X_i| > a_n/r_n\,,\, |X_1| > a_n\right)$$
$$\leq n\,r_n\,P(|X_1| > a_n)\,P(|X_1| > a_n/r_n)$$
$$= O(r_n\,P(|X_1| > a_n/r_n)) = o(1).$$

Thus, in the m_0-dependent case we have the following special case of Theorem 1.

Proposition 1 *Assume that (X_t) is a strictly stationary m_0-dependent sequence for some $m_0 \geq 1$ which also satisfies condition (**RV**) for some $\alpha \in (0, 2)$. Moreover, assume $\mathbb{E}(X_1) = 0$ for $\alpha > 1$ and X_1 is symmetric for $\alpha = 1$. Then the conclusions of Theorem 1 hold with $c_+ = b_+(m_0 + 1) - b_+(m_0)$ and $c_- = b_-(m_0 + 1) - b_-(m_0)$.*

Proof We have already verified conditions (**MX**) and (**AC**) of Theorem 1. Following the lines of the proof of Theorem 1 with $e_{nd} = 0$, we arrive at (3.5) for every $d \geq 1$. In view of the second part of Lemma 1, the right-hand side of (3.5) is independent of d for $d > m_0$ as the limit of $k_n(\varphi_{nm}(x) - 1)$ as $n \to \infty$. This finishes the proof by taking $d = m_0 + 1$.

We mention in passing that we may conclude from the proof of Proposition 1 that condition (**TB**) is satisfied since $c_+ = b_+(d + 1) - b_+(d)$ and $c_- = b_-(d + 1) - b_-(d)$ for $d > m_0$. This is a fact which is not easily seen by direct calculation on the tails of S_d, $d > m_0$.

4.2 The stochastic volatility model

The stochastic volatility model is one of the standard econometric models for financial returns of the form

$$X_t = \sigma_t \, Z_t,$$

where the volatility sequence (σ_t) is strictly stationary independent of the iid noise sequence (Z_t). See e.g. Andersen et al. [1] for a recent reference on stochastic volatility models or the collection of papers [51].

*Conditions (**RV**), (**TB**) and (**CT**)*

We assume that Z is regularly varying with index $\alpha > 0$, implying that (Z_t) is regularly varying. We also assume that $\mathbb{E}\sigma^p < \infty$ for some $p > \alpha$. Under these assumptions it is known (see Davis and Mikosch [14]) that (X_t) is regularly varying with index α, and the limit measure μ_d in (2.1) is given by

$$\mu_d(dx_1, \ldots, dx_d) = \sum_{i=1}^{d} \lambda_\alpha(dx_i) \prod_{i \neq j} \varepsilon_0(dx_j), \tag{4.1}$$

where ε_x is Dirac measure at x,

$$\lambda_\alpha(x, \infty) = \widetilde{p}\, x^{-\alpha} \quad \text{and} \quad \lambda_\alpha(-\infty, -x] = \widetilde{q}\, x^{-\alpha}, \quad x > 0,$$

and

$$\widetilde{p} = \lim_{x \to \infty} \frac{\mathbb{P}(Z > x)}{\mathbb{P}(|Z| > x)} \quad \text{and} \quad \widetilde{q} = \lim_{x \to \infty} \frac{\mathbb{P}(Z \leq -x)}{\mathbb{P}(|Z| > x)}, \tag{4.2}$$

are the tail balance parameters of Z. This means that the measures μ_d are supported on the axes as if the sequence (X_t) were iid regularly varying with tail balance parameters $\widetilde{p}$ and $\widetilde{q}$. By virtue of (4.1) and (2.4) we have $b_+(d) = \widetilde{p}\, d$ and $b_-(d) = \widetilde{q}\, d$, hence $c_+ = \widetilde{p}$ and $c_- = \widetilde{q}$. We also assume $\mathbb{E}Z = 0$ for $\alpha > 1$. Then $\mathbb{E}X = 0$. If $\alpha = 1$ we assume Z symmetric. Then S_d is symmetric for every $d \geq 1$ and (**CT**) is satisfied.

*Conditions (**MX**) and (**AC**)*

In order to meet (**MX**) we assume that (σ_t) is strongly mixing with rate function (α_h). It is well known (e.g. Doukhan [23]) that (X_t) is then strongly mixing with rate function $(4\alpha_h)$.

It is common use in financial econometrics to assume that $(\log \sigma_t)$ is a Gaussian linear process. The mixing rates for Gaussian linear processes are well studied. For example, if $(\log \sigma_t)$ is a Gaussian ARMA process then (α_h) decays exponentially fast. We will assume this condition in the sequel. Then we may apply Lemma 3 with

$r_n = n^{\gamma_1}$, $m_n = n^{\gamma_2}$, $0 < \gamma_1 < \gamma_2 < 1$ for sufficiently small γ_1 and γ_2, to conclude that (**MX**) holds and $n\alpha_{r_n} \to 0$.

Next we verify (**AC**). We have by Markov's inequality for small $\epsilon > 0$,

$$n \mathbb{P} \left(\max_{i=d+1,\ldots,r_n} |X_i| > a_n/r_n, |X_1| > a_n \right)$$

$$\leq n \sum_{i=d+1}^{r_n} \mathbb{P}(|X_i| > a_n/r_n, |X_1| > a_n)$$

$$\leq n \sum_{i=d+1}^{r_n} \mathbb{P}(\max(\sigma_i, \sigma_1) \min(|Z_i|, |Z_1|) > a_n/r_n)$$

$$\leq n (r_n/a_n)^{\alpha+\epsilon} \sum_{i=d+1}^{r_n} \mathbb{E}(\max(\sigma_i^{\alpha+\epsilon}, \sigma_1^{\alpha+\epsilon}))$$

$$\leq c \, n \, r_n^{1+\alpha+\epsilon} \, a_n^{-\alpha-\epsilon}.$$

The right-hand side converges to zero if we choose γ_1 and ϵ sufficiently small. This proves (3.8) and by Lemma 2 also (**AC**).

Proposition 2 *Assume that (X_t) is a stochastic volatility model satisfying the following additional conditions*:

(a) *(Z_t) is iid regularly varying with index $\alpha \in (0, 2)$ and tail balance parameters $\widetilde{p}$ and $\widetilde{q}$.*
(b) *For $\alpha \in (1, 2)$, $\mathbb{E}Z = 0$, and for $\alpha = 1$, Z is symmetric.*
(c) *$(\log \sigma_t)$ is a Gaussian ARMA process.*

Then the stochastic volatility process (X_t) satisfies the conditions of Theorem 1 with parameters $c_+ = \widetilde{p}$ and $c_- = \widetilde{q}$ defined in (4.2).

Hence a stochastic volatility model with Gaussian ARMA log-volatility sequence satisfies the same stable limit relation as an iid regularly varying sequence with index $\alpha \in (0, 2)$ and tail balance parameters $\widetilde{p}$ and $\widetilde{q}$.

In applications it is common to study powers of the absolute values, $(|X_t|^p)$, most often for $p = 1, 2$. We assume the conditions of Proposition 2. Then the sequence (X_t^2) is again a stochastic volatility process which is regularly varying with index $\alpha/2 \in (0, 1)$. It is not difficult to see that the conditions of Proposition 2 are satisfied for this sequence with $b_-(d) = 0$ and $b_+(d) = d$, hence $c_+ = 1$ and $c_- = 0$.

A similar remark applies to $(|X_t|)$ with one exception: the centering condition (**CT**) cannot be satisfied. This case requires special treatment. However, the cases $\alpha \neq 1$ are similar. For $\alpha < 1$, $(|X_t|)$ is a stochastic volatility model satisfying all conditions

of Proposition 2. For $\alpha \in (1, 2)$ we observe that

$$a_n^{-1} \sum_{i=1}^{n} (|X_t| - \mathbb{E}|X|) = a_n^{-1} \sum_{t=1}^{n} \sigma_t (|Z_t| - \mathbb{E}|Z|) + a_n^{-1} \mathbb{E}|Z| \sum_{t=1}^{n} (\sigma_t - \mathbb{E}\sigma)$$

$$= a_n^{-1} \sum_{t=1}^{n} \sigma_t (|Z_t| - \mathbb{E}|Z|) + o_{\mathbb{P}}(1).$$

In the last step we applied the central limit theorem to (σ_t). Then the process $(\sigma_t(|Z_t| - \mathbb{E}|Z|))$ is a stochastic volatility model satisfying the conditions of Proposition 2 with $c_+ = 1$ and $c_- = 0$.

4.3 Solutions to stochastic recurrence equations

We consider the stochastic recurrence equation

$$X_t = A_t X_{t-1} + B_t, \quad t \in \mathbb{Z}, \tag{4.3}$$

where $((A_t, B_t))$ constitutes an iid sequence of non-negative random variables A_t and B_t. Various econometric time series models (X_t) have this form, including the squared ARCH(1) process and the volatility sequence of a GARCH(1, 1) process; see Sect. 4.4. The conditions $\mathbb{E} \log A < 0$ and $\mathbb{E}|\log B| < \infty$ are sufficient for the existence of a strictly stationary causal solution (X_t) to (4.3) such that $(X_n)_{n \leq 0}$ and $((A_n, B_n))_{n \geq 1}$ are independent; see Kesten [38].

Condition (**RV**)

Kesten [38] and Goldie [28] showed under general conditions that X has almost precise power law tail in the sense that

$$\mathbb{P}(X > x) \sim c_0 x^{-\alpha} \tag{4.4}$$

for some constant $c_0 > 0$, where the value α is given by the unique positive solution to the equation

$$\mathbb{E} A^\kappa = 1, \quad \kappa > 0.$$

We quote Theorem 4.1 in Goldie [28] to get the exact conditions for (4.4).

Theorem 2 *Assume that A is a non-negative random variable such that the conditional law of A given $A \neq 0$ is non-arithmetic and there exists $\alpha > 0$ such that $\mathbb{E}A^\alpha = 1, \mathbb{E}(A^\alpha \log^+ A) < \infty$. Then $-\infty \leq \mathbb{E} \log A < 0$ and $\mathbb{E}(A^\alpha \log A) \in (0, \infty)$. Moreover, if $\mathbb{E}B^\alpha < \infty$, then a unique strictly stationary causal solution (X_t) to (4.3)*

exists such that (4.4) *holds with constant*

$$c_0 = \frac{\mathbb{E}\left[(B_1 + A_1 X_0)^\alpha - (A_1 X_0)^\alpha\right]}{\alpha\, \mathbb{E}(A^\alpha \log A)}. \tag{4.5}$$

The condition of non-arithmeticity of the distribution of A is satisfied if A has a Lebesgue density. *In what follows, we assume that the conditions of Theorem 2 are satisfied.*

Iterating the defining Eq. (4.3) and writing

$$\Pi_t = A_1 \cdots A_t, \quad t \geq 1,$$

we see that

$$(X_1, \ldots, X_d) = X_0\,(\Pi_1, \Pi_2, \ldots, \Pi_d) + R_d, \tag{4.6}$$

where R_d is independent of X_0. Under the assumptions of Theorem 2, the moments $\mathbb{E}A^\alpha$ and $\mathbb{E}B^\alpha$ are finite, hence $\mathbb{E}(R_d^\alpha) < \infty$ and $\mathbb{P}(|R_d| > x) = o(\mathbb{P}(|X_0| > x))$. By a multivariate version of a result of Breiman [8] (see Basrak et al. [5]) it follows that the first term on the right-hand side of (4.6) inherits the regular variation from X_0 with index α and by a standard argument (see Jessen and Mikosch [36, Lemma 3.12]) it follows that $(X_1, \ldots, X_d)$ and the first term on the right-hand side of (4.6) have the same limit measure μ_d. Hence the sequence (X_t) is regularly varying with index α, i.e., condition (**RV**) is satisfied for $\alpha > 0$ with $\mathbb{E}A^\alpha = 1$.

*Condition (**TB**)*

Next we want to determine the quantities $b_+(d)$. Choose (a_n) such that $n\,\mathbb{P}(X > a_n) \sim 1$, i.e., $a_n = (c_0\,n)^{1/\alpha}$, and write

$$T_d = \sum_{i=1}^{d} \Pi_i, \quad d \geq 1.$$

We obtain for every $d \geq 1$, by (4.6),

$$n\,\mathbb{P}(S_d > a_n) \sim n\,\mathbb{P}(X_0\,T_d > a_n) \sim n\,\mathbb{P}(X_0 > a_n)\,\mathbb{E}(T_d^\alpha) \sim \mathbb{E}(T_d^\alpha) = b_+(d).$$

Here we again used Breiman's result [8] for $\mathbb{P}(X_0\,T_d > x) \sim \mathbb{E}(T_d^\alpha)\mathbb{P}(X_0 > x)$ in a modified form. In general, this result requires that $\mathbb{E}(T_d^{\alpha+\delta}) < \infty$ for some $\delta > 0$. However, if $\mathbb{P}(X > x) \sim c_0 x^{-\alpha}$, Breiman's result is applicable under the weaker condition $\mathbb{E}A^\alpha < \infty$; see Jessen and Mikosch [36, Lemma 4.2(3)]. Of course, $b_-(d) = 0$. We mention that the values $b_{\pm}(d)$ do not change if S_d is centered by a constant.

Our next goal is to determine c_+. Since $\mathbb{E}A^\alpha = 1$ we have

$$b_+(d + 1) - b_+(d) = \mathbb{E}\left[(1 + T_d)^\alpha - T_d^\alpha\right]. \tag{4.7}$$

The condition $\mathbb{E}A^\alpha = 1$ and convexity of the function $g(\kappa) = \mathbb{E}A^\kappa$, $\kappa > 0$, imply that $\mathbb{E} \log A < 0$ and therefore

$$T_d \stackrel{\text{a.s.}}{\to} T_\infty = \sum_{i=1}^{\infty} \Pi_i < \infty.$$

Therefore the question arises as to whether one may let $d \to \infty$ in (4.7) and replace T_d in the limit by T_∞. This is indeed possible as the following dominated convergence argument shows.

If $\alpha \in (0, 1]$ concavity of the function $f(x) = x^\alpha$ yields that $(1 + T_d)^\alpha - T_d^\alpha \le 1$ and then Lebesgue dominated convergence applies. If $\alpha \in (1, 2)$, the mean value theorem yields that

$$(1 + T_d)^\alpha - T_d^\alpha = \alpha (T_d + \xi)^{\alpha - 1},$$

where $\xi \in (0, 1)$. Hence $(1 + T_d)^\alpha - T_d^\alpha$ is dominated by the function $\alpha[T_d^{\alpha-1} + 1]$. By convexity of $g(\kappa)$, $\kappa > 0$, we have $\mathbb{E}(A^{\alpha-1}) < 1$ and therefore

$$\mathbb{E}(T_\infty^{\alpha-1}) \le \sum_{i=1}^{\infty} \mathbb{E}(\Pi_i^{\alpha-1}) = \sum_{i=1}^{\infty} (\mathbb{E}(A^{\alpha-1}))^i = \mathbb{E}(A^{\alpha-1})(1 - \mathbb{E}(A^{\alpha-1}))^{-1} < \infty.$$

An application of Lebesgue dominated convergence yields for any $\alpha \in (0, 2)$ that

$$c_+ = \lim_{d \to \infty} \left[b_+(d + 1) - b_+(d) \right] = \mathbb{E} \left[(1 + T_\infty)^\alpha - T_\infty^\alpha \right] \in (0, \infty). \quad (4.8)$$

Remark 6 The quantity T_∞ has the stationary distribution of the solution to the stochastic recurrence equation

$$Y_t = A_t Y_{t-1} + 1, \quad t \in \mathbb{Z}.$$

This solution satisfies the conditions of Theorem 2 and therefore

$$\mathbb{P}(Y_0 > x) = \mathbb{P}(T_\infty > x) \sim c_1 x^{-\alpha},$$

with constant

$$c_1 = \frac{\mathbb{E}[Y_1^\alpha - (A_1 Y_0)^\alpha]}{\alpha \mathbb{E}(A^\alpha \log A)} = \frac{\mathbb{E}[(1 + A_1 Y_0)^\alpha - (A_1 Y_0)^\alpha]}{\alpha \mathbb{E}(A^\alpha \log A)}.$$

In particular, $\mathbb{E}(T_\infty^\alpha) = \infty$. This is an interesting observation in view of $c_+ \in (0, \infty)$. It is also interesting to observe that the limit relation (4.8) implies that

$$\frac{b_+(d)}{d} = \frac{\mathbb{E}(T_d^\alpha)}{d} = \mathbb{E} \left[d^{-1/\alpha} T_d \right]^\alpha \to \mathbb{E} \left[(1 + T_\infty)^\alpha - T_\infty^\alpha \right],$$

although $d^{-1/\alpha} T_d \overset{a.s.}{\to} 0$. This relation yields some information about the rate at which $T_d \overset{a.s.}{\to} T_\infty$.

Condition (**MX**)

The stationary solution (X_t) to the stochastic recurrence equation (4.3) is strongly mixing with geometric rate provided that some additional conditions are satisfied. For example, Basrak et al. [5, Theorem 2.8], assume that the Markov chain (X_t) is μ-irreducible, allowing for the machinery for Feller chains with drift conditions as for example explained in Feigin and Tweedie [25] or Meyn and Tweedie [42]. The drift condition can be verified if one assumes that A_t has polynomial structure; see Mokkadem [45]. The latter conditions can be calculated for GARCH and bilinear processes, assuming some positive Lebesgue density for the noise in a neighborhood of the origin; see Basrak et al. [5], Straumann and Mikosch [53].

  In what follows, we will assume that (X_t) is strongly mixing with geometric rate. Then, by Lemma 3, we may assume that we can choose $r_n = n^{\gamma_1}$, $m_n = n^{\gamma_2}$ for sufficiently small values $0 < \gamma_1 < \gamma_2 < 1$. Then (**MX**) holds and $n\alpha_{r_n} \to 0$.

Condition (**AC**)

We verify condition (3.9) and apply Lemma 2. It suffices to bound the quantities

$$I_n(d) = \mathbb{P}\left(|S_{r_n} - S_d| > a_n \mid X_0 > a_n\right).$$

Writing $\Pi_{s,t} = \prod_{i=s}^{t} A_i$ for $s \leq t$ and $\Pi_{st} = 1$ for $s > t$, we obtain

$$X_i = X_0 \, \Pi_i + \sum_{l=1}^{i} \Pi_{l+1,i} \, B_l = X_0 \, \Pi_i + C_i, \quad i \geq 1.$$

Then, using the independence of X_0 and C_i, $i \geq 1$, applying Markov's inequality for $\kappa < \alpha \wedge 1$ and Karamata's theorem (see Bingham et al. [7]),

$$
\begin{aligned}
I_n(d) &\leq \mathbb{P}\left(X_0 \sum_{i=d+1}^{r_n} \Pi_i > a_n/2 \mid X_0 > a_n\right) + \mathbb{P}\left(\sum_{i=d+1}^{r_n} C_i > a_n/2\right) \\
&\leq c\, \frac{\mathbb{E}(X_0^\kappa I_{\{X_0 > a_n\}})}{a_n^\kappa \, \mathbb{P}(X_0 > a_n)} \sum_{i=d+1}^{r_n} E(\Pi_i^\kappa) + c\, a_n^{-\kappa} \sum_{i=d+1}^{r_n} \sum_{l=1}^{i} (\mathbb{E}A^\kappa)^{i-l} \, \mathbb{E}B^\kappa \\
&\leq c \sum_{i=d+1}^{\infty} (\mathbb{E}(A^\kappa))^i + c\, r_n^{1+\kappa} a_n^{-\kappa} (1 - \mathbb{E}A^\kappa)^{-1} \mathbb{E}B^\kappa \\
&\leq c \left((\mathbb{E}(A^\kappa))^d + r_n^{1+\kappa} a_n^{-\kappa}\right).
\end{aligned}
$$

Here we also used the fact that $\mathbb{E}(A^\kappa) < 1$ by convexity of the function $g(\kappa) = \mathbb{E}(A^\kappa)$, $\kappa > 0$, and $g(\alpha) = 1$. Choosing $r_n = n^{\gamma_1}$ for γ_1 sufficiently small, we see that

$$\lim_{d \to \infty} \limsup_{n \to \infty} I_n(d) = 0.$$

This proves (3.9).

Condition 5

Since X is non-negative, (**CT**) cannot be satisfied; the case $\alpha = 1$ needs special treatment. We focus on the case $\alpha \in (1, 2)$. It is not difficult to see that all calculations given above remain valid if we replace X_t by $X_t - \mathbb{E}X$, provided γ_1 in $r_n = n^{\gamma_1}$ is chosen sufficiently small.

We summarize our results.

Proposition 3 *Under the conditions of Theorem* 2 *the stochastic recurrence equation* (4.3) *has a strictly stationary solution* (X_t) *which is regularly varying with index* $\alpha > 0$ *given by* $\mathbb{E}[A^\alpha] = 1$. *If* $\alpha \in (0, 1) \cup (1, 2)$ *and* (X_t) *is strongly mixing with geometric rate the conditions of Theorem* 1 *are satisfied. In particular,*

$$(c_0 n)^{-1/\alpha} (S_n - b_n) \xrightarrow{d} Z_\alpha, \quad \text{where } b_n = \begin{cases} 0 & \text{for } \alpha \in (0, 1), \\ \mathbb{E}S_n = n\mathbb{E}X & \text{for } \alpha \in (1, 2), \end{cases}$$

where the constant c_0 *is given in* (4.5) *and the* α-*stable random variable* Z_α *has characteristic function* $\psi_\alpha(t) = \exp(-|t|^\alpha \chi_\alpha(t, c_+, 0))$, *where*

$$c_+ = \mathbb{E}[(1 + T_\infty)^\alpha - T_\infty^\alpha] \in (0, \infty) \quad \text{and} \quad T_\infty = \sum_{i=1}^\infty A_1 \cdots A_i.$$

Remark 7 Analogs of of Proposition 3 have recently been proved in Guivarc'h and Le Page [30] in the one-dimensional case and in Buraczewski et al. [9], Theorem 1.6, also in the multivariate case. The results are formulated for a non-stationary version of the process (X_n) starting at some fixed value $X_0 = x$. (This detail is not essential for the limit theorem.) The proofs are tailored for the situation of stochastic recurrence equations and therefore different from those in this paper where the proofs do not depend on some particular structure of the underlying stationary sequence.

4.4 ARCH(1) and GARCH(1,1) processes

In this section we consider the model

$$X_t = \sigma_t Z_t,$$

where (Z_t) is an iid sequence with $\mathbb{E}Z = 0$ and $\text{var}(Z) = 1$ and

$$\sigma_t^2 = \alpha_0 + \left(\alpha_1 Z_{t-1}^2 + \beta_1\right)\sigma_{t-1}^2. \tag{4.9}$$

We assume that $\alpha_0 > 0$ and the non-negative parameters α_1, β_1 are chosen such that a strictly stationary solution to the stochastic recurrence equation (4.9) exists, namely,

$$-\infty \le \mathbb{E}\log\left(\alpha_1 Z^2 + \beta_1\right) < 0, \tag{4.10}$$

see Goldie [28], cf. Mikosch and Stărică [43]. Then the process (X_t) is strictly stationary as well. It is called a GARCH(1,1) process if $\alpha_1\beta_1 > 0$ and an ARCH(1) process if $\beta_1 = 0$ and $\alpha_1 > 0$. Notice that condition (4.10) implies that $\beta_1 \in [0, 1)$.

As a matter of fact, these classes of processes fit nicely into the class of stochastic recurrence equations considered in Sect. 4.3. Indeed, the squared volatility process (σ_t^2) satisfies the stochastic recurrence equation (4.3) with $X_t = \sigma_t^2$, $A_t = \alpha_1 Z_{t-1}^2 + \beta_1$ and $B_t = \alpha_0$. Moreover, the squared ARCH(1) process (X_t^2) satisfies (4.3) with $Y_t = X_t^2$, $A_t = \alpha_1 Z_t^2$ and $B_t = \alpha_0 Z_t^2$.

A combination of the results in Davis and Mikosch [13] for ARCH(1) and in Mikosch and Stărică [43] for GARCH(1, 1) with Proposition 3 above yields the following.

Proposition 4 *Let (X_t) be a strictly stationary* GARCH(1, 1) *process. Assume that Z has a positive density on $\mathbb{R}$ and that there exists $\alpha > 0$ such that*

$$\mathbb{E}\left[(\alpha_1 Z^2 + \beta_1)^\alpha\right] = 1 \quad and \quad \mathbb{E}\left[(\alpha_1 Z^2 + \beta_1)^\alpha \log(\alpha_1 Z^2 + \beta_1)\right] < \infty. \tag{4.11}$$

Then the following statements hold.

(1) *The stationary solution (σ_t^2) to (4.9) is regularly varying with index α and strongly mixing with geometric rate. In particular, there exists a constant $c_1 > 0$, given in (4.5) with $A_1 = \alpha_1 Z_0^2 + \beta_1$ and $B_1 = \alpha_0$ such that*

$$\mathbb{P}(\sigma^2 > x) \sim c_1 x^{-\alpha}. \tag{4.12}$$

For $\beta_1 = 0$, the squared ARCH(1) *process (X_t^2) is regularly varying with index α and strongly mixing with geometric rate. In particular, there exists a constant $c_0 > 0$ given in (4.5) with $A_1 = \alpha_1 Z_1^2$ and $B_1 = \alpha_0 Z_1^2$ such that*

$$\mathbb{P}(X^2 > x) \sim c_0 x^{-\alpha}.$$

(2) *Assume $\alpha \in (0, 1) \cup (1, 2)$ in the* GARCH(1, 1) *case. Then*

$$(c_1 n)^{-1/\alpha}\left(\sum_{t=1}^n \sigma_t^2 - b_n\right) \xrightarrow{d} Z_\alpha, \quad where \quad b_n = \begin{cases} 0 & \alpha \in (0, 1) \\ n\,\mathbb{E}(\sigma^2) & \alpha \in (1, 2), \end{cases}$$

and the α-stable random variable Z_α has characteristic function $\psi_\alpha(t) = \exp(-|t|^\alpha \chi_\alpha(t, c_+, 0))$, where

$$c_+ = \mathbb{E}\left[(1 + T_\infty)^\alpha - T_\infty^\alpha\right] \quad \text{and} \quad T_\infty = \sum_{t=1}^\infty \prod_{i=1}^t \left(\alpha_1 Z_i^2 + \beta_1\right).$$

(3) *Assume $\alpha \in (0, 1) \cup (1, 2)$ in the ARCH(1) case. Then*

$$(c_0 n)^{-1/\alpha} \left(\sum_{t=1}^n X_t^2 - b_n\right) \xrightarrow{d} \widetilde{Z}_\alpha, \quad \text{where} \quad b_n = \begin{cases} 0 & \alpha \in (0, 1) \\ n\,\mathbb{E}(\sigma^2) & \alpha \in (1, 2), \end{cases}$$

and the α-stable random variable $\widetilde{Z}_\alpha$ has characteristic function $\psi_\alpha(t) = \exp(-|t|^\alpha \chi_\alpha(t, c_+, 0))$, where

$$c_+ = \mathbb{E}\left[(1 + \widetilde{T}_\infty)^\alpha - \widetilde{T}_\infty^\alpha\right] \quad \text{and} \quad \widetilde{T}_\infty = \sum_{t=1}^\infty \alpha_1^t \prod_{i=1}^t Z_i^2.$$

The limit results above require that we know the constants c_1 and c_0 appearing in the tails of σ^2 and X^2. For example, in the ARCH(1) case,

$$c_0 = \frac{\mathbb{E}[(\alpha_0 + \alpha_1 X^2)^\alpha - (\alpha_1 X^2)^\alpha]\,\mathbb{E}|Z|^{2\alpha}}{\alpha\,\mathbb{E}[(\alpha_1 Z^2)^\alpha \log(\alpha_1 Z^2)]}.$$

Moreover, (8.66) in Embrechts et al. [24] yields that $X^2 \overset{d}{=} (\alpha_0/\alpha_1)\widetilde{T}_\infty$. Hence the constant c_0 can be written in the form

$$c_0 = \frac{\alpha_0^\alpha\,\mathbb{E}[(1 + \widetilde{T}_\infty)^\alpha - \widetilde{T}_\infty^\alpha]\,\mathbb{E}|Z|^{2\alpha}}{\alpha\,\mathbb{E}[(\alpha_1 Z^2)^\alpha \log(\alpha_1 Z^2)]} = \frac{\alpha_0^\alpha\,c_+\,\mathbb{E}|Z|^{2\alpha}}{\alpha\,\mathbb{E}[(\alpha_1 Z^2)^\alpha \log(\alpha_1 Z^2)]}.$$

The moments of $|Z|$ can be evaluated by numerical methods given that one assumes that Z has a tractable Lebesgue density, such as the standard normal or student densities. Using similar numerical techniques, the value α can be derived from (4.11). The evaluation of the quantity $\mathbb{E}[(\alpha_0 + \alpha_1 X^2)^\alpha - (\alpha_1 X^2)^\alpha]$ is a hard problem; Monte-Carlo simulation of the ARCH(1) process is an option. In the general GARCH(1, 1) case, similar remarks apply to the constants c_+ and c_1 appearing in the stable limits of the partial sum processes of (σ_t^2). Various other financial time series models fit into the framework of stochastic recurrence equations, such as the AGARCH and EGARCH models; see e.g. the treatment in Straumann and Mikosch [53] and the lecture notes by Straumann [52].

In what follows, we consider the GARCH(1, 1) case and prove stable limits for the partial sums of (X_t) and (X_t^2).

Conditions (**RV**)*,* (**MX**) *and* (**AC**)

In what follows, we assume that Z is symmetric, has a positive Lebesgue density on $\mathbb{R}$ *and there exists* $\alpha > 0$ *such that* (4.11) *holds.* Under these assumptions, it follows from Mikosch and Stărică [43] that (X_t^2) is regularly varying with index α and strongly mixing with geometric rate. By Breiman's [8] result we have in particular,

$$\mathbb{P}(X^2 > x) \sim \mathbb{E}|Z|^{2\alpha}\,\mathbb{P}(\sigma^2 > x) \sim \mathbb{E}|Z|^{2\alpha}\,c_1\,x^{-\alpha}.$$

By definition of multivariate regular variation, the sequence $(|X_t|)$ inherits regular variation with index 2α from (X_t^2). By symmetry of Z the sequences $(\text{sign}(Z_t))$ and $(|Z_t|)$, hence $(\text{sign}(X_t))$ and $(|X_t|)$, are independent. Then an application of the multivariate Breiman result in Basrak et al. [5] shows that (X_t) is regularly varying with index 2α and

$$\mathbb{P}(X > x) = 0.5\,\mathbb{P}(|X| > x) \sim 0.5\,\mathbb{E}|Z|^{2\alpha}\,c_1\,x^{-2\alpha}.$$

Thus both sequences (X_t) and (X_t^2) are regularly varying with indices 2α and α, respectively. Moreover, (**MX**) is satisfied for both sequences and we may choose $r_n = n^{\gamma_1}$, $m_n = n^{\gamma_2}$ for sufficiently small $0 < \gamma_1 < \gamma_2 < 1$. An application of Lemma 2 yields (**AC**). We omit details.

4.4.1 Condition (**TB**) *for the squared* GARCH(1, 1) *process*

Recall the notation $A_t = \alpha_1 Z_{t-1}^2 + \beta_1$, $B_t = \alpha_0$, $\Pi_t = \prod_{i=1}^{t} A_i$ and that (a_n) satisfies $n\,\mathbb{P}(X^2 > a_n) \sim 1$. The same arguments as for (4.6) yield

$$\begin{aligned} X_1^2 + \cdots + X_d^2 &= Z_1^2 \sigma_1^2 + \cdots + Z_d^2 \sigma_d^2 \\ &= \sigma_0^2\,(Z_1^2 \Pi_1 + \cdots + Z_d^2 \Pi_d) + R_d. \end{aligned}$$

Under the assumption (4.11), $\mathbb{E}(R_d^\alpha) < \infty$, hence $P(R_d > a_n) = o(P(X^2 > a_n))$. This fact and Breiman's result [8] ensure that

$$\begin{aligned} n\,\mathbb{P}(X_1^2 + \cdots + X_d^2 > a_n) &\sim n\,\mathbb{P}(\sigma_0^2\,(Z_1^2 \Pi_1 + \cdots + Z_d^2 \Pi_d) > a_n) \\ &\sim [n\,\mathbb{E}|Z|^{2\alpha}\,\mathbb{P}(\sigma_0^2 > a_n)]\,\frac{\mathbb{E}|Z_1^2 \Pi_1 + \cdots + Z_d^2 \Pi_d|^\alpha}{\mathbb{E}|Z|^{2\alpha}} \\ &\sim [n\,\mathbb{P}(X^2 > a_n)]\,\frac{\mathbb{E}|Z_1^2 \Pi_1 + \cdots + Z_d^2 \Pi_d|^\alpha}{\mathbb{E}|Z|^{2\alpha}} \\ &\sim \frac{\mathbb{E}|Z_1^2 \Pi_1 + \cdots + Z_d^2 \Pi_d|^\alpha}{\mathbb{E}|Z|^{2\alpha}} \\ &= \frac{\mathbb{E}|Z_0^2 + Z_1^2 \Pi_1 + \cdots + Z_{d-1}^2 \Pi_{d-1}|^\alpha}{\mathbb{E}|Z|^{2\alpha}} = b_+(d). \end{aligned}$$

In the last step we used that A_1 is independent of $Z_1, \ldots, Z_d$ and that $\mathbb{E}A^\alpha = 1$. Write

$$T_d = Z_1^2 \Pi_1 + \cdots + Z_d^2 \Pi_d.$$

Observe that

$$T_d \le \alpha_1^{-1} \left[\Pi_2 + \cdots + \Pi_{d+1} \right].$$

The same argument as in Sect. 4.3 proves that the right-hand side converges a.s. to a finite limit. Hence $T_d \xrightarrow{\text{a.s.}} T_\infty$ for some finite limit $T_\infty = \sum_{t=1}^{\infty} Z_t^2 \Pi_t$. If $\alpha \in (0, 1]$, we have by concavity of the function $f(x) = x^\alpha$, $x > 0$,

$$\mathbb{E}\left[T_{d+1}^\alpha - T_d^\alpha \right] = \mathbb{E}\left[(Z_0^2 + T_d)^\alpha - T_d^\alpha \right] \le \mathbb{E}|Z|^{2\alpha} < \infty.$$

If $\alpha \in (1, 2)$ we have by the mean value theorem for some $\xi \in (0, Z_0^2)$ and using the concavity of the function $f(x) = x^{\alpha-1}$, $x > 0$,

$$
\begin{aligned}
\mathbb{E}\left[(Z_0^2 + T_d)^\alpha - T_d^\alpha \right] &= \alpha \mathbb{E}\left[(T_d + \xi)^{\alpha-1} \right] \\
&\le \alpha \left[\mathbb{E}(T_d^{\alpha-1}) + \mathbb{E}(|Z|^{2(\alpha-1)}) \right] \\
&\le \alpha \, \mathbb{E}(|Z|^{2(\alpha-1)}) \left[1 + \mathbb{E}(A^{\alpha-1}) + \cdots + (\mathbb{E}(A^{\alpha-1}))^d \right] \\
&= \alpha \, \mathbb{E}(|Z|^{2(\alpha-1)}) \, (1 - \mathbb{E}A^{\alpha-1})^{-1} < \infty.
\end{aligned}
$$

An application of Lebesgue dominated convergence yields in the general case $\alpha \in (0, 2)$ that

$$
\begin{aligned}
c_+ &= \lim_{d \to \infty} [b_+(d+1) - b_+(d)] = \lim_{d \to \infty} \frac{\mathbb{E}[(Z_0^2 + T_d)^\alpha - T_d^\alpha]}{\mathbb{E}|Z|^{2\alpha}} \\
&= \frac{\mathbb{E}[(Z_0^2 + T_\infty)^\alpha - T_\infty^\alpha]}{\mathbb{E}|Z|^{2\alpha}} \in (0, \infty).
\end{aligned}
$$

4.4.2 Condition (TB) for the GARCH(1, 1) process

Next we calculate the corresponding value c_+ for the GARCH(1, 1) sequence (X_t). By the assumed symmetry of Z, we have $c_+ = c_-$. Slightly abusing notation, we use the same symbols $b_\pm(d)$, (a_n), T_d, etc., as for (X_t^2). We choose (a_n) such that $n\,\mathbb{P}(|X| > a_n) \sim 1$. We have

$$S_d = Z_1 \sigma_1 + \cdots + Z_d \sigma_d.$$

Since $\mathbb{E}|Z|^{2\alpha} < \infty$ we have for any $\epsilon > 0$,

$$n\,\mathbb{P}(\|(Z_1\sigma_1, \ldots, Z_d\sigma_d) - (Z_1\sigma_1, Z_2|A_2|^{0.5}\sigma_1, \ldots, Z_d|A_d|^{0.5}\sigma_{d-1}\| > \epsilon\,a_n)$$

$$\leq n\,\mathbb{P}\left(\sqrt{\alpha_0}\left(\sum_{i=2}^{d} Z_i^2\right)^{1/2} > \epsilon\,a_n\right) \to 0.$$

Hence (see Jessen and Mikosch [36, Lemma 3.12])

$$n\,\mathbb{P}(S_d > a_n) \sim n\,\mathbb{P}\left(Z_1\sigma_1 + Z_2 A_2^{0.5}\sigma_1 + Z_3 A_3^{0.5}\sigma_2 \cdots + Z_d A_d^{0.5}\sigma_{d-1} > a_n\right).$$

Proceeding by induction, using the same argument as above and in addition Breiman's result, and writing $\Pi_{s,t} = \prod_{i=s}^{t} A_i$ for $s \leq t$ and $\Pi_{st} = 1$ for $s > t$, we see that

$$n\,\mathbb{P}(S_d > a_n) \sim n\,\mathbb{P}\left(\sigma_1\left(Z_1 + A_2^{0.5} Z_2 + \cdots + \Pi_{2,d}^{0.5} Z_d\right) > a_n\right)$$

$$\sim [n\,\mathbb{P}(\sigma_1 > a_n)]\,\mathbb{E}\left[(Z_1 + A_2^{0.5} Z_2 + \cdots + \Pi_{2,d}^{0.5} Z_d)_+^{2\alpha}\right]$$

$$\sim \frac{\mathbb{E}[(Z_1 + A_2^{0.5} Z_2 + \cdots + \Pi_{2,d}^{0.5} Z_d)_+^{2\alpha}]}{\mathbb{E}|Z|^{2\alpha}}$$

$$= \frac{\mathbb{E}[|Z_1 + A_2^{0.5} Z_2 + \cdots + \Pi_{2,d}^{0.5} Z_d|^{2\alpha}]}{2\,\mathbb{E}|Z|^{2\alpha}} = b_+(d).$$

In the last step we used the symmetry of the Z_t's. Writing $T_d = \sum_{i=1}^{d} Z_i \Pi_{2i}^{0.5}$, we have

$$b_+(d) = \frac{\mathbb{E}[|T_d|^{2\alpha}]}{2\mathbb{E}|Z|^{2\alpha}},$$

and

$$|T_d| \leq \sum_{i=1}^{\infty} |Z_i|\,\Pi_{2,i}^{0.5} \leq \alpha_1^{-1/2} \sum_{i=1}^{\infty} \Pi_{2,i+1}^{0.5}. \tag{4.13}$$

Since $\mathbb{E}[(\log A)^{1/2}] = 0.5\mathbb{E}\log A < 0$, the right-hand side converges a.s. to a finite limit. By a Cauchy sequence argument,

$$T_d \overset{\text{a.s.}}{\to} T_\infty = \sum_{i=1}^{\infty} Z_i\,\Pi_{2,i}^{0.5}$$

for some a.s. finite T_∞.

Let (Z_1', A_2') be an independent copy of (Z_1, A_2), independent of T_d. Assume $2\alpha \in (0, 1]$. Then by symmetry of the Z_t's and since $\mathbb{E}[(A_2')^\alpha] = 1$, using the concavity of the function $f(x) = x^{2\alpha}, x > 0$,

$$
\begin{aligned}
&\mathbb{E}\left[\left||T_{d+1}|^{2\alpha} - |T_d|^{2\alpha}\right|\right] \\
&= \mathbb{E}\left[\left||Z_1' + (A_2')^{0.5} T_d|^{2\alpha} - |(A_2')^{0.5} T_d|^{2\alpha}\right|\right] \\
&= \mathbb{E}\left[\left||Z_1'| + (A_2')^{0.5} T_d|^{2\alpha} - |(A_2')^{0.5} T_d|^{2\alpha}\right|\right] \\
&= \mathbb{E}[(|Z_1'| + (A_2')^{0.5} (T_d)_+)^{2\alpha} - ((A_2')^{0.5}(T_d)_+)^{2\alpha}] \\
&\quad + \mathbb{E}\left[\left|(|Z_1'| - (A_2')^{0.5} (T_d)_-)^{2\alpha} - ((A_2')^{0.5}(T_d)_-)^{2\alpha}\right| I_{\{(A_2')^{0.5}(T_d)_- \leq |Z_1'|\}}\right] \\
&\quad + \mathbb{E}\left[\left|((A_2')^{0.5} (T_d)_- - |Z_1'|)^{2\alpha} - ((A_2')^{0.5}(T_d)_-)^{2\alpha}\right| I_{\{(A_2')^{0.5}(T_d)_- > |Z_1'|\}}\right] \\
&\leq \mathbb{E}|Z|^{2\alpha}.
\end{aligned}
$$

For $2\alpha \in (1, 2)$ we use the same decomposition as above and the mean value theorem to obtain

$$
\begin{aligned}
\mathbb{E}\left[\left||T_{d+1}|^{2\alpha} - |T_d|^{2\alpha}\right|\right] &\leq 2\alpha\, \mathbb{E}\left[(A_2')^{0.5}|T_d| + |Z_1'|\right]^{2\alpha-1} \\
&\leq 2\alpha\, \mathbb{E}\left[(A_2')^{0.5}|T_d|\right]^{2\alpha-1} + 2\alpha\mathbb{E}|Z|^{2\alpha-1}.
\end{aligned}
$$

The right-hand side is bounded since $\mathbb{E}|Z|^{2\alpha} < \infty$ and, using (4.13) and $\mathbb{E}[A^{\alpha-0.5}] < 1$,

$$
\mathbb{E}|T_d|^{2\alpha-1} \leq c \sum_{i=1}^{\infty} \mathbb{E}[\Pi_{2,i+1}^{\alpha-0.5}] = c \sum_{i=1}^{\infty} (\mathbb{E}[A^{\alpha-0.5}])^i < \infty.
$$

Now we may apply Lebesgue dominated convergence to conclude that the limit

$$
c_+ = \lim_{d\to\infty} [b_+(d+1) - b_+(d)] = \frac{\mathbb{E}[|Z_1' + (A_2')^{0.5} T_\infty|^{2\alpha} - |(A_2')^{0.5} T_\infty|^{2\alpha}]}{2\mathbb{E}|Z|^{2\alpha}}
$$

exists and is finite.

Since T_∞ and Z_1' assume positive and negative values we have to show that $c_+ > 0$. First we observe that

$$
c_+ = \lim_{d\to\infty} d^{-1}\mathbb{E}[|T_d|^{2\alpha}]. \tag{4.14}
$$

Applying Khintchine's inequality (see Ledoux and Talagrand [41]) conditionally on $(|Z_t|)$, we obtain for some constant $c_\alpha > 0$, all $d \geq 1$,

$$
\begin{aligned}
\mathbb{E}[|T_d|^{2\alpha}] &\geq c_\alpha \, \mathbb{E}\left(\sum_{i=1}^d Z_i^2 \Pi_{2,i}\right)^\alpha \\
&= c_\alpha \, \alpha_1^{-\alpha} \mathbb{E}\left(\sum_{i=1}^d (\alpha_1 Z_i^2 + \beta_1)\Pi_{2,i} - \beta_1 \sum_{i=1}^d \Pi_{2,i}\right)^\alpha \\
&\geq c_\alpha \, \alpha_1^{-\alpha} \left[\mathbb{E}\left(\sum_{i=1}^d \Pi_{2,i+1}\right)^\alpha - \beta_1^\alpha \, \mathbb{E}\left(\sum_{i=1}^d \Pi_{2,i}\right)^\alpha\right] \\
&= c_\alpha \, \alpha_1^{-\alpha}(1 - \beta_1^\alpha)\mathbb{E}\left(\sum_{i=1}^d \Pi_{2,i}\right)^\alpha .
\end{aligned}
$$

Now, Remark 6 and (4.14) imply that $c_+ > 0$.

4.4.3 Condition 5

We assume Z symmetric. Then $\mathbb{E}X = 0$ for $2\alpha \in (1, 2)$ and (**CT**) holds for (X_t). For (X_t^2), (**CT**) cannot be satisfied and needs special treatment. If $\alpha \in (1, 2)$ all arguments above remain valid when X_t^2 is replaced by $\mathbb{E}(X_t^2) - \mathbb{E}(X^2)$.

We summarize our results for the GARCH(1, 1) process (X_t) and its squares.

Proposition 5 *Let (X_t) be a strictly stationary* GARCH(1, 1) *process with symmetric iid unit variance noise (Z_t). Assume that Z has a positive density on $\mathbb{R}$ and that (4.11) holds for some positive α. Then the following statements hold.*

(1) *The sequences (X_t) and (X_t^2) are regularly varying with indices 2α and α, respectively, and both are strongly mixing with geometric rate. In particular,*

$$
\mathbb{P}(X > x) \sim \frac{1}{2}\mathbb{E}|Z|^{2\alpha} \, c_1 \, x^{-2\alpha},
$$

where c_1 is defined in (4.12)*.*

(2) *Assume $2\alpha \in (0, 2)$. Then*

$$
\left(c_1 \, E|Z|^{2\alpha} \, n\right)^{-1/(2\alpha)} S_n \xrightarrow{d} Z_{2\alpha},
$$

where $Z_{2\alpha}$ is symmetric 2α-stable with characteristic function $\psi_{2\alpha}(t) = \exp(-|t|^{2\alpha}\chi_{2\alpha}(t, c_+, c_+))$, where

$$
c_+ = \frac{\mathbb{E}[|Z_0 + |\alpha_1 Z_0^2 + \beta_1|^{0.5}T_\infty|^{2\alpha} - ||\alpha_1 Z_0^2 + \beta_1|^{0.5}T_\infty|^{2\alpha}]}{2\mathbb{E}|Z|^{2\alpha}},
$$

and

$$T_\infty = \sum_{t=1}^{\infty} Z_t \prod_{i=1}^{t-1} \left(\alpha_1 Z_i^2 + \beta_1 \right)^{0.5}.$$

(3) *Assume $\alpha \in (0, 1) \cup (1, 2)$. Then*

$$\left(c_1 E|Z|^{2\alpha} n \right)^{-1/\alpha} \left(\sum_{t=1}^{n} X_t^2 - b_n \right) \xrightarrow{d} \widetilde{Z}_\alpha,$$

where

$$b_n = \begin{cases} 0 & \text{if } \alpha \in (0, 1) \\ n\, \mathbb{E}(X^2) & \text{if } \alpha \in (1, 2), \end{cases}$$

and $\widetilde{Z}_\alpha$ is α-stable with characteristic function

$$\psi_\alpha(t) = \exp\left(-|t|^\alpha \chi_\alpha(t, c_+, 0) \right),$$

where

$$c_+ = \frac{\mathbb{E}[(Z_0^2 + \widetilde{T}_\infty)^\alpha - \widetilde{T}_\infty^\alpha]}{\mathbb{E}|Z|^{2\alpha}}.$$

In the above relation,

$$\widetilde{T}_\infty = \sum_{t=1}^{\infty} Z_{t+1}^2 \prod_{i=1}^{t} \left(\alpha_1 Z_i^2 + \beta_1 \right).$$

Remark 8 For ARCH(1) processes the above technique of identification of parameters of the limiting law was developed in [3].

4.5 Stable stationary sequence

In this section we consider a strictly stationary symmetric α-stable (sαs) sequence (X_t), $\alpha \in (0, 2)$, having the integral representation

$$X_n = \int_E f_n(x)\, M(dx), \quad n \in \mathbb{Z}.$$

Here M is an sαs random measure with control measure μ on the σ-field $\mathcal{E}$ on E and (f_n) is a suitable sequence of deterministic functions $f_n \in L^\alpha(E, \mathcal{E}, \mu)$. We refer to Samorodnitsky and Taqqu [50] for an encyclopedic treatment of stable processes

and to Rosiński [49] for characterizing the classes of stationary (X_t) in terms of their integral representations.

Then for some sαs random variable Y_α,

$$
S_n = \int_E (f_1(x) + \cdots + f_n(x)) M(dx)
$$

$$
\stackrel{d}{=} Y_\alpha \left(\int_E |f_1(x) + \cdots + f_n(x)|^\alpha \, \mu(dx) \right)^{1/\alpha}. \tag{4.15}
$$

Since $P(Y_\alpha > x) \sim 0.5 c_0 x^{-\alpha}$ for some $c_0 > 0$ (see Feller [26]), we have with $n\, P(|X| > a_n) \sim 1$,

$$
n\, P(S_d > a_n) \sim \frac{\int_E |f_1(x) + \cdots + f_d(x)|^\alpha \, \mu(dx)}{\int_E |f_1(x)|^\alpha \mu(dx)} = b_+(d), \quad d \geq 1.
$$

Moreover, it follows from (4.15) that $a_n^{-1} S_n \stackrel{d}{\to} Z_\alpha$ for some Z_α if and only if

$$
n^{-1} b_+(n) \to c_+ \tag{4.16}
$$

for some constant c_+ and the limit Z_α is sαs, possibly zero.

Since we know the distribution of $a_n^{-1} S_n$ for every fixed n we do not need Theorem 1 to determine a sαs limit. In the examples considered above we are not in this fortunate situation. In the sαs case we will investigate which of the conditions in Theorem 1 are satisfied in order to see how restrictive they are. Since the finite-dimensional distributions of (X_t) are α-stable, (**RV**) is satisfied. Conditions (**CT**) and $\mathbb{E}X = 0$ for $\alpha \in (1, 2)$ are automatic. Under (4.16), using the special form of the characteristic function of a sαs random variable, (**MX**) holds for any sequence $m_n \to \infty$. Condition (**AC**) is difficult to be checked. In particular, it does not seem to be known when (X_t) is strongly mixing. An inspection of the proof of Lemma 1, using the particular form of the characteristic functions of the sαs random variables, shows that (**AC**) can be replaced by (**TB**) which implies (4.16). Thus (**TB**) is the only additional restriction in this case.

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