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Spectral estimates and stable processes

Claudia Klüppelberg and Thomas Mikosch
Department of Mathematics, ETH-Zürich, Switzerland

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Let $X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$ be a discrete time moving average process based on i.i.d. symmetric random variables $\{Z_t\}$ with a common distribution function from the domain of normal attraction of a p-stable law $(0<p<2)$. We derive the limit distribution of the normalized periodogram $I_{n,X}(\lambda) = |n^{-1/p} \sum_{t=1}^{n} X_t \,\mathrm{e}^{-\mathrm{i}t\lambda}|^2$, $-\pi \le \lambda \le \pi$. This generalizes the classical result for $p=2$. In contrast to the classical case, for values $0<\lambda_1<\cdots<\lambda_m<\pi$ the periodogram ordinates $I_{n,X}(\lambda_i)$, $i=1,\ldots,m$, are not asymptotically independent.

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moving average processes * general linear model * stable processes * stable laws * spectral estimate * periodogram * characteristic function * spectral measure

1. Introduction

We consider the discrete time moving average process

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \quad t=0,\pm1,\pm2,\ldots, \tag{1.1}$$

where $(Z_t)_{t\in\mathbb{Z}}$ is a noise sequence of independent identically distributed (i.i.d.) symmetric random variables (r.v.'s) with common distribution function (d.f.) in the domain of normal attraction of a p-stable d.f. for some $p\in(0,2)$ $(Z_1 \in \mathrm{DNA}(p))$. Equivalently, there exists a r.v. Y_1 with characteristic function (ch.f.) $E\,\mathrm{e}^{\mathrm{i}tY_1} = \mathrm{e}^{-d|t|^p}$ for some $p\in(0,2)$ and some $d>0$ such that

$$n^{-1/p} \sum_{t=1}^{n} Z_t \xrightarrow{\mathrm{d}} Y_1, \quad n\to\infty, \tag{1.2}$$

where $\xrightarrow{\mathrm{d}}$ denotes convergence in distribution. This implies for the tail probabilities of Z_1 that $P(|Z_1|>x)\sim cx^{-p}$ as $x\to\infty$ for some positive constant c and hence Z_1 has infinite variance. For more information on p-stable d.f.'s and their domains of attraction we refer to Ibragimov and Linnik (1971) or Petrov (1975).

Correspondence to: Dr Claudia Klüppelberg, Department of Mathematics, ETH-Zürich, CH-8092 Zürich, Switzerland.

During the last years there has been an increasing interest in modelling time series phenomena by ARMA processes with heavy tailed noise variables. Davis and Resnick (1986) give detailed references of examples occurring in engineering and economics. Davis and Resnick (1985, 1986) studied the weak limit behaviour of the sample covariance and sample correlation function for the sequence (1.1) in the infinite variance case. It is remarkable that their results are quite analogous to the finite variance case.

In this paper we investigate the probabilistic properties of the sequence $(X_t)_{t\in\mathbb{Z}}$ in the frequency domain. We derive the limit distribution of the renormalized periodogram ordinates

$$I_{n,X}(\lambda) = \left| n^{-1/p} \sum_{t=1}^{n} X_t \, \mathrm{e}^{-\mathrm{i}\lambda t} \right|^2, \quad -\pi \leqslant \lambda \leqslant \pi,$$

for time series (1.1) satisfying (1.2). Notice that for $p = 2$ this is the usual periodogram which is a fundamental tool of statistical inference based on frequency domain properties of the underlying time series. In that case the following result is well-known and can be found e.g. in Brockwell and Davis (1987, Chapter 10). Because of the symmetry of $I_{n,X}$ we can restrict ourselves to frequencies in $[0, \pi]$.

Theorem A. *Suppose $(X_t)_{t\in\mathbb{Z}}$ satisfies (1.1) where $(Z_t)_{t\in\mathbb{Z}}$ is a sequence of i.i.d. mean zero r.v.'s with $EZ_1^2 = \sigma^2 > 0$ and $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$. Then for $\lambda \in (0, \pi)$,*

$$I_{n,X}(\lambda) = |\psi(\lambda)|^2 I_{n,Z}(\lambda) + o_P(1) \xrightarrow{\mathrm{d}} 2\pi f_X(\lambda)(N_1^2 + N_2^2), \quad n \to \infty,$$

where

$$\psi(\lambda) = \sum_{j=-\infty}^{\infty} \psi_j \, \mathrm{e}^{-\mathrm{i}\lambda j},$$

$I_{n,Z}$ denotes the periodogram of $(Z_t)_{t\in\mathbb{Z}}$, f_X is the spectral density of $(X_t)_{t\in\mathbb{Z}}$ and N_1, N_2 are i.i.d. normal r.v.'s with mean zero and variance $\frac{1}{2}$. Moreover, for any values $0 < \lambda_1 < \cdots < \lambda_m < \pi$ the periodogram ordinates $I_{n,X}(\lambda_i)$ are asymptotically independent with mean $2\pi f_X(\lambda_i)$ for $i = 1, \ldots, m$. □

In classical theory Theorem A is applied to estimate the spectral density $f_X(\lambda) = |\psi(\lambda)|^2 \sigma^2/(2\pi)$. This is done via smoothing techniques where the asymptotic independence of $I_{n,X}(\lambda)$ for different frequencies λ plays a central role.

In the infinite variance case the classical spectral density does not exist. For so-called *harmonizable processes* Hosoya (1978) introduced a spectral density via the ch.f.'s of the finite-dimensional distributions of the process. These processes permit a representation

$$X_t = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\lambda} \, \mathrm{d}Y(\lambda)$$

with respect to a p-stable motion $(Y(\lambda))_{-\pi\le\lambda\le\pi}$. Thus harmonizable processes are comparable with stationary processes having finite variance. Unfortunately, for $0<p<2$, harmonizable processes and moving average processes are distinct classes (Schilder, 1970; see also Cambanis and Soltani, 1984).

In the infinite variance case the so-called *power transfer function* $|\psi(\lambda)|^2$ corresponds to the classical spectral density. From a statistical point of view $|\psi(\lambda)|^2$ is of central interest since it determines the model completely. In a forthcoming paper (Klüppelberg and Mikosch, 1992a) we develop a unifying theory for the normalized periodogram $\tilde{I}_{n,X}(\lambda)=I_{n,X}(\lambda)/\sum_{t=1}^{n}X_t^2$ for any $p\in(0,2]$, independent of p. Although for $p<2$ again different normalized periodogram ordinates are not asymptotically independent they at least are pairwise uncorrelated and have exponentially fast decreasing tails. This, as can be shown, suffices to construct consistent estimators for $|\psi(\lambda)|^2$. We apply these results to derive a Whittle type estimator for parameter estimation in finite ARMA models (Klüppelberg and Mikosch, 1992b).

For the present paper we have decided to investigate the mathematical properties of $I_{n,X}(\lambda)$.

Our paper is organized as follows: In Section 2 we formulate our main results which generalize Theorem A to moving average processes $(X_t)_{t\in\mathbb{Z}}$ satisfying (1.1) and (1.2). For instance we shall show that for any frequency λ which is an irrational multiple of 2π,

$$I_{n,X}(\lambda)\xrightarrow{\mathrm{d}}|\psi(\lambda)|^2S(N_1^2+N_2^2),\quad n\to\infty,$$

where S, N_1, N_2 are independent r.v.'s, S is positive $(\frac{1}{2}p)$-stable and N_1, N_2 are i.i.d. normal. In Section 3 we introduce some concepts which will be needed due to the fact that we cannot use Hilbert space methods as in the finite variance case. We shall prove our results in Section 4. Finally, in order to demonstrate the dependency structure in the limit vector of several periodogram ordinates we discuss in detail the case of two frequencies in Section 5.

2. Main results

Throughout this section we consider the moving average process $(X_t)_{t\in\mathbb{Z}}$ with a noise sequence $(Z_t)_{t\in\mathbb{Z}}$ such that $Z_1\in\mathrm{DNA}(p)$ for some $p\in(0,2)$. We also introduce a sequence $(Y_t)_{t\in\mathbb{Z}}$ of i.i.d. p-stable r.v.'s with ch.f. $E\,\mathrm{e}^{\mathrm{i}tY_1}=\mathrm{e}^{-d|t|^p}$ for some $d>0$. According to the 3-series theorem (e.g. Petrov, 1975) the series $\sum_{j=-\infty}^{\infty}\psi_jZ_j$ converges a.s. if and only if $\sum_{j=-\infty}^{\infty}|\psi_j|^p<\infty$. On the other hand, to prove Proposition 2.1 below we need that $\sum_{j=-\infty}^{\infty}|\psi_j|<\infty$. Hence we shall always assume that

$$\sum_{j=-\infty}^{\infty}|\psi_j|^{p\wedge 1}<\infty. \tag{2.1}$$

For any sequence of r.v.'s $(A_t)_{t\in\mathbb{Z}}$, $A_1\in\mathrm{DNA}(p)$, $p\in(0,2]$, we define its *renormalized*

Fourier transform by

$$J_{n,A}(\lambda)=n^{-1/p}\sum_{t=1}^{n}A_t\,\mathrm{e}^{-\mathrm{i}\lambda t},\quad -\pi\leqslant\lambda\leqslant\pi.$$

Then the periodogram of $(A_t)_{t\in\mathbb{Z}}$ is given by

$$I_{n,A}(\lambda)=|J_{n,A}(\lambda)|^2=J_{n,A}(\lambda)J_{n,A}(-\lambda).$$

Again by symmetry of $I_{n,A}$ we can restrict ourselves to frequencies $\lambda\in[0,\pi]$.

The following is analogous to the finite variance case (cf. Theorem A).

Proposition 2.1. *Suppose* $(X_t)_{t\in\mathbb{Z}}$ *satisfies* (1.1) *with* (1.2) *and* (2.1). *Then*

$$I_{n,X}(\lambda)=|\psi(\lambda)|^2 I_{n,Z}(\lambda)+R_n(\lambda),$$

where $\max_{0\leqslant\lambda\leqslant\pi}P(|R_n(\lambda)|>\varepsilon)\to 0$ *as* $n\to\infty$ *for every* $\varepsilon>0$.

Since we are interested in the asymptotic behaviour of $I_{n,X}$ we have to investigate the limit distribution of $I_{n,Z}$. The next result is a first step in this direction.

Proposition 2.2. *Suppose* $(Z_t)_{t\in\mathbb{Z}}$ *satisfies* (1.2). *Then for any frequencies*

$$0<\lambda_1<\cdots<\lambda_m<\pi,$$

$$(I_{n,Z}(\lambda_1),\ldots,I_{n,Z}(\lambda_m))\stackrel{\mathrm{d}}{=}(I_{n,Y}(\lambda_1)+o_P(1),\ldots,I_{n,Y}(\lambda_m)+o_P(1)),\quad n\to\infty.$$

The periodogram $I_{n,Y}(\lambda)$ can be explicitly calculated as follows:

$$\begin{aligned}I_{n,Y}(\lambda)&=|J_{n,Y}(\lambda)|^2=n^{-2/p}\sum_{t=1}^{n}\sum_{s=1}^{n}Y_tY_s\cos\lambda(t-s)\\&=\left(n^{-1/p}\sum_{t=1}^{n}Y_t\cos\lambda t\right)^2+\left(n^{-1/p}\sum_{t=1}^{n}Y_t\sin\lambda t\right)^2\\&=:\alpha_n^2(\lambda)+\beta_n^2(\lambda).\end{aligned}$$

By p-stability and independence of the Y_t, $t\in\mathbb{Z}$, we conclude that

$$\alpha_n(\lambda)=n^{-1/p}\sum_{t=1}^{n}Y_t\cos\lambda t\stackrel{\mathrm{d}}{=}\left(\frac{1}{n}\sum_{t=1}^{n}|\cos\lambda t|^p\right)^{1/p}Y_1,$$

$$\beta_n(\lambda)=n^{-1/p}\sum_{t=1}^{n}Y_t\sin\lambda t\stackrel{\mathrm{d}}{=}\left(\frac{1}{n}\sum_{t=1}^{n}|\sin\lambda t|^p\right)^{1/p}Y_1,$$

so that we might expect that the vector $(\alpha_n(\lambda),\beta_n(\lambda))$ is p-stable in $\mathbb{R}^2$ for each $n\in\mathbb{N}$. This is indeed true as will be seen later; but first we recall the definition of a p-stable random vector:

Definition 2.3. An $\mathbb{R}^d$-valued symmetric random vector $\boldsymbol{Y}$ is p-stable for some $p \in (0, 2)$ if there exists a unique finite symmetric measure Γ on the unit sphere $\mathbb{S}^{d-1}$ of $\mathbb{R}^d$ such that $\boldsymbol{Y}$ has the ch.f.

$$E\,\mathrm{e}^{\mathrm{i}(\boldsymbol{\Theta}, \boldsymbol{Y})} = \exp\left\{-\int_{\mathbb{S}^{d-1}} |(\boldsymbol{\Theta}, s)|^p \,\mathrm{d}\Gamma(s)\right\}.$$

The measure Γ is called the spectral measure of $\boldsymbol{Y}$.

The notion spectral measure should not be confused with the notion spectral density mentioned before. The main aim of this paper is to derive the limit distribution of the vector of the periodogram ordinates $(I_{n,X}(\lambda_1), \ldots, I_{n,X}(\lambda_m))$, $0 < \lambda_1 < \cdots < \lambda_m < \pi$. There will be a crucial difference between frequencies λ which are rational multiples of π and those which are irrational multiples of π. Therefore we write in the sequel $\lambda = 2\pi\omega$ where $\omega \in (0, \frac{1}{2})$ for any frequency $\lambda \in (0, \pi)$. Moreover, for the vector $(\lambda_1, \ldots, \lambda_m)$, $\lambda_t \in (0, \pi)$, $t = 1, \ldots, m$, we rearrange the corresponding vector $(\omega_1, \ldots, \omega_m)$ such that the first q $(0 \leqslant q \leqslant m)$ components $\omega_1, \ldots, \omega_q$ are irrational and $\omega_{q+1}, \ldots, \omega_m$ are rational numbers. We say that the real numbers $(a_1, \ldots, a_m)$ are *linearly dependent over* $\mathbb{Q}$ (the rational numbers) if there exist $q_i \in \mathbb{Q}$, $i = 1, \ldots, m$, such that $0 = q_1 a_1 + \cdots + q_m a_m$, and *linearly independent over* $\mathbb{Q}$, otherwise.

Theorem 2.4. *Suppose $(\lambda_1, \ldots, \lambda_m) = 2\pi(\omega_1, \ldots, \omega_m)$ is a vector of distinct frequencies such that $\omega_1, \ldots, \omega_q$ are irrational and $\omega_{q+1}, \ldots, \omega_m$ are rational numbers in $(0, \frac{1}{2})$ for some $q \in [0, m]$. Denote by g the least common denominator of $\omega_{q+1}, \ldots, \omega_m$ such that $\omega_t = u_t/g$ for certain integers u_t and $t = q+1, \ldots, m$. Then*

$$(I_{n,X}(\lambda_t))_{t=1,\ldots,m} \xrightarrow{\mathrm{d}} (|\psi(\lambda_t)|^2(\alpha^2(\lambda_t) + \beta^2(\lambda_t)))_{t=1,\ldots,m}$$

and the vector $(\alpha(\lambda_t), \beta(\lambda_t))_{t=1,\ldots,m}$ has the ch.f.

$$E\exp\left\{\mathrm{i}\sum_{t=1}^{m}(\delta_t\alpha(\lambda_t) + \vartheta_t\beta(\lambda_t))\right\} = \exp\{-\mathrm{d}K_{\boldsymbol{\lambda}}(\boldsymbol{\delta}, \boldsymbol{\vartheta})\},$$

where $K_{\boldsymbol{\lambda}}(\boldsymbol{\delta}, \boldsymbol{\vartheta})$ is defined below.

(i) *Suppose $1, \omega_1, \ldots, \omega_q$ are linearly independent over $\mathbb{Q}$. Then*

$$\begin{aligned} &K_{\boldsymbol{\lambda}}(\boldsymbol{\delta}, \boldsymbol{\vartheta}) \\ &\quad= \frac{1}{g}\sum_{h=0}^{g-1}\int_{[0,1)^q}\left|\sum_{t=1}^{q}(\delta_t\cos(2\pi x_t) + \vartheta_t\sin(2\pi x_t))\right. \\ &\qquad\qquad \left. + \sum_{t=q+1}^{m}\left(\delta_t\cos\left(2\pi\frac{h}{g}u_t\right) + \vartheta_t\sin\left(2\pi\frac{h}{g}u_t\right)\right)\right|^p \mathrm{d}x_1\cdots\mathrm{d}x_q. \end{aligned}$$

(ii) *Suppose* $1, \omega_1, \ldots, \omega_q$ *are linearly dependent over* $\mathbb{Q}$. *Then there exist natural numbers* $G \geq 1$ *and* v *with* $0 < v < q$, *such that*

$$K_{\boldsymbol{\lambda}}(\boldsymbol{\delta}, \boldsymbol{\vartheta}) = \frac{1}{gG\xi_1^{q-v}(\mathscr{L}_1)} \sum_{h=0}^{g-1} \sum_{r=1}^{G} \int_{\mathscr{L}_r} \Bigg| \sum_{t=1}^{q} (\delta_t \cos(2\pi x_t) + \vartheta_t \sin(2\pi x_t)) + \sum_{t=q+1}^{m} \left(\delta_t \cos\left(2\pi \frac{h}{g} u_t\right) + \vartheta_t \sin\left(2\pi \frac{h}{g} u_t\right)\right)\Bigg|^p \mathrm{d}\xi_r^{q-v}(\boldsymbol{x}),$$

where $\mathscr{L}_1, \ldots, \mathscr{L}_G$ *are the* $(q-v)$*-dimensional linear manifolds in* $[0,1)^q$ *defined in Section* 3.3, *and* ξ_r^{q-v} *denotes the* $(q-v)$*-dimensional Lebesgue measure on* $\mathscr{L}_r$, $r = 1, \ldots, G$. *Furthermore,* $\mathscr{L}_1, \ldots, \mathscr{L}_G$ *have the same* $(q-v)$*-dimensional Lebesgue measure.*

For $q = 0$ *or* $q = m$ *the quantities* $K_{\boldsymbol{\lambda}}(\boldsymbol{\delta}, \boldsymbol{\vartheta})$ *are defined in the natural way.*

It is not difficult to see that, after an appropriate coordinate transformation in $K_{\boldsymbol{\lambda}}(\boldsymbol{\delta}, \boldsymbol{\vartheta})$, the ch.f. of the random vector $(\alpha(\lambda_t), \beta(\lambda_t))_{t=1,\ldots,m}$ satisfies Definition 2.3 and hence is a p-stable vector in $\mathbb{R}^{2m}$. Moreover, $(\alpha(\lambda), \beta(\lambda))_{0 \leq \lambda \leq \pi}$ constitutes a stochastic process whose finite dimensional distributions are p-stable; hence it is a p-stable process (cf. Hosoya, 1978, or Samorodnitski and Taqqu, 1991). From the ch.f. representation it is not difficult to see that both processes $(\alpha(\lambda))_{0 \leq \lambda \leq \pi}$ and $(\beta(\lambda))_{0 \leq \lambda \leq \pi}$ are non-stationary, non-selfsimilar, they do not have independent or stationary increments. If $(\lambda_1, \ldots, \lambda_m)$ is a set of frequencies such that the components of the corresponding vector $(\omega_1, \ldots, \omega_m)$ are irrational and linearly independent over $\mathbb{Q}$, then $\alpha(\lambda_t)$, $t = 1, \ldots, m$, and $\beta(\lambda_t)$, $t = 1, \ldots, m$, are exchangeable sets of r.v.'s. Moreover,

$$(\alpha(\lambda_t), \beta(\lambda_t))_{t=1,\ldots,m} \stackrel{\mathrm{d}}{=} (\alpha(\lambda_t + h), \beta(\lambda_t + h))_{t=1,\ldots,m}$$

provided h is a rational multiple of 2π or the components of the ω-vector corresponding to $(\lambda_1 + h, \ldots, \lambda_m + h)$ are also linearly independent over $\mathbb{Q}$.

Now we want to derive the limit distribution of the one-dimensional periodogram ordinate $I_{n,X}(\lambda)$. Suppose $\lambda = 2\pi\omega$, $\omega \in (0, \frac{1}{2})$. From Theorem 2.4 we obtain

$$I_{n,X}(\lambda) \xrightarrow{\mathrm{d}} |\psi(\lambda)|^2 (\alpha^2(\lambda) + \beta^2(\lambda)), \quad n \to \infty.$$

If ω is irrational, then the vector $(\alpha(\lambda), \beta(\lambda))$ has the ch.f.

$$E \exp\{\mathrm{i}(\delta\alpha(\lambda) + \vartheta\beta(\lambda))\} = \exp\left\{-d \int_0^1 |\delta \cos 2\pi x + \vartheta \sin 2\pi x|^p \, \mathrm{d}x\right\}.$$

Introducing spherical coordinates we may write

$$(\delta, \vartheta) = \sqrt{\delta^2 + \vartheta^2} (\cos\varphi, \sin\varphi) \quad \text{for some } \varphi = \varphi(\delta, \vartheta).$$

Then, using sum formulas for trigonometric functions we obtain

$$\int_0^1 |\delta \cos 2\pi x + \vartheta \sin 2\pi x|^p \, dx = (\delta^2 + \vartheta^2)^{p/2} \frac{1}{\pi} \int_0^\pi |\cos x|^p \, dx.$$

If we denote by $\mathbb{S}^1$ the unit circle in $\mathbb{R}^2$ and by Γ the uniform distribution on $\mathbb{S}^1$, we can rewrite the above integral

$$\int_0^1 |\delta \cos 2\pi x + \vartheta \sin 2\pi x|^p \, dx = \int_{\mathbb{S}^1} \left| \begin{pmatrix} \delta \\ \vartheta \end{pmatrix}^T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right|^p d\Gamma(y_1, y_2).$$

This implies that $(\alpha(\lambda), \beta(\lambda))$ is an isotropic p-stable vector in $\mathbb{R}^2$; i.e. the distribution of $(\alpha(\lambda), \beta(\lambda))$ is invariant under rotation. Consequently, there exist independent r.v.'s S, N_1 and N_2 where S is positive $(\frac{1}{2}p)$-stable and N_1, N_2 are i.i.d. normal r.v.'s such that

$$(\alpha(\lambda), \beta(\lambda)) \stackrel{d}{=} (S^{1/2} N_1, S^{1/2} N_2).$$

The vector $(S^{1/2} N_1, S^{1/2} N_2)$ is also called sub-Gaussian (Samorodnitski and Taqqu, 1991).

Next we consider ω rational, say $\omega = u/g$ with $u, g \in \mathbb{N}$ relatively prime. Then the vector $(\alpha(\lambda), \beta(\lambda))$ has the ch.f.

$$\begin{aligned} &E \exp\{\mathrm{i}(\delta\alpha(\lambda) + \vartheta\beta(\lambda))\} \\ &\quad = \exp\left\{ -\frac{d}{g} \sum_{t=0}^{g-1} \left| \delta \cos\left(2\pi \frac{t}{g}\right) + \vartheta \sin\left(2\pi \frac{t}{g}\right) \right|^p \right\}. \end{aligned} \tag{2.2}$$

This ch.f. corresponds to a discrete uniform spectral measure Γ on the circle $\mathbb{S}^1$ in $\mathbb{R}^2$ with mass d/g at the points $(\cos 2\pi t/g, \sin 2\pi t/g)$, $t = 0, \ldots, g-1$. Moreover, (2.2) is the ch.f. of the convolution of the independent random vectors $(Y_t g^{-1/p} \cos 2\pi t/g,\ Y_t g^{-1/p} \sin 2\pi t/g)$, $t = 0, \ldots, g-1$, where Y_t, $t = 0, \ldots, g-1$, are i.i.d. p-stable r.v.'s with ch.f. $E\,\mathrm{e}^{\mathrm{i}\delta Y_0} = \mathrm{e}^{-d|\delta|^p}$.

We summarize the above facts in a theorem which should be compared with Theorem A in Section 1 (see Remark 1 below).

Theorem 2.5. *Suppose* $\lambda = 2\pi\omega$, $\omega \in (0, \frac{1}{2})$. *Then*

$$I_{n,X}(\lambda) \xrightarrow{d} |\psi(\lambda)|^2 (\alpha^2(\lambda) + \beta^2(\lambda)), \quad n \to \infty,$$

where $(\alpha(\lambda), \beta(\lambda))$ *is a p-stable random vector in* $\mathbb{R}^2$.

(i) *If* ω *is irrational then* $(\alpha(\lambda), \beta(\lambda))$ *has a uniform spectral-measure on the unit circle* $\mathbb{S}^1$. *Moreover, we can write*

$$I_{n,X}(\lambda) \xrightarrow{d} |\psi(\lambda)|^2 S(N_1^2 + N_2^2), \quad n \to \infty,$$

where S, N_1 *and* N_2 *are independent r.v.'s,* S *is positive* $(\frac{1}{2}p)$*-stable with Laplace transform* $E\,\mathrm{e}^{-tS} = \mathrm{e}^{-t^{p/2}}$, $t \geq 0$, *and* N_1, N_2 *are i.i.d.* $\mathrm{N}(0, 2((d/\pi)\int_0^\pi |\cos x|^p \, dx)^{2/p})$ *r.v.'s.*

(ii) *If $\omega = u/g$ with $u, g \in \mathbb{N}$ relatively prime, then $(\alpha(\lambda), \beta(\lambda))$ has a discrete uniform spectral-measure on the unit circle $\mathbb{S}^1$. Moreover, we can write*

$$I_{n,X}(\lambda) \xrightarrow{d} |\psi(\lambda)|^2 \left\{ \left(g^{-1/p} \sum_{t=0}^{g-1} Y_t \cos\left(2\pi \frac{t}{g}\right)\right)^2 + \left(g^{-1/p} \sum_{t=0}^{g-1} Y_t \sin\left(2\pi \frac{t}{g}\right)\right)^2 \right\},$$

where Y_t, $t = 0, \ldots, g-1$, are i.i.d. p-stable r.v.'s with ch.f. $E\, \mathrm{e}^{\mathrm{i}\delta Y_0} = \mathrm{e}^{-d|\delta|^p}$. □

Notice that Theorem 2.5. formally includes Theorem A of Section 1 in the sense that for $p = 2$ for irrational ω the stable variable S degenerates to 1 and for rational ω the limit r.v. is exponentially distributed (see also (2.3) below).

Remark 1. Theorem A and Theorem 2.5 show very clearly the differences but also the similarities in the limit behaviour of $I_{n,X}(\lambda)$ for $p = 2$ and $p < 2$:

In both cases the weak limit of $I_{n,X}(\lambda)$ can be represented as $|\psi(\lambda)|^2(\alpha^2(\lambda) + \beta^2(\lambda))$. The distribution of the vector $(\alpha(\lambda), \beta(\lambda))$ is for $p = 2$ independent of λ; and this also holds for $p < 2$ for every λ such that $\omega = \lambda/(2\pi)$ is irrational. But for $p < 2$ there is a countable exceptional set, the rational numbers $\omega = \lambda/(2\pi) \in (0, \frac{1}{2})$, where $I_{n,X}(\lambda)$ have different distributional limits.

On the other hand, the process $(\alpha(\lambda), \beta(\lambda))_{0<\lambda<\pi}$ is continuous in λ in the following sense: Using spherical coordinates we can rewrite the exponent in (2.2),

$$\begin{aligned}\frac{1}{g} \sum_{t=0}^{g-1} &\left| \delta \cos\left(2\pi \frac{t}{g}\right) + \vartheta \sin\left(2\pi \frac{t}{g}\right)\right|^p \\ &= (\delta^2 + \vartheta^2)^{p/2} \frac{1}{g} \sum_{t=0}^{g-1} \left| \cos\left(2\pi \frac{t}{g} - \varphi\right)\right|^p, \end{aligned} \tag{2.3}$$

where $\varphi = \varphi(\delta, \vartheta)$ is the argument of (δ, ϑ). If we consider a sequence $(\omega_k)_{k \in \mathbb{N}}$ of rational numbers in $(0, \frac{1}{2})$, $\omega_k = u_k/g_k$, with $u_k, g_k \in \mathbb{N}$ relatively prime, and $\omega_k \to \omega \in (0, \frac{1}{2})$ as $k \to \infty$, for ω irrational, then necessarily $g_k \to \infty$ as $k \to \infty$ and the following ergodic theorem holds:

$$\frac{1}{g_k} \sum_{t=0}^{g_k-1} \left| \cos\left(2\pi \frac{t}{g_k} - \varphi\right)\right|^p \to \int_0^1 |\cos 2\pi x|^p \,\mathrm{d}x = \frac{1}{\pi} \int_0^\pi |\cos x|^p \,\mathrm{d}x.$$

Moreover,

$$\begin{aligned} E \exp\{\mathrm{i}(\delta\alpha(2\pi\omega_k) + \vartheta\beta(2\pi\omega_k))\} &\to E \exp\{\mathrm{i}(\delta\alpha(2\pi\omega) + \vartheta\beta(2\pi\omega))\} \\ &= \exp\left\{ -(\delta^2 + \vartheta^2)^{p/2} \frac{d}{\pi} \int_0^\pi |\cos x|^p \,\mathrm{d}x \right\}. \end{aligned}$$

In the case $p = 2$ the random vectors $(\alpha(\lambda), \beta(\lambda))$ are for distinct frequencies independent. This is not true for $p < 2$, but for any set $\{\omega_1, \ldots, \omega_m\}$ of irrational numbers, linearly independent over $\mathbb{Q}$, the corresponding vector $(\alpha(\lambda_k), \beta(\lambda_k))_{k=1,\ldots,m}$ is exchangeable. Using Kolmogorov's consistency theorem we see immediately that this vector can be embedded in an infinite sequence of exchangeable r.v.'s such that the finite-dimensional distributions are just the ones of the vectors $(\alpha(\lambda_k), \beta(\lambda_k))_{k=1,\dots,m}$, $m=1,2,\dots$. Thus for any sequence $\{\omega_k\}$ of irrational numbers, linearly independent over $\mathbb{Q}$, the corresponding sequence $(\alpha(\lambda_k), \beta(\lambda_k))_{k=1,2,\dots}$ is infinitely exchangeable, hence conditionally independent.

Remark 2. It should be noted that the vectors $(\alpha(\lambda_t), \beta(\lambda_t))_{t=1,\dots,m}$, $m>1$, are not isotropic p-stable vectors because they do not have a uniform spectral measure on $\mathbb{S}^{2m-1}$ but they are concentrated on lower dimensional submanifolds of $\mathbb{S}^{2m-1}$. Nevertheless, results by Schilder (1970) and Samorodnitski (1988), see also Samorodnitski and Taqqu (1991), yield an integral representation of the stable vector $(\alpha(\lambda_t), \beta(\lambda_t))_{t=1,\dots,m}$. We present it in Theorem 5.2. For illustration purpose we restrict ourselves to $m=2$. The structure is perfectly demonstrated in two dimensions and it should become clear how to extract the higher dimensional results from Theorem 2.4.

Remark 3. For ease of representation we restrict ourselves to symmetric $Z_1 \in \mathrm{DNA}(p)$ for $p<2$. By a refinement of our methods, analogous results for any Z_1 in the domain of attraction of a symmetric p-stable law can be proved.

Suppose Z_1 is in the domain of attraction of a symmetric p-stable r.v. Y_1. This implies that there exist norming constants $a_n>0$, $n\in\mathbb{N}$, such that

$$a_n^{-1}\sum_{i=1}^{n} Z_i \xrightarrow{\mathrm{d}} Y_1 .$$

Then $a_n = n^{1/p}L(n)$ for some slowly varying function L which depends on the tail of Z_1. The appropriate normalization for the periodogram yields

$$I_{n,X}(\lambda) = \left| a_n^{-1}\sum_{t=1}^{n} X_t \,\mathrm{e}^{-\mathrm{i}\lambda t}\right|^2$$

and the results of Section 2 remain valid also for this more general setup. This can be seen by a careful analysis of the proofs in Section 4: Of course neither the contraction principle of Section 3.1 nor the gauge function of Section 3.2 can be applied. This makes the proofs of Section 4 notationally much more complicated. But nevertheless, a consequent application of the invariance principle of Simons and Stout (1978) (see Lemma 3.3) and Ottaviani's maximal inequality (e.g. Petrov, 1975) yield the results of Section 4. An example of this method of proof can be found in the proof of Proposition 2.2.

3. Auxiliary results

In this section we introduce several conceptional tools that will be needed for the proofs in Section 4. They replace the Hilbert space methods which are basic in the finite variance case. For functions f and g such that $\lim_{x\to\infty} f(x)/g(x)=1$ we shall write $f(x)\sim g(x)$ as $x\to\infty$.

3.1. The contraction principle

Lemma 3.1 (Sztencel, 1981). *Let $A_1,\ldots,A_n$ be independent symmetric r.v.'s. Then for any real numbers $a_1,\ldots,a_n$,*

$$P\left(\left|\sum_{i=1}^{n} a_i A_i\right|>\varepsilon\right)\leqslant 2P\left(\max_{1\leqslant i\leqslant n}|a_i|\left|\sum_{i=1}^{n} A_i\right|>\varepsilon\right)$$

holds for any $\varepsilon>0$. □

3.2. Stable random variables and their DNA

Lemma 3.2 (Ibragimov and Linnik, 1971, Theorems 2.6.1 and 2.6.3). *Suppose $Z_1\in$ DNA(p) for some $p\in(0,2)$. Then there exist real constants $c_1, c_2>0$, such that*

$$P(|Z_1|>x)\sim\frac{c_1}{x^p}\quad\textit{and}\quad\frac{x^2P(|Z_1|>x)}{EZ_1^2 I(|Z_1|\leqslant x)}\to c_2,\quad x\to\infty.\qquad\square$$

If we denote by F_A the d.f. of a r.v. A and by $F_A^{\leftarrow}$ its generalized inverse, then immediately by right continuity,

$$\begin{aligned}F_A^{\leftarrow}(y)\leqslant t \;&\Leftrightarrow\; y\leqslant F_A(t),\\ F_A^{\leftarrow}(y)>t \;&\Leftrightarrow\; y>F_A(t).\end{aligned}\tag{3.1}$$

Next we consider a sequence $(U_t)_{t\in\mathbb{Z}}$ of independent r.v.'s uniformly distributed on $(0,1)$, then immediately by (3.1),

$$(F_{Y_1}^{\leftarrow}(U_t))_{t\in\mathbb{Z}}\stackrel{\mathrm{d}}{=}(Y_t)_{t\in\mathbb{Z}}\quad\text{and}\quad(F_{Z_1}^{\leftarrow}(U_t))_{t\in\mathbb{Z}}\stackrel{\mathrm{d}}{=}(Z_t)_{t\in\mathbb{Z}}.\tag{3.2}$$

Furthermore, we conclude from Simons and Stout (1978), Theorem 3 and Corollary 3, that the following weak invariance principle holds.

Lemma 3.3. *Suppose Z_1 and Y_1 satisfy* (1.2). *Then*

$$n^{-1/p}\sum_{t=1}^{n}(F_{Y_1}^{\leftarrow}(U_t)-F_{Z_1}^{\leftarrow}(U_t))=o_P(1),\quad n\to\infty.\qquad\square$$

For any r.v. A we introduce the *gauge function*

$$\Lambda_p(A)=\left(\sup_{t>0} t^pP(|A|>t)\right)^{1/p}$$

and denote by L_0^p the set of r.v.'s A on a common probability space with $\Lambda_p(A)<\infty$.

We list some properties of Λ_p:

Lemma 3.4. (a) *For any r.v.* $A \in L_0^p$ *and* $a \in \mathbb{R}$,

$$\Lambda_p(a\,A) = |a| \Lambda_p(A).$$

(b) *If* $Z_1 \in \mathrm{DNA}(p)$ *for some* $p \in (0, 2)$ *then* $Z_1 \in L_0^p$.

(c) *Suppose* $(A_t)_{t \in \mathbb{N}}$ *are independent symmetric r.v.'s in* L_0^p *and* $(a_t)_{t \in \mathbb{N}}$ *is a sequence of real numbers. Then for all* $n \in \mathbb{N}$ *there exists a constant* $c > 0$ *(independent of* n*) such that*

$$\Lambda_p^p\left(\sum_{i=1}^n a_i A_i\right) \leq c \max_{1 \leq i \leq n} |a_i|^p \sum_{i=1}^n \Lambda_p^p(A_i).$$

(d) *Suppose* $(A_t)_{t \in \mathbb{N}}$ *is a sequence of r.v.'s in* L_0^p. *If* $\lim_{t \to \infty} \Lambda_p(A_t) = 0$ *then* $A_t \to_{\mathrm{P}} 0$ *as* $n \to \infty$.

Proof. (a) and (b) are easy consequences of Lemma 3.2. Rosinski (1980) proved that

$$\Lambda_p^p\left(\sum_{i=1}^n A_i\right) \leq \tfrac{1}{2} c \sum_{t=1}^n \Lambda_p^p(A_t).$$

We combine this with the contraction principle in Lemma 3.1 to obtain (c). To prove (d) note that by definition of Λ_p immediately $\sup_{t>0} t^p P(|A_n| > t) \to 0$ as $n \to \infty$; hence for all $\varepsilon > 0$ we get $P(|A_n| > \varepsilon) \to 0$. □

3.3. Uniform distribution of sequences

The development of this theory started with a celebrated paper by Weyl (1916) and all results we use from this theory can already be found there. We also refer to a book on this subject by Kuipers and Niederreiter (1974). Furthermore, the one-dimensional result has also been considered as an ergodic theorem by Cornfeld, Fomin and Sinai (1982) and it has been applied by Hosoya (1982) in a different context for stable processes.

For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^k$, $k \in \mathbb{N}$, we say that $\boldsymbol{a} < \boldsymbol{b}$ ($\boldsymbol{a} \leq \boldsymbol{b}$) if the inequality holds componentwise. The set of points $\boldsymbol{y} \in \mathbb{R}^k$ such that $\boldsymbol{a} \leq \boldsymbol{y} < \boldsymbol{b}$ will be denoted by $[\boldsymbol{a}, \boldsymbol{b})$ and is called an *interval*. The k-dimensional unit cube I^k is the interval $[\boldsymbol{0}, \boldsymbol{1})$ where $\boldsymbol{0} = (0, \ldots, 0)$ and $\boldsymbol{1} = (1, \ldots, 1)$.

The integral part of a vector $\boldsymbol{x} = (x_1, \ldots, x_k)$ is $[\boldsymbol{x}] = ([x_1], \ldots, [x_k])$ and the fractional part of $\boldsymbol{x}$ is $\{\boldsymbol{x}\} = (\{x_1\}, \ldots, \{x_k\})$. Now let $(\boldsymbol{x}_i)_{i \in \mathbb{N}}$ be a sequence in $\mathbb{R}^k$. For a subset E of I^k, let $\mathrm{card}\{\{\boldsymbol{x}_i\} \in E;\ i = 1, \ldots, n\}$ denote the number of points $\{\boldsymbol{x}_i\}$, $i = 1, \ldots, \boldsymbol{n}$, that fall into E.

Definition 3.5. Let $\mathscr{L}_1, \ldots, \mathscr{L}_r$ be s-dimensional linear manifolds in I^k, $1 \leq s \leq k$. ξ_i^s denote the (s-dimensional) Lebesgue measure on $\mathscr{L}_i$ and

$$\Xi^s(B) := \sum_{i=1}^r \xi_i^s(B \cap \mathscr{L}_i) \Big/ \sum_{i=1}^r \xi_i^s(\mathscr{L}_i)$$

for every interval B on $\mathscr{L}=\bigcup_{i=1}^{r}\mathscr{L}_i$. The sequence $(\{\boldsymbol{x}_i\})_{i\in\mathbb{N}}$ in I^k is said to be uniformly distributed on $\mathscr{L}_1,\ldots,\mathscr{L}_r$ if

$$\lim_{n\to\infty}\frac{1}{n}\operatorname{card}\{\{\boldsymbol{x}_i\}\in B;\ i=1,\ldots,n\}=\Xi^s(B)$$

for every interval B on $\mathscr{L}$.

According to Weyl (1916) the uniform distribution of $(\{\boldsymbol{x}_i\})_{i\in\mathbb{N}}$ can be expressed by an equivalent condition.

Proposition 3.6. *The sequence $(\{\boldsymbol{x}_i\})_{i\in\mathbb{N}}$ in $\mathscr{L}$ is uniformly distributed on the s-dimensional linear manifolds $\mathscr{L}_1,\ldots,\mathscr{L}_r$ if and only if*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f(\{\boldsymbol{x}_i\})=\int_{\mathscr{L}}f(\boldsymbol{x})\Xi^s(\boldsymbol{x})$$

holds for every bounded Riemann-integrable function f on $\mathscr{L}$. □

We shall apply this result to prove Theorem 2.4. Hereby the important effect of the irrational components $\omega_1,\ldots,\omega_q$ relates to the next result which follows easily from Satz 14 in Weyl (1916):

Lemma 3.7. *Let $\omega_1,\ldots,\omega_q$ be irrational numbers such that $1,\omega_1,\ldots,\omega_q$ are independent over $\mathbb{Q}$. Furthermore, let g and h be integers with $g\neq 0$. Then the sequence*

$$(\{\boldsymbol{\omega}(h+gn)\})_{n\in\mathbb{N}}=(\{\omega_1(h+gn)\},\ldots,\{\omega_q(h+gn)\})_{n\in\mathbb{N}}$$

is uniformly distributed on I^q. □

Now suppose that $1,\omega_1,\ldots,\omega_q$ are linearly dependent over $\mathbb{Q}$. This means that there exists a non-zero vector $\boldsymbol{l}=(l_1,\ldots,l_q)\in\mathbb{Z}^q$ such that

$$\sum_{i=1}^{q}l_i\omega_i\equiv 0\pmod 1. \tag{3.3}$$

This causes the sequence $(\{\boldsymbol{\omega}(h+gn)\})_{n\in\mathbb{N}}$ to show a certain pattern of periodicity which prevents the sequence to be uniformly distributed on the whole of I^q. Instead uniformity will occur on certain parallel submanifolds. Equation (3.3) implies in particular that for any fixed $g\in\mathbb{N}$ there exist non-zero vectors $\boldsymbol{l}\in\mathbb{Z}^q$ such that

$$g\sum_{i=1}^{q}l_i\omega_i=:r(\boldsymbol{l})\in\mathbb{Q}. \tag{3.4}$$

All points $\boldsymbol{l}\in\mathbb{Z}^q$ satisfying (3.4) constitute a lattice L in the sense that with $\boldsymbol{l}',\boldsymbol{l}''\in L$ also $-\boldsymbol{l}'\in L$ and $\boldsymbol{l}'+\boldsymbol{l}''\in L$. Now there exists a basis of L which is a set of points $\boldsymbol{l}_1,\ldots,\boldsymbol{l}_v$ in $\mathbb{Z}^q$, $v\leq q$, such that each $\boldsymbol{l}\in L$ has a representation

$$\boldsymbol{l}=\sum_{j=1}^{v}y_j\boldsymbol{l}_j,\quad y_j\in\mathbb{Z},\ j=1,\ldots,v,$$

and $l_1, \ldots, l_v$ are linearly independent over $\mathbb{Q}$. Denote by G the least common denominator of $r(l_1), \ldots, r(l_v)$ for the basis points of L. This implies in particular that

$$Gg \sum_{i=1}^{q} l_i \omega_i \equiv 0 \pmod 1 \tag{3.5}$$

for all $l \in L$, hence G is the least common denominator of all rational numbers $r(l)$, $l \in L$.

Then for any fixed $n \in \mathbb{N}$ the congruencies

$$g \sum_{i=1}^{q} l_i x_i \equiv r(l) n \pmod 1, \quad l \in L, \quad x \in \mathbb{R}^q, \tag{3.6}$$

define a $(q-v)$-dimensional linear manifold $\mathscr{L}_n$ in I^q (more precisely one should say in $\mathbb{R}^q$ (mod 1)). Because of (3.5) there exist at most G different linear manifolds $\mathscr{L}_1, \ldots, \mathscr{L}_G$ in I^q and it is possible that some of them coincide. From Satz 18 in Weyl (1916) we immediately derive the following property.

Lemma 3.8. *Let $\omega_1, \ldots, \omega_q$ be irrational numbers such that $1, \omega_1, \ldots, \omega_q$ are linearly dependent over $\mathbb{Q}$; furthermore, let g and h be integers with $g \neq 0$. Then the sequence $(\{\boldsymbol{\omega}(h+gn)\})_{n \in \mathbb{N}}$ is uniformly distributed on the linear manifolds $\mathscr{L}_1, \ldots, \mathscr{L}_G$ of I^q defined above.* $\square$

4. Proofs of the results in Section 2

Proof of Proposition 2.1. Analogous to the finite variance case (cf. Brockwell and Davis, 1987, p. 336), we write

$$\begin{aligned} J_{n,X}(\lambda) &= n^{-1/p} \sum_{t=1}^{n} X_t \, \mathrm{e}^{-\mathrm{i}\lambda t} \\ &= n^{-1/p} \sum_{j=-\infty}^{\infty} \psi_j \, \mathrm{e}^{-\mathrm{i}\lambda j} \left(\sum_{t=1}^{n} Z_t \, \mathrm{e}^{-\mathrm{i}\lambda t} + U_{nj} \right) = \psi(\lambda) J_{n,Z}(\lambda) + Y_n(\lambda), \end{aligned}$$

where

$$U_{nj} = \sum_{t=1-j}^{n-j} Z_t \, \mathrm{e}^{-\mathrm{i}\lambda t} - \sum_{t=1}^{n} Z_t \, \mathrm{e}^{-\mathrm{i}\lambda t}, \qquad Y_n(\lambda) = n^{-1/p} \sum_{j=-\infty}^{\infty} \psi_j \, \mathrm{e}^{-\mathrm{i}\lambda j} \, U_{nj}.$$

Hence

$$I_{n,X}(\lambda) = |J_{n,X}(\lambda)|^2 = |\psi(\lambda)|^2 |J_{n,Z}(\lambda)|^2 + R_n(\lambda) = |\psi(\lambda)|^2 I_{n,Z}(\lambda) + R_n(\lambda),$$

where

$$R_n(\lambda) = \psi(\lambda) J_{n,Z}(\lambda) Y_n(-\lambda) + \psi(-\lambda) J_{n,Z}(-\lambda) Y_n(\lambda) + |Y_n(\lambda)|^2.$$

According to the contraction principle of Lemma 3.1, for $\varepsilon > 0$ arbitrary

$$\begin{aligned} &P(|J_{n,Z}(\lambda)| > \varepsilon) \\ &\quad \leq P\left(n^{-1/p}\left|\sum_{t=1}^{n} Z_t \cos\lambda t\right| > \tfrac{1}{2}\varepsilon\right) + P\left(n^{-1/p}\left|\sum_{t=1}^{n} Z_t \sin\lambda t\right| > \tfrac{1}{2}\varepsilon\right) \\ &\quad \leq 4P\left(n^{-1/p}\left|\sum_{t=1}^{n} Z_t\right| > \tfrac{1}{2}\varepsilon\right) \to 4P(|Y_1| > \tfrac{1}{2}\varepsilon), \quad n \to \infty, \end{aligned} \tag{4.1}$$

so that $J_{n,Z}(\lambda)$ is bounded in probability, uniformly for $\lambda \in [0, \pi]$. Thus it remains to show that $Y_n(\lambda) \xrightarrow{P} 0$. We prove this by a suitable decomposition of the sum $Y_n(\lambda)$,

$$Y_n(\lambda) = n^{-1/p} \sum_{|j|>n} \psi_j \,\mathrm{e}^{-\mathrm{i}\lambda j}\, U_{nj} + n^{-1/p} \sum_{|j|\leq n} \psi_j \,\mathrm{e}^{-\mathrm{i}\lambda j}\, U_{nj} =: S_1 + S_2.$$

Furthermore

$$\begin{aligned} S_1 &= n^{-1/p} \sum_{|j|>n} \psi_j \,\mathrm{e}^{-\mathrm{i}\lambda j}\left(-\sum_{t=1}^{n} Z_t \,\mathrm{e}^{-\mathrm{i}\lambda t}\right) + n^{-1/p} \sum_{|j|>n} \psi_j \,\mathrm{e}^{-\mathrm{i}\lambda j} \sum_{t=1-j}^{n-j} Z_t \,\mathrm{e}^{-\mathrm{i}\lambda t} \\ &=: S_{11} + S_{12}. \end{aligned}$$

We shall prove that $S_{11} \xrightarrow{P} 0$ and $S_{12} \xrightarrow{P} 0$ as $n \to \infty$. First note that

$$|S_{11}| \leq \sum_{|j|>n} |\psi_j|\,|J_{n,Z}(\lambda)|.$$

By (4.1), $J_{n,Z}(\lambda)$ is bounded in probability. Furthermore, condition (2.1) implies that $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ and hence $\sum_{|j|>n} |\psi_j| \to 0$ as $n \to \infty$ which implies $S_{11} \xrightarrow{P} 0$ as $n \to \infty$. Next we turn to

$$\begin{aligned} S_{12} &= n^{-1/p} \sum_{j=n+1}^{\infty} \psi_j \,\mathrm{e}^{-\mathrm{i}\lambda j} \sum_{t=1-j}^{n-j} Z_t \,\mathrm{e}^{-\mathrm{i}\lambda t} + n^{-1/p} \sum_{j=-\infty}^{-n-1} \psi_j \,\mathrm{e}^{-\mathrm{i}\lambda j} \sum_{t=1-j}^{n-j} Z_t \,\mathrm{e}^{-\mathrm{i}\lambda t} \\ &=: S_{121} + S_{122}. \end{aligned}$$

Now write

$$\begin{aligned} S_{121} &= n^{-1/p}\left(\sum_{t=-n}^{-1} Z_t \,\mathrm{e}^{-\mathrm{i}\lambda t} \sum_{j=n+1}^{n-t} \psi_j \,\mathrm{e}^{-\mathrm{i}\lambda j}\right) \\ &\quad + n^{-1/p}\left(\sum_{t=-\infty}^{-n-1} Z_t \,\mathrm{e}^{-\mathrm{i}\lambda t} \sum_{j=1-t}^{n-t} \psi_j \,\mathrm{e}^{-\mathrm{i}\lambda j}\right) \\ &=: S_{1211} + S_{1212}. \end{aligned}$$

Lemma 3.4 ensures the existence of some $c > 0$ such that

$$\begin{aligned} \Lambda_p^p(\mathrm{Re}(S_{1211})) &\leq \max_{-1 \leq t \leq -n} \left|\sum_{j=n+1}^{n-t} \psi_j \cos\lambda(t+j)\right|^p \\ &\leq c\left(\sum_{j=n+1}^{\infty} |\psi_j|\right)^p \to 0, \quad n \to \infty, \end{aligned}$$

and hence $S_{1211} \xrightarrow{P} 0$. For S_{1212} we obtain for some $c>0$,

$$\Lambda_p^p(\mathrm{Re}(S_{1212})) \leqslant cn^{-1} \sum_{t=-\infty}^{-n-1} \left| \sum_{j=1-t}^{n-t} \psi_j \cos\lambda(t+j) \right|^p .$$

For $0<p\leqslant 1$ we have

$$\Lambda_p^p(\mathrm{Re}(S_{1212})) \leqslant cn^{-1} \left(\sum_{j=n+2}^{2n+1} (j-n-1)|\psi_j|^p + n \sum_{j=2n+2}^{\infty} |\psi_j|^p \right)$$

$$\leqslant c \sum_{j=n+2}^{\infty} |\psi_j|^p \to 0, \quad n\to\infty.$$

Now suppose $1<p<2$. W.l.o.g. we assume that $\sum_{j=-\infty}^{\infty} |\psi_j| \leqslant 1$ by taking the coefficients $\psi_j' = \psi_j / \sum_{j=-\infty}^{\infty} |\psi_j|$ instead of ψ_j, then

$$\Lambda_p^p(\mathrm{Re}(S_{1212})) \leqslant \left(\sum_{j=-\infty}^{\infty} |\psi_j| \right)^p cn^{-1} \sum_{t=-\infty}^{-n-1} \left| \sum_{j=1-t}^{n-t} \psi_j' \cos\lambda(t+j) \right|^p$$

$$\leqslant \left(\sum_{j=-\infty}^{\infty} |\psi_j| \right)^p cn^{-1} \sum_{t=-\infty}^{-n-1} \sum_{j=1-t}^{n-t} |\psi_j'| \to 0, \quad n\to\infty,$$

by applying the above for the case $p=1$.

This proves that $\mathrm{Re}(S_{1212}) \xrightarrow{P} 0$ as $n\to\infty$ for $p\in(0,2)$. A similar argument for $\mathrm{Im}(S_{1212})$ concludes the proof of $S_{121} \xrightarrow{P} 0$, and analogously, $S_{122} \xrightarrow{P} 0$ which implies $S_{12} \xrightarrow{P} 0$ and, finally, $S_1 \xrightarrow{P} 0$ as $n\to\infty$.

To prove that also $S_2 \xrightarrow{P} 0$ as $n\to\infty$ we write

$$S_2 = n^{-1/p} \sum_{j=1}^{n} \psi_j \,\mathrm{e}^{-\mathrm{i}\lambda j} \sum_{t=1-j}^{0} Z_t \,\mathrm{e}^{-\mathrm{i}\lambda t} - n^{-1/p} \sum_{j=1}^{n} \psi_j \,\mathrm{e}^{-\mathrm{i}\lambda j} \sum_{t=n-j+1}^{n} Z_t \,\mathrm{e}^{-\mathrm{i}\lambda t}$$

$$- n^{-1/p} \sum_{j=-n}^{-1} \psi_j \,\mathrm{e}^{-\mathrm{i}\lambda j} \sum_{t=1}^{-j} Z_t \,\mathrm{e}^{-\mathrm{i}\lambda t} + n^{-1/p} \sum_{j=-n}^{-1} \psi_j \,\mathrm{e}^{-\mathrm{i}\lambda j} \sum_{t=n+1}^{n-j} Z_t \,\mathrm{e}^{-\mathrm{i}\lambda t}$$

$$= S_{21} - S_{22} - S_{23} - S_{24}.$$

Again we prove that the moduli of these four terms tend to 0 in probability as $n\to\infty$.

First we consider

$$S_{21} = n^{-1/p} \sum_{t=1-n}^{0} Z_t \,\mathrm{e}^{-\mathrm{i}\lambda t} \sum_{j=1-t}^{n} \psi_j \,\mathrm{e}^{-\mathrm{i}\lambda j}.$$

By Lemma 3.4 we obtain for some constant $c>0$,

$$\Lambda_p^p(\mathrm{Re}(S_{21})) \leqslant cn^{-1} \sum_{t=1-n}^{0} \left| \sum_{j=1-t}^{n} \psi_j \cos\lambda(j+t) \right|^p$$

$$\leqslant cn^{-1} \sum_{t=1-n}^{0} \left(\sum_{j=1-t}^{n} |\psi_j| \right)^p \to 0, \quad n\to\infty,$$

as a Cesaro limit. A similar argument for $\mathrm{Im}(S_{21})$ yields $S_{21} \xrightarrow{P} 0$ as $n\to\infty$.

To estimate S_{22} we fix $j_0>1$ and obtain

$$S_{22} = n^{-1/p} \sum_{j=1}^{j_0} \psi_j \,\mathrm{e}^{-\mathrm{i}\lambda j} \sum_{t=n-j+1}^{n} Z_t \,\mathrm{e}^{-\mathrm{i}\lambda t} + n^{-1/p} \sum_{j=j_0+1}^{n} \psi_j \,\mathrm{e}^{-\mathrm{i}\lambda j} \sum_{t=n-j+1}^{n} Z_t \,\mathrm{e}^{-\mathrm{i}\lambda t},$$

$$=: S_{221} + S_{222}.$$

Obviously, $S_{221} \xrightarrow{P} 0$ as $n \to \infty$. Furthermore,

$$\Lambda_p^p(\mathrm{Re}(S_{222})) \leq cn^{-1} \sum_{t=1}^{n} \left| \sum_{j=(j_0+1)\vee(n-t+1)}^{n} \psi_j \cos \lambda (j+t) \right|^p$$
$$\leq c\left(\sum_{j=j_0+1}^{\infty} |\psi_j| \right)^p < \varepsilon$$

for arbitrary $\varepsilon > 0$ and j_0 sufficiently large. Now a diagonalization argument yields the existence of a sequence $j_0(n)\uparrow\infty$ such that $S_{221} \xrightarrow{P} 0$ and $S_{222} \xrightarrow{P} 0$ as $n \to \infty$. One shows analogously that $S_{223} \xrightarrow{P} 0$ and $S_{224} \xrightarrow{P} 0$ as $n \to \infty$. Combining all the relations above we find that $Y_n(\lambda) \xrightarrow{P} 0$. Following the lines of the proof it is not difficult to see that we even have $\max_{0 \leq \lambda \leq \pi} P(|R_n(\lambda)| > \varepsilon) \to 0$ as $n \to \infty$ for every $\varepsilon > 0$. □

Proof of Proposition 2.2. According to (3.2) and Lemma 3.3 there exists a probability space with $(Y_t)_{t\in\mathbb{Z}}$, $(Z_t)_{t\in\mathbb{Z}}$ redefined on it such that $n^{-1/p} \sum_{t=1}^{n} (Y_t - Z_t) = o_P(1)$ as $n \to \infty$ and $(Y_t - Z_t)_{t\in\mathbb{Z}}$ is a sequence of i.i.d. r.v.'s. We denote by $\|\cdot\|_2$ the Euclidean norm in $\mathbb{R}^m$ and obtain for any $\varepsilon > 0$,

$$P(\|(J_{n,Z}(\lambda_1) - J_{n,Y}(\lambda_1), \ldots, J_{n,Z}(\lambda_m) - J_{n,Y}(\lambda_m))\|_2 > \varepsilon)$$
$$\leq P\left(\sum_{i=1}^{m} |J_{n,Z}(\lambda_i) - J_{n,Y}(\lambda_i)| > \varepsilon \right)$$
$$\leq \sum_{i=1}^{m} P\left(|J_{n,Z}(\lambda_i) - J_{n,Y}(\lambda_i)| > \frac{\varepsilon}{m} \right).$$

Thus it suffices to show that $P(|J_{n,Z}(\lambda) - J_{n,Y}(\lambda)| > \varepsilon) \to 0$, $n \to \infty$, for any choice of $\lambda \in (0, \pi)$ and $\varepsilon > 0$. We restrict ourselves to the real part and prove that

$$K_n = P\left(\left| n^{-1/p} \sum_{t=1}^{n} (Z_t - Y_t) \cos \lambda t \right| > \varepsilon \right) \to 0, \quad n \to \infty,$$

for fixed λ and ε. To this end we order the values $\cos \lambda t$, $t = 1, \ldots, n$, in such a way that $\cos \lambda t_1 \leq \cdots \leq \cos \lambda t_n$, and $\{t_1, \ldots, t_n\} = \{1, \ldots, n\}$, then

$$K_n = P\left(\left| n^{-1/p} \sum_{j=1}^{n} (Z_j - Y_j) \cos \lambda t_j \right| > \varepsilon \right).$$

We set $S_j = \sum_{t=1}^{j} (Z_t - Y_t)$ and $S_0 \equiv 0$ and obtain

$$\left| \sum_{j=1}^{n} (Z_j - Y_j) \cos \lambda t_j \right| = \left| \sum_{j=1}^{n} (S_j - S_{j-1}) \cos \lambda t_j \right|$$
$$= \left| S_n \cos \lambda t_n + \sum_{j=1}^{n-1} (\cos \lambda t_j - \cos \lambda t_{j+1}) S_j \right|$$
$$\leq |S_n| + 2 \max_{1 \leq j \leq n-1} |S_j|$$
$$\leq 3 \max_{1 \leq j \leq n} |S_j|.$$

Hence

$$K_n \leqslant P\left(n^{-1/p} \max_{1\leqslant j\leqslant n} |S_j| > \tfrac{1}{3}\varepsilon\right)$$

and an application of Ottaviani's inequality (e.g. Petrov, 1975) yields that

$$K_n \leqslant cP(n^{-1/p}|S_n| > \tfrac{1}{4}\varepsilon)$$

for n sufficiently large. This combined with Lemma 3.3 proves that $K_n \to 0$. □

Proof of Theorem 2.4. According to Proposition 2.2 and the succeeding remark

$$(I_{n,X}(\lambda_t))_{t=1,\ldots,m} \stackrel{\mathrm{d}}{=} (|\psi(\lambda_t)|^2(\alpha_n^2(\lambda_t)+\beta_n^2(\lambda_t))+o_P(1))_{t=1,\ldots,m}, \quad n\to\infty.$$

So it suffices to derive the asymptotic distribution of the vector $(\alpha_n(\lambda_t), \beta_n(\lambda_t))_{t=1,\ldots,m}$. For this end we consider the joint ch.f.

$$\begin{aligned}
&E \exp\left\{i \sum_{t=1}^{m} (\delta_t\alpha_n(\lambda_t)+\vartheta_t\beta_n(\lambda_t))\right\}\\
&= E \exp\left\{i n^{-1/p} \sum_{j=1}^{n}\sum_{t=1}^{m} Y_j(\delta_t \cos(2\pi\omega_t j)+\vartheta_t \sin(2\pi\omega_t j))\right\}\\
&= \exp\left\{-\frac{d}{n}\sum_{j=1}^{n}\left|\sum_{t=1}^{m} (\delta_t \cos(2\pi\omega_t j)+\vartheta_t \sin(2\pi\omega_t j))\right|^p\right\}\\
&= \exp\left\{-\frac{d}{n}\sum_{j=1}^{n}\left|\sum_{t=1}^{q} (\delta_t \cos(2\pi\{\omega_t j\})+\vartheta_t \sin(2\pi\{\omega_t j\}))\right.\right.\\
&\qquad\qquad \left.\left.+\sum_{t=q+1}^{m}\left(\delta_t \cos\left(2\pi\frac{u_t}{g} j\right)+\vartheta_t \sin\left(2\pi\frac{u_t}{g} j\right)\right)\right|^p\right\}\\
&=: \exp\{-dK_{n,\boldsymbol{\lambda}}(\boldsymbol{\delta}, \boldsymbol{\vartheta})\}.
\end{aligned}$$

Put $h = j \pmod g$, then $j = h + gk$ where $h = 0, 1, \ldots, g-1$ and $k = 0, 1, 2, \ldots$. Then by periodicity of the sine and cosine functions we get

$$\begin{aligned}
&K_{n,\boldsymbol{\lambda}}(\boldsymbol{\delta}, \boldsymbol{\vartheta})\\
&= \frac{1}{g}\sum_{h=0}^{g-1}\frac{g}{n}\sum_{\{k;h+kg\leqslant n\}}\left|\sum_{t=1}^{q} (\delta_t \cos(2\pi\{\omega_t(h+gk)\})+\vartheta_t \sin(2\pi\{\omega_t(h+gk)\}))\right.\\
&\qquad\qquad \left.+\sum_{t=q+1}^{m}\left(\delta_t \cos\left(2\pi\frac{u_t}{g} h\right)+\vartheta_t \sin\left(2\pi\frac{u_t}{g} h\right)\right)\right|^p.
\end{aligned}$$

Under the assumptions of part (i), in view of Lemma 3.7, the sequence $(\{\omega_1(h+gk)\}, \ldots, \{\omega_q(h+gk)\})_{k\in\mathbb{N}}$ is uniformly distributed in I^q. Furthermore, notice that

$\mathrm{card}\{k;\ h+gk\leqslant n\}\sim n/g$ as $n\to\infty$, uniformly for $h=0,1,\ldots,g-1$. Hence by Proposition 3.6,

$$\lim_{n\to\infty} K_{n,\boldsymbol{\lambda}}(\boldsymbol{\delta},\boldsymbol{\vartheta})$$
$$=\frac{1}{g}\sum_{h=0}^{g-1}\int_{I^q}\left|\sum_{t=1}^{q}(\delta_t\cos 2\pi x_t+\vartheta_t\sin 2\pi x_t)\right.$$
$$\left.+\sum_{t=q+1}^{m}\left(\delta_t\cos\left(2\pi\frac{u_t}{g}i\right)+\vartheta_t\sin\left(2\pi\frac{u_t}{g}i\right)\right)\right|^p \mathrm{d}x_1\cdots\mathrm{d}x_q.$$

Under the assumptions of part (ii) and in view of Lemma 3.8, the sequence $(\{\boldsymbol{\omega}(h+gk)\})_{k\in\mathbb{N}}$ is uniformly distributed on the $(q-v)$-dimensional linear manifolds $\mathscr{L}_1,\ldots,\mathscr{L}_G$ of I^q defined by (3.6). Hence by Proposition 3.6,

$$\lim_{n\to\infty} K_{n,\boldsymbol{\lambda}}(\boldsymbol{\delta},\boldsymbol{\vartheta})$$
$$=\frac{1}{g}\sum_{h=0}^{g-1}\int_{\mathscr{S}}\left|\sum_{t=1}^{q}(\delta_t\cos(2\pi x_t)+\vartheta_t\sin(2\pi x_t))\right.$$
$$\left.+\sum_{t=q+1}^{m}\left(\delta_t\cos\left(2\pi\frac{h}{g}u_t\right)+\vartheta_t\sin\left(2\pi\frac{h}{g}u_t\right)\right)\right|^p \mathrm{d}\Xi^{q-v}(\boldsymbol{x})$$
$$=\frac{1}{g}\sum_{h=0}^{g-1}\left\{\sum_{r=1}^{G}\xi_r^{q-v}(\mathscr{L}_r)\right\}^{-1}$$
$$\times\sum_{r=1}^{G}\int_{\mathscr{S}_r}\left|\sum_{t=1}^{q}(\delta_t\cos(2\pi x_t)+\vartheta_t\sin(2\pi x_t))\right.$$
$$\left.+\sum_{t=q+1}^{m}\left(\delta_t\cos\left(2\pi\frac{h}{g}u_t\right)+\vartheta_t\sin\left(2\pi\frac{h}{g}u_t\right)\right)\right|^p \mathrm{d}\xi_r^{q-v}(\boldsymbol{x}).$$

Then we use the property that $\mathscr{L}_1,\ldots,\mathscr{L}_G$ have the same $(q-v)$-dimensional Lebesgue measure. □

5. The dependence structure of the periodogram ordinates

One of the statistical advantages of classical spectral theory is the asymptotic independence of periodogram ordinates for different frequencies. Unfortunately, this breaks down in case of a heavy tailed innovation process. To illustrate the dependence structure given in terms of the characteristic function in Theorem 2.4 we devote this section to a detailed investigation of the case of two frequencies λ_1 and λ_2.

Suppose $\lambda_t=2\pi\omega_t$, $\omega_t\in(0,\frac{1}{2})$, $t=1,2$. We have to distinguish four different cases:

(a) ω_1,ω_2 are irrational and $1,\omega_1,\omega_2$ are linearly independent over $\mathbb{Q}$.

(b) ω_1,ω_2 are irrational and $1,\omega_1,\omega_2$ are linearly dependent over $\mathbb{Q}$.

(c) ω_1 is irrational, ω_2 is rational.

(d) ω_1,ω_2 are rational.

Proposition 5.1. *The random vector* $(\alpha(\lambda_1), \beta(\lambda_1), \alpha(\lambda_2), \beta(\lambda_2))$ *has a ch.f. as in Theorem* 2.4 *with function* $K_{\boldsymbol\lambda}(\boldsymbol\delta, \boldsymbol\vartheta)$ *corresponding to the four cases above*:

(a) $$K_{\boldsymbol\lambda}(\boldsymbol\delta, \boldsymbol\vartheta) = \int_{[0,1)^2} \left| \sum_{t=1}^{2} (\delta_t \cos(2\pi x_t) + \vartheta_t \sin(2\pi x_t)) \right|^p \mathrm{d}x_1 \, \mathrm{d}x_2 .$$

(b) *Suppose* $k_1\omega_1 + k_2\omega_2 = k_3$ *for certain integers* k_1, k_2 *and* k_3 *relatively prime and* $k_3/k_2 = k_3'/G$ *for relatively prime integers* k_3', G. *Then*

$$K_{\boldsymbol\lambda}(\boldsymbol\delta, \boldsymbol\vartheta) = \frac{1}{G} \sum_{h=0}^{G-1} \int_0^1 \left| \delta_1 \cos(2\pi x) + \vartheta_1 \sin(2\pi x) + \delta_2 \cos\left(2\pi(k_1 x - k_3')\frac{h}{G}\right) + \vartheta_2 \sin\left(2\pi(k_1 x - k_3')\frac{h}{G}\right)\right|^p \mathrm{d}x.$$

(c) *Suppose* $\omega_2 = u/g$ *for relatively prime integers* u, g. *Then*

$$K_{\boldsymbol\lambda}(\boldsymbol\delta, \boldsymbol\vartheta) = \frac{1}{g} \sum_{h=0}^{g-1} \int_0^1 \left| \delta_1 \cos(2\pi x) + \vartheta_1 \sin(2\pi x) + \delta_2 \cos\left(2\pi \frac{u}{g} h\right) + \vartheta_2 \sin\left(2\pi \frac{u}{g} h\right)\right|^p \mathrm{d}x.$$

(d) *Suppose* $\omega_t = u_t/g$, $t = 1, 2$, *where* g *is the least common denominator of* ω_1, ω_2. *Then*

$$K_{\boldsymbol\lambda}(\boldsymbol\delta, \boldsymbol\vartheta) = \frac{1}{g} \sum_{h=0}^{g-1} \left| \sum_{t=1}^{2} \delta_t \cos\left(2\pi \frac{u_t}{g} h\right) + \vartheta_t \sin\left(2\pi \frac{u_t}{g} h\right)\right|^p .$$

Proof. The cases (a), (c) and (d) follow immediately from Theorem 2.4(i). It remains to derive $K_{\boldsymbol\lambda}(\boldsymbol\delta, \boldsymbol\vartheta)$ in the case (b).

According to Section 3.3 we have to consider the lattice L of vectors $\boldsymbol l = (l_1, l_2)^{\mathrm T}$ satisfying

$$l_1\omega_1 + l_2\omega_2 = r(\boldsymbol l) \in \mathbb{Q}.$$

This implies

$$\omega_1\left(l_1 - l_2\frac{k_1}{k_2}\right) + l_2\frac{k_3}{k_2} = r(\boldsymbol l) \in \mathbb{Q},$$

but this relation is only possible if $l_1 = l_2 k_1/k_2$. Hence $L = \{\boldsymbol l = K(k_1, k_2),\ K \in \mathbb{Z}\}$ and $r(\boldsymbol l) = l_2 k_3/k_2 = l_2 k_3'/G$. Now the defining congruencies for the linear manifolds $\mathscr{L}_n$ are

$$K(k_1x_1 + k_2x_2) \equiv nKk_3 \pmod 1 \quad \forall K \in \mathbb{Z}.$$

Equivalently,

$$K(k_1x_1 + k_2x_2 - nk_3) \equiv 0 \pmod 1 \quad \forall K \in \mathbb{Z}.$$

Hence the relations

$$k_1x_1 + k_2x_2 - nk_3 = q, \quad x_1, x_2 \in [0, 1) \tag{5.1}$$

for suitable integers q (such that $x_1, x_2 \in [0,1)$) define the 1-dimensional linear manifold $\mathscr{L}_n$ in $[0,1)^2$. Since G is the least common denominator of the rational numbers $r(\boldsymbol{l})$, $\boldsymbol{l} \in L$, we conclude that there exist exactly G different parallel manifolds $\mathscr{L}_i$. Combining these facts with the formula for $K_{\boldsymbol{\lambda}}(\boldsymbol{\delta}, \boldsymbol{\vartheta})$ in Theorem 2.4(ii) we get the asssertion. □

Example. We illustrate the subcase (b) of Proposition 5.1. Put $\omega_1 = \frac{1}{10}(\pi + 1)$; $\omega_2 = \frac{1}{10}(\pi - 1)$. This implies that $1, \omega_1, \omega_2$ are linearly dependent over $\mathbb{Q}$ and

$$5(l_1\omega_1 + l_2\omega_2) \equiv 0 \pmod 1$$

for all $\boldsymbol{l} \in L = \{\boldsymbol{l} \in \mathbb{Z}^2;\ (l_1, l_2) = K(1,-1),\ K \in \mathbb{Z}\}$. Moreover, for the basis $\boldsymbol{l} = (1,-1)$ of L we obtain

$$\omega_1 - \omega_2 = \tfrac{1}{5}.$$

According to (5.1) the relations

$$5x_1 - 5x_2 - n = q$$

define for each $n \in \mathbb{N}$ and suitably chosen $q \in \mathbb{Z}$ a one-dimensional linear manifold $\mathscr{L}_n$ in $[0,1)^2$. Here we obtain five different ones, namely

$$\mathscr{L}_1 = \{(x_1, x_2) \in [0,1)^2;\ x_1 - x_2 = \tfrac{1}{5} \text{ or } x_1 - x_2 = -\tfrac{4}{5}\},$$
$$\mathscr{L}_2 = \{(x_1, x_2) \in [0,1)^2;\ x_1 - x_2 = \tfrac{2}{5} \text{ or } x_1 - x_2 = -\tfrac{3}{5}\},$$
$$\mathscr{L}_3 = \{(x_1, x_2) \in [0,1)^2;\ x_1 - x_2 = \tfrac{3}{5} \text{ or } x_1 - x_2 = -\tfrac{2}{5}\},$$
$$\mathscr{L}_4 = \{(x_1, x_2) \in [0,1)^2;\ x_1 - x_2 = \tfrac{4}{5} \text{ or } x_1 - x_2 = -\tfrac{1}{5}\},$$
$$\mathscr{L}_5 = \{(x_1, x_2) \in [0,1)^2;\ x_1 = x_2\}.$$

Notice that $\mathscr{L}_1, \ldots, \mathscr{L}_5$ are parallel (mod 1) and have the same length $\sqrt{2}$. Then

$$\begin{aligned}K_{\boldsymbol{\lambda}}(\boldsymbol{\delta}, \boldsymbol{\vartheta}) &= \frac{1}{5\sqrt{2}} \sum_{r=1}^{5} \int_{\mathscr{L}_r} \left| \sum_{t=1}^{2} \delta_t \cos(2\pi x_t) + \vartheta_t \sin 2\pi x_t \right|^p \mathrm{d}\xi_r(x_1, x_2) \\ &= \frac{1}{5} \sum_{r=1}^{5} \int_0^1 \Big| \delta_1 \cos(2\pi x) + \vartheta_1 \sin(2\pi x) \\ &\qquad + \delta_2 \cos(2\pi(x - \tfrac{1}{5}r)) + \vartheta_2 \sin(2\pi(x - \tfrac{1}{5}r)) \Big|^p \,\mathrm{d}x,\end{aligned}$$

where ξ_r is the Lebesgue measure on $\mathscr{L}_r$.

According to Schilder (1970) or Samorodnitski (1988), see also Samorodnitski and Taqqu (1991), Chapter 3, any p-stable symmetric random vector $\boldsymbol{A} = (A_1, \ldots, A_4)$ has a ch.f.

$$E \exp\left\{ \mathrm{i} \sum_{j=1}^{4} t_j A_j \right\} = \exp\left\{ -\int_E \left| \sum_{j=1}^{4} t_j f_j(\boldsymbol{x}) \right|^p \mathrm{d}m(\boldsymbol{x}) \right\} > 0$$

and permits the integral representation

$$\boldsymbol{A} = \left(\int_E f_1(\boldsymbol{x})\, \mathrm{d}M(\boldsymbol{x}), \ldots, \int_E f_4(\boldsymbol{x})\, \mathrm{d}M(\boldsymbol{x}) \right),$$

where M is a p-stable symmetric random measure on the Borel measurable space $(E, \mathscr{B}(E))$ and m is a measure on the finite sets of $(E, \mathscr{B}(E))$, the so-called *control measure*.

From Proposition 5.1 we derive the following result.

Theorem 5.2. *Suppose $\lambda_t = 2\pi\omega_t$ with $\omega_t = (0, \frac{1}{2})$ for $t = 1, 2$. Set $E = [0, 1)^2$ and denote by $\mathscr{B}(E)$ the Borel-σ-algebra in E and $\boldsymbol{x} = (x_1, x_2) \in E$. Then*

$$\begin{aligned}&(\alpha(\lambda_1), \beta(\lambda_1), \alpha(\lambda_2), \beta(\lambda_2))\\ &\quad \stackrel{\mathrm{d}}{=} \left(\int_E \cos(2\pi x_1)\, \mathrm{d}M(\boldsymbol{x}), \int_E \sin(2\pi x_1)\, \mathrm{d}M(\boldsymbol{x}),\right.\\ &\qquad\quad \left.\int_E \cos(2\pi x_2)\, \mathrm{d}M(\boldsymbol{x}), \int_E \sin(2\pi x_2)\, \mathrm{d}M(\boldsymbol{x})\right),\end{aligned}$$

where M is a p-stable random measure M on $(E, \mathscr{B}(E))$ with control measure m corresponding to the four cases above:

(a) *$d^{-1}m$ is the Lebesgue-measure on E.*

(b) $$m = \frac{d}{G\xi(\mathscr{L}_1)} \sum_{h=0}^{G-1} \xi_{\mathscr{L}_h},$$

where $\mathscr{L}_0, \ldots, \mathscr{L}_{G-1}$ are G parallel linear manifolds in E, $\xi_{\mathscr{L}_h}(A) = \xi(\mathscr{L}_h \cap A)$, $A \in \mathscr{B}(E)$, $h = 0, \ldots, G-1$, and ξ is the 1-dimensional Lebesgue-measure on $\mathscr{L} = \mathscr{L}_0 \cup \cdots \cup \mathscr{L}_{G-1}$.

(c) $$m = \frac{d}{g} \sum_{h=0}^{g-1} \xi_{\mathscr{L}_h},$$

where $\mathscr{L}_h = \{(x_1, x_2)\colon 0 \leqslant x_1 < 1, x_2 = h/g\}$, $h = 0, \ldots, g-1$, and $\xi_{\mathscr{L}_h}$, $h = 0, \ldots, g-1$, are defined as in part (b).

(d) *m is the discrete measure with mass d/g at the points $((u_1/g)h \pmod 1), (u_2/g)h \pmod 1))$, $h = 0, \ldots, g-1$.* □

The integral representation of Theorem 5.2 is of course available for any set of frequencies. For the sake of an intuitive representation of the control measure we restricted ourselves to the case of two frequencies.

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References

P.J. Brockwell and R.A. Davis, Times Series: Theory and Methods (Springer, Berlin, 1987).

S. Cambanis and A.R. Soltani, Prediction of stable processes: spectral and moving average representations, Z. Wahrsch. Verw. Gebiete 66 (1984) 593-612.

I.P. Cornfeld, S.V. Fomin and Ya.G. Sinai, Ergodic Theory (Springer, New York, 1982).

R.A. Davis and S.I. Resnick, Limit theory for moving averages of random variables with regularly varying tail probabilities, Ann. Probab. 13 (1985) 179-195.

R.A. Davis and S.I. Resnick, Limit theory for the sample covariance and correlation functions of moving averages, Ann. Statist. 14 (1986) 533-558.

Y. Hosoya, Discrete-time stable processes and their certain properties, Ann. Probab. 6 (1978) 94-105.

Y. Hosoya, Harmonizable stable processes, Z. Wahrsch. Verw. Gebiete 60 (1982) 517-533.

I.A. Ibragimov and Yu.V. Linnik, Independent and Stationary Sequences of Random Variables (Wolters-Noordhoff, Groningen, 1971).

C. Klüppelberg and T. Mikosch, Some limit theory for the normalized periodogram of p-stable moving average processes, ETH preprint (Zürich, 1992a).

C. Klüppelberg and T. Mikosch, Parameter estimation for ARMA models with infinite variance innovations, ETH preprint (Zürich, 1992b).

L. Kuipers and H. Niederreiter, Uniform Distribution of Sequences (Wiley, New York, 1974).

V.V. Petrov, Sums of Independent Random Variables (Springer, Berlin, 1975).

J. Rosinski, Remarks on Banach spaces of stable type, Probab. Math. Statist. 1 (1980) 67-71.

G. Samorodnitski, Extrema of skewed stable processes, Stochastic Process. Appl. 30 (1988) 17-39.

G. Samorodnitski and M.S. Taqqu, Non-Gaussian stable processes, preprint (1991).

M. Schilder, Some structure theorems for the symmetric stable laws, Ann. Math. Statist. 41 (1970) 412-421.

G. Simons and W.F. Stout, A weak invariance principle with applications to domains of attraction, Ann. Probab. 6 (1978) 294-315.

R. Sztencel, On boundedness and convergence of some Banach space valued random series, Probab. Math. Statist. 2 (1981) 83-88.

H. Weyl, Über die Gleichverteilung von Zahlen mod, Eins. Math. Ann. 77 (1916) 313-352.