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TAIL BEHAVIOR OF RANDOM PRODUCTS AND STOCHASTIC EXPONENTIALS

SERGE COHEN AND THOMAS MIKOSCH

Abstract. In this paper we study the distributional tail behavior of the solution to a linear stochastic differential equation driven by infinite variance α-stable Lévy motion. We show that the solution is regularly varying with index α. An important step in the proof is the study of a Poisson number of products of independent random variables with regularly varying tail. The study of these products deserves its own interest because it involves interesting saddle-point approximation techniques.

1. Introduction

In this paper we study the distributional tail behavior of the unique strong solution to the linear Itô stochastic differential equation

$$d\eta_t = 1 + \eta_{t-}\, d\xi_t\,, \quad t \in [0,1]\,, \tag{1.1}$$

where the driving process $(\xi_t)_{t\in[0,1]}$ is α-stable Lévy motion on $[0,1]$ for some given $\alpha \in (0,2)$, see Samorodnitsky and Taqqu [15] as a general reference to stable distributions and processes.

α-stable Lévy motion is a pure jump process; jumps occur on a dense set in every interval. This can be seen from the following series representation ([15], p. 151). Let $(\Gamma_n)_{n=1,2,\ldots}$ be the sequence of increasing points of a unit rate Poisson process on $(0,\infty)$, $(\gamma_n)_{n=1,2,\ldots}$ an iid sequence of Bernoulli random variables with distribution

$$p = P(\gamma_i = 1) = 1 - P(\gamma_i = -1) = 1 - q = \frac{1+\beta}{2} \tag{1.2}$$

for some number $\beta \in [-1,1]$, the skewness parameter of the stable distribution, and $(U_n)_{n=1,2,\ldots}$ be an iid sequence of uniform $U(0,1)$ random variables. Moreover, the sequences (U_n), (γ_n) and (Γ_n) are supposed to be independent.[1] A series representation of a standardized version of ξ is then given by

$$\xi_t = \begin{cases} \sum_{i=1}^\infty \gamma_i \Gamma_i^{-1/\alpha} I_{\{U_i\le t\}}\,, & \alpha \in (0,1)\,,\\ \sum_{i=1}^\infty \left(\gamma_i \Gamma_i^{-1/\alpha} I_{\{U_i\le t\}} - \beta\, t\, b_i^{(1)}\right) + \beta\, t\, \log\left(\frac{2}{\pi}\right)\,, & \alpha = 1\,,\\ \sum_{i=1}^\infty \left(\gamma_i \Gamma_i^{-1/\alpha} I_{\{U_i\le t\}} - \beta\, t\, b_i^{(\alpha)}\right)\,, & \alpha \in (1,2)\,. \end{cases} \tag{1.3}$$

Here $b_i^{(\alpha)}$ are certain constants; see [15] for details. The convergence of the series on the right sides is uniform for $t \in [0,1]$ with probability 1. We always assume that ξ has representation

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[1] In what follows, we will use the convention that the symbols U, γ, etc., represent a generic element of the iid sequences (U_i), (γ_i), etc.

(1.3). Then ξ is automatically right-continuous and has limits from the left at every point. In what follows, we always assume that either $\alpha < 1$ or $\alpha \in [1,2)$ and $\beta = 0$. The case $\beta = 0$ corresponds to symmetric α-stable Lévy motion, in particular, $p = q = 0.5$ in (1.2). In both cases, $\alpha < 1$ and $\beta = 0$, centering is not needed in the series representation (1.2).

The stochastic differential equation (1.1) and its solution have attracted a lot of attention in the case when ξ is 2-stable Lévy motion or, in other words, ξ is Brownian motion. Then the solution to (1.1) is often taken as generating model for speculative prices in financial mathematics. It is a well known fact from Itô calculus that the unique strong solution to (1.1) is given by the so-called *Dooleans-Dade exponential*:

$$\eta_t = \mathrm{e}^{\xi_t} \prod_{0\le s\le t} (1+\Delta\xi_s)\,\mathrm{e}^{-\Delta\xi_s}\,, \quad t\in[0,1]\,,$$

where $\Delta\xi_s = \xi_s - \xi_{s-}$, see Protter [12], Theorem 36. In the proof of the latter result it is shown that the infinite product converges absolutely with probability 1.

With this convention, notice that for $\delta > 0$,

$$\prod_{0\le s\le 1, |\Delta\xi_s|>\delta} (1+\Delta\xi_s) = \mathrm{e}^{\xi_1}\left(\prod_{0\le s\le 1, |\Delta\xi_s|>\delta} (1+\Delta\xi_s)\,\mathrm{e}^{-\Delta\xi_s}\right)\exp\left\{\sum_{0\le s\le 1, |\Delta\xi_s|>\delta} \Delta\xi_s - \xi_1\right\}.$$

This identity is justified by the fact that the Lévy process ξ has only finitely many jump sizes $|\Delta\xi_s| > \delta$ in $[0,1]$, see Sato [16], cf. the series representation (1.3). Letting $\delta \downarrow 0$, the right side product converges absolutely a.s. and the right side sum converges a.s. to ξ_1 because of the Lévy-Itô representation of a Lévy process. Therefore the left-hand side converges a.s. as $\delta\downarrow 0$ as well and the limits on both sides represent η_1. By virtue of the series representation (1.3) and the additional assumptions $\alpha < 1$ or $\beta = 0$ the jump sizes $\Delta\xi_s$ are of the form $\gamma_i\Gamma_i^{-1/\alpha}$. Hence

$$\eta_1 \;=\; \lim_{\delta\downarrow 0} \prod_{i\ge 1:\Gamma_i^{-1/\alpha}>\delta} (1+\gamma_i\,\Gamma_i^{-1/\alpha}) = \prod_{i=1}^{\infty}\left(1+\gamma_i\Gamma_i^{-1/\alpha}\right) \quad \text{a.s.} \tag{1.4}$$

The right hand infinite product will always be interpreted as the limit of the left hand expressions as $\delta\downarrow 0$.

It is the aim of this paper to study the asymptotic behavior of the tails $P(\eta_1 > x)$ and $P(\eta_1 \le -x)$ as $x\to\infty$. In the case of a driving standard Brownian motion ξ (corresponding to the case $\alpha = 2$) with $\mu = 0$ and $\sigma = 1$ the solution of the stochastic differential equation (1.1) is geometric Brownian motion $\eta_t = \exp\{-(t/2)+\xi_t\}$, $0\le t\le 1$. The distributional properties of η_1 and ξ_1 are, of course, very different. In particular, ξ_1 has finite moment generating function on the entire real line, whereas the (lognormal) distribution of η_1 is subexponential, hence a moment generating function does not exist, see Embrechts et al. [6].

In the cases $\alpha < 2$ it turns out that the tail behaviors of ξ_1 and η_1 are quite similar. Recall that for an α-stable distribution $P(|\xi_1| > x) \sim c_\alpha\, x^{-\alpha}$ as $x\to\infty$, for some positive constant c_α, see Zolotarev [17]. The results of this paper indicate that $|\eta_1|$ is regularly varying with the same index α as ξ_1.[2] Thus, up to a slowly varying multiple, the asymptotic behaviors of the tails of

[2] We say that a random variable A is regularly varying with index $\delta \ge 0$ if there exist constants $c_\pm \ge 0$ and a slowly varying function L such that as $x\to\infty$,

$$P(A > x) = c_+\, x^{-\delta}\, L(x) \quad\text{and}\quad P(A\le -x) \sim c_-\, x^{-\delta}\, L(x)\,.$$

For the notion of regularly varying function and distribution and their properties we refer to Bingham et al. [1].

$|\eta_1|$ and $|\xi_1|$ are the same. Although the process η is the "stochastic exponential" of the stable process ξ this does not imply that η_1 is "log-stable" in the straightforward meaning of the word.

The tail behavior of η_1 can be understood by a closer inspection of the infinite product in (1.4). By its definition, we can decompose it into two parts:

$$\prod_{i=1}^{\infty}\Big(1+\gamma_i\Gamma_i^{-1/\alpha}\Big)=\prod_{i\geq 1:\Gamma_i\leq 1}\Big(1+\gamma_i\Gamma_i^{-1/\alpha}\Big)\prod_{i\geq 1:\Gamma_i>1}\Big(1+\gamma_i\Gamma_i^{-1/\alpha}\Big)=P_{\leq 1}\,P_{>1}\,. \tag{1.5}$$

The points (Γ_i,γ_i) constitute a Poisson random measure on the state space $(0,\infty)\times\{1,-1\}$ with mean measure Leb $\times F_\gamma$, where Leb denotes Lebesgue measure on $(0,\infty)$ and F_γ is the distribution of γ, see Resnick [13] for an introduction to Poisson random measures. Since the Poisson points in $P_{\leq 1}$ and $P_{>1}$ come from disjoint sets, the random variables $P_{\leq 1}$ and $P_{>1}$ are independent. Moreover, we will show that $P_{>1}$ has moments of order $\alpha+\epsilon$ for some $\epsilon>0$ and that $P_{\leq 1}$ is regularly varying with index α. Then a result of Breiman [3] is applicable: if A and B are independent non-negative random variables with $EA^{\alpha+\epsilon}<\infty$ and B is regularly varying with index α, then the product AB is regularly varying with the same index. In particular,

$$P(A\,B>x)\sim EA^{\alpha}\,P(B>x)\,,\quad x\to\infty\,. \tag{1.6}$$

Hence we conclude, for example, that

$$P(|\eta_1|>x)\sim E(P_{>1}^{\alpha})\,P(|P_{\leq 1}|>x)\,. \tag{1.7}$$

The tail behavior of stochastic integrals driven by Lévy processes with a regularly varying Lévy measure has recently attracted some attention. If ξ is a Lévy process on $[0,1]$ whose Lévy measure is regularly varying with index $\delta>0$[3] Hult and Lindskog [9] prove that the stochastic integrals $(\int_0^t V_{s-}d\xi_s)_{0\leq t\leq 1}$ are regularly varying with index δ in a functional sense, provided the predictable process V has moment $\delta+\epsilon$ for some $\epsilon>0$. The notion of regular variation for a stochastic process is defined similarly to weak convergence of stochastic processes via its finite-dimensional distributions and a tightness condition, see Hult and Lindskog [8, 10]. The result in [9] can be understood as a stochastic integral analog of Breiman's one-dimensional result. The result is not applicable in our situation: it will be shown in Theorem 2.1 that the integrand η in the linear stochastic differential equation (1.1) does not have a moment of order $\alpha+\epsilon$.

The proof of the regular variation of $P_{\leq 1}$ crucially depends on a representation as a random product

$$P_{\leq 1}\stackrel{d}{=}\prod_{i=1}^{N_1}(1+\gamma_i\,U_i^{-1/\alpha})\,, \tag{1.8}$$

where $(N_t)_{t\geq 0}$ denotes the unit rate Poisson process of the points Γ_i, (U_i) is an iid sequence of uniform $U(0,1)$ random variables, and N, (U_i) and (γ_i) are mutually independent. It is not difficult to see that $1+\gamma\,U^{-1/\alpha}$ is regularly varying with index α. Then it is a well known fact that the product of iid factors $\prod_{i=1}^n|1+\gamma_i\,U_i^{-1/\alpha}|$ is regularly varying with index α for every $n\geq 1$, see Embrechts and Goldie [5]. The slowly varying function in the tail of such products is in general unknown. Moreover, the random index N_1 of the product does not make one's life easier. We will apply a saddle-point approximation in order to get the precise tail asymptotics of the random product on the right side of (1.8). This approximation will show the exact deviation of the tails of η_1 from those of ξ_1.

[3]This means that the tail of the Lévy measure of ξ_1 is regularly varying with index $-\delta$.

The paper is organized as follows. In Section 2 we study the decomposition of η_1 into the factors $P_{\le 1}$ and $P_{>1}$. We will give the main results of this paper (Theorem 2.1 and 2.5) on the asymptotic tail behavior of the solution η of the stochastic differential equation (1.1). Since the result crucially depends on the understanding of random products of iid regularly varying factors we devote Section 3 to the study of those objects.

2. Main result

In this section we formulate the main results of this paper on the regular variation of the solution η to the stochastic differential equation (1.1). We will use the notation of Section 1 without recalling it.

Theorem 2.1. *Let ξ be α-stable Lévy motion on $[0,1]$ and assume that $\alpha < 1$ or $\alpha \in [1,2)$ and ξ_1 is symmetric, i.e., $p = q = 0.5$ in (1.2). Then the solution η to the stochastic differential equation (1.1) has the product representation (1.4) and $|\eta_1|$ is regularly varying with index α. In particular,*

$$P(|\eta_1| > x) \sim C_{>1}\, C_{\le 1}\, x^{-\alpha}\, \mathrm{e}^{2\sqrt{\alpha \log x}}\, (\log x)^{-3/4}\,, \quad x \to \infty\,. \tag{2.1}$$

with the constants

$$\begin{aligned} C_{>1} &= \exp\left\{ \int_0^1 u^{-2} \left(p\,(1+u^{1/\alpha})^\alpha + q\,(1-u^{1/\alpha})^\alpha - 1 \right) dy \right\}\,, \\ C_{\le 1} &= \frac{\mathrm{e}^{K(\alpha)-1}}{\alpha^{3/4}}\, \frac{1}{2\sqrt{\pi}}\,, \\ K(\alpha) &= p \int_0^1 u^{-1} \left((1+u^{1/\alpha})^\alpha - 1 \right) du + q \int_0^1 u^{-1} \left((1-u^{1/\alpha})^\alpha - 1 \right) du\,. \end{aligned}$$

Remark 2.2. The restrictions $\alpha < 1$ and $\alpha \in [0,1)$, $p = q = 0.5$ are needed to ensure that the integral in $C_{>1}$ is well defined. In contrast, the integrals in the definition of $K(\alpha)$ exist for any $\alpha > 0$ and $p, q \ge 0$.

Remark 2.3. It is interesting to observe (see Lemmas 3.1 and 2.4) that, up to a constant multiple, the tails of $|\eta_1|$ and $\prod_{i=1}^{N_1} \Gamma_i^{-1/\alpha} \stackrel{d}{=} \prod_{i=1}^{N_1} U_i^{-1/\alpha}$ are the same, where (U_i) is an iid sequence of uniform random variables on $(0,1)$ independent of N.

Proof. Recall the decomposition $\eta_1 = P_{\le 1} P_{>1}$ from (1.5) and that the factors are independent. We first show that

$$E(P_{>1}^l) < \infty \text{ for every } l > 0. \tag{2.2}$$

Using the order statistics property and the homogeneity of the Poisson process N, we see that for any $T > 0$,

$$C_T(l) = E\left[\prod_{i=N_1+1}^{N_{T+1}} (1 + \gamma_i \Gamma_i^{-1/\alpha})^l \right] = E\left[\prod_{i=1}^{N_T} (1 + \gamma_i (1 + T U_i)^{-1/\alpha})^l \right]\,.$$

where (U_i) is an iid sequence of uniform random variables on $(0,1)$, independent of (γ_i) and N. Straightforward calculation of the right hand expectation yields

$$C_T(l) = \exp\left\{ - \int_0^T \left[1 - p|1 + (1+y)^{-1/\alpha}|^l - q|1 - (1+y)^{-1/\alpha}|^l \right] dy \right\} .$$

A Taylor series argument shows that under the assumptions $\alpha < 1$ or $\alpha \in [1,2)$ and symmetry of γ_i, i.e., $p = q = 0.5$, the integral in the exponent converges as $T \to \infty$ to a finite limit. Then, by Fatou's lemma, (2.2) holds. Moreover, for every $l > 0$, the family $\prod_{i=N_1+1}^{N_{T+1}} (1 + \gamma_i \Gamma_i^{-1/\alpha})^l$, $T > 0$, is uniformly integrable. Hence we also have $C_T(l) \to E(P_{>1}^l)$ for every $l > 0$, in particular for $l = \alpha$.

Set $C_{>1} = E(P_{>1}^\alpha)$. It follows from Lemma 2.4 below that

$$P(|P_{\le 1}| > x) \sim C_{\le 1}\, x^{-\alpha}\, \mathrm{e}^{2\sqrt{\alpha \log x}}\, (\log x)^{-3/4} .$$

Then relation (2.1) is an immediate consequence of Breiman's [3] result, see (1.6) and (1.7). □

Here we provide the auxiliary result mentioned in the proof. Notice that the proof holds for general $\alpha > 0$ and $p, q \ge 0$ with $p + q = 1$.

Lemma 2.4. *Consider the product $P_{\le 1}$ in (1.8) for any $\alpha > 0$. Then*

$$P(|P_{\le 1}| > x) \sim \mathrm{e}^{K(\alpha)}\, P(\Pi_1 > x)\,, \quad x \to \infty\,, \tag{2.3}$$

where the asymptotic behavior of $P(\Pi_1 > x)$ is given in (3.4), the function

$$K(\theta) = E(|1 + \gamma\, U^{-1/\alpha}|^\theta) - \frac{\alpha}{\alpha - \theta} \tag{2.4}$$

is continuous on $(0, \alpha]$ with

$$K(\alpha) = \lim_{\theta \uparrow \alpha} K(\theta) = p \int_0^1 u^{-1} \left((1 + u^{1/\alpha})^\alpha - 1 \right) du + q \int_0^1 u^{-1} \left((1 - u^{1/\alpha})^\alpha - 1 \right) du \,.$$

Proof. We have with $y = \log x$ and $Y_i = \log|1 + \gamma_i U_i^{-1/\alpha}|$

$$P(|P_{\le 1}| > x) = P\left(\sum_{i=1}^{N_1} Y_i > y \right) .$$

It is easily seen that the density of Y is gamma-like in the sense of (4.1), that $\int_{\mathbb{R}} f_Y^s(x)dx < \infty$ for $s \in (1,2)$. Proposition 4.1 with the corresponding notation can be applied. Calculation yields for $\theta < \alpha$,

$$\begin{aligned} \phi_Y(\theta) &= p\, E(1 + U^{-1/\alpha})^\theta + q\, E(U^{-1/\alpha} - 1)^\theta \\ &= E(U^{-\theta/\alpha}) + K(\theta) = \frac{\alpha}{\alpha - \theta} + K(\theta) \,. \end{aligned}$$

The function K is twice continuously differentiable on $[0, \alpha]$. Moreover,

$$\phi_Y'(\theta) = \alpha^{-1}(1 - \theta/\alpha)^{-2} + K'(\theta)\,,$$

The saddle-point equation $\phi_Y'(\theta) = y$ takes on the form

$$\alpha^{-1}(1 - \theta/\alpha)^{-2} = y - K'(\alpha)(1 + o(1))$$

as $\theta \uparrow \alpha$ or, equivalently, $y \uparrow \infty$. Similar arguments yield

$$\theta\,\sigma_S(\theta) = \theta\sqrt{\phi''_Y(\theta)} \sim (\alpha\,y)^{3/4}\,\sqrt{2}\,. \tag{2.5}$$

and

$$\theta = \alpha - \sqrt{\alpha/y}\left[1 + 0.5K'(\alpha)\,y^{-1} + o(y^{-1})\right]\,. \tag{2.6}$$

Then, as $y \to \infty$,

$$\begin{aligned} \mathrm{e}^{-\theta y}\phi_S(\theta) &= \mathrm{e}^{-\theta y}\mathrm{e}^{-(1-\phi_Y(\theta))} \\ &\sim \mathrm{e}^{-\alpha\,y+\sqrt{\alpha y}}\mathrm{e}^{-(1-K(\alpha)-\sqrt{\alpha\,y})} \\ &= \mathrm{e}^{-\alpha y+2\sqrt{\alpha y}}\mathrm{e}^{-1+K(\alpha))}\,. \end{aligned}$$

Now we combine the latter relation with (2.5) and plug them into Proposition 4.1. Also recall that $y = \log x$. Then we obtain from Proposition 4.1

$$P(|P_{\le 1}| > x) \quad \sim \quad \mathrm{e}^{K(\alpha)}\,\mathrm{e}^{-1}\,\frac{1}{2\sqrt{\pi}}\,x^{-\alpha}\,\mathrm{e}^{2\sqrt{\alpha\,\log x}}\,(\alpha\,\log x)^{-3/4}\,.$$

By Lemma 3.1 we may conclude that we finally arrived at the desired relation (2.3). □

It is immediate from the proof of Theorem 2.1 that the tail behavior of η_1 is determined by the tails of $P_{\le 1}$. In what follows, we study the tails of $P_{\le 1}$ more in detail. We observe that

$$P_{\le 1} = \prod_{i\ge 1:\Gamma_i\le 1\,,\gamma_i>0}(1+\Gamma_i^{-1/\alpha})\prod_{i\ge 1:\Gamma_i\le 1\,,\gamma_i<0}(1-\Gamma_i^{-1/\alpha}) = P_+P_-\,.$$

Since P_+ and P_- are constructed from disjoint sets of the points (Γ_i, γ_i) of a Poisson random measure, P_+ and P_- are independent, and one can represent them because of independent thinning as

$$P_\pm \stackrel{d}{=} \prod_{i=1}^{N_\pm}(1 \pm U_i^{-1/\alpha})\,,$$

where N_+ and N_- are Poisson with mean p and q, respectively, and both are independent of the iid sequence (U_i) of uniform random variables on $(0, 1)$. To get the exact asymptotic behavior of $P(P_{\le 1} > x)$ one can again use the saddlepoint approximation for the sum $\log P_+ + \log P_-$ of two independent random variables $\log P_\pm$. Results in this spirit can be found in the literature, see e.g. Feigin and Yashchin [7], Davis and Resnick [4] or Breidt and Davis [2], or one can follow the lines of the saddlepoint approximation for compound Poisson sums advocated in Jensen [11]. Although $\log\max(0, P_-)$ is not compound Poisson, its structure is only slightly different.

Theorem 2.5. *Assume the conditions of Theorem 2.1 hold and that $q > 0$. Then, with the notation of Theorem 2.1,*

$$P(\eta_1 > x) \sim 0.5\,P(|\eta_1| > x) \sim C_{>1}\mathrm{e}^{K(\alpha)}\mathrm{e}^{-1}\frac{1}{4\,\sqrt{\pi}}\,x^{-\alpha}\mathrm{e}^{2\sqrt{\alpha\log x}}\,(\alpha\log x)^{-3/4}\,.$$

Remark 2.6. If $q = 0$ then $\eta_1 > 0$ a.s. If $q > 0$,

$$P(\eta_1 > x) \sim P(\eta_1 \le -x) \sim 0.5\,P(|\eta_1| > x)\,, \tag{2.7}$$

Thus, for $q > 0$, large values of η_1 may occur symmetrically in the asymptotic sense described in (2.7) even though η_1 is not symmetric.

Proof. As in the proof of Theorem 2.1 it follows from Breiman's result and regular variation of $P_{\le 1}$ that

$$P(\eta_1 > x) \sim C_{>1}\, P(P_{\le 1} > x) = C_{>1}\, P(P_+ P_- > x) = P(\log P_+ + \log(P_- I_{\{P_->0\}}) > \log x)\,.$$

Corollary 1 in [7], p. 39, yields the following approximation:

$$P(P_{\le 1} > x) \sim \mathrm{e}^{-\theta\, y}\, \frac{\phi(\theta)}{\theta\sigma(\theta)} \frac{1}{\sqrt{2\pi}}\,,$$

if the so-called Condition B in [7] is fulfilled. Here

$$\phi(\theta) = E\mathrm{e}^{\theta(\log P_+ + \log(P_- I_{\{P_->0\}}))} = E(P_-^\theta I_{\{P_->0\}})\, E(P_+^\theta)\,,$$

θ solves the equation $(\log\phi(\theta))' = y = \log x$ and $\sigma^2(\theta) = (\log\phi(\theta))''$. In our case Condition B can be rewritten

$$\exists h \quad \text{such that } \frac{\sigma(\theta+\epsilon/\sigma(\theta)}{\sigma(\theta)} \to 1 \text{ for } |\epsilon| < h, \text{ when } \theta \uparrow \alpha. \tag{2.8}$$

It will be a consequence of asymptotic behavior of $\log\phi(\theta)$ and its two first derivatives when $\theta \uparrow \alpha$.

We observe that

$$\begin{aligned}
\phi(\theta) &= \mathrm{e}^{-p\,[1-E((1+U^{-1/\alpha})^\theta)]} \sum_{k=1}^\infty P(N_- = 2k)\,[E((U^{-1/\alpha}-1)^\theta)]^{2k} \\
&= \mathrm{e}^{-p\,[1-E((1+U^{-1/\alpha})^\theta)]}\,\mathrm{e}^{-q} \sum_{k=1}^\infty \frac{q^{2k}}{(2k)!}\,[E((U^{-1/\alpha}-1)^\theta)]^{2k} \\
&= \mathrm{e}^{-p\,[1-E((1+U^{-1/\alpha})^\theta)]-q\,[1-E((U^{-1/\alpha}-1)^\theta)]}\,\frac{1}{2}\left[1+\mathrm{e}^{-2\,q\,E((U^{-1/\alpha}-1)^\theta)}\right].
\end{aligned}$$

Then for $\theta < \alpha$, with $K(\theta)$ defined in (2.4),

$$\log\phi(\theta) = -1 + \frac{\alpha}{\alpha-\theta} + K(\theta) - \log 2 + \log(1+\mathrm{e}^{-2\,q\,E((U^{-1/\alpha}-1)^\theta})\,.$$

Notice that $E((U^{-1/\alpha}-1)^\theta) \to \infty$ as $\theta \uparrow \alpha$ and therefore, if $q > 0$,

$$\log\phi(\theta) = -1 - \log 2 + \frac{\alpha}{\alpha-\theta} + K(\alpha)(1+o(1))\,.$$

By similar arguments, the saddlepoint equation is then given by

$$(\log\phi(\theta))' = \alpha^{-1}(1-\theta/\alpha)^{-2} + K'(\alpha)(1+o(1)) = y.$$

The same arguments as in the proof of Lemma 2.4 lead to θ given in (2.6) and to the approximation

$$P(P_{\le 1} > x) \sim \mathrm{e}^{K(\alpha)}\mathrm{e}^{-1}\frac{1}{4\sqrt{\pi}}\, x^{-\alpha}\mathrm{e}^{2\sqrt{\alpha\log x}}\,(\alpha\log x)^{-3/4}\,.$$

Together with Theorem 2.1 this proves the theorem. □

Remark 2.7. *Actually one can also prove with the same technique that*

$$\begin{aligned}
P(P_- > x) &\sim \mathrm{e}^{q\,(K_-(\alpha)-1)}\,(q\,/\alpha^3)^{1/4}\,x^{-\alpha}(\log x)^{-3/4}\mathrm{e}^{2\sqrt{q\,\alpha\log x}}\,, \\
P(P_+ > x) &\sim \mathrm{e}^{p\,(K_+(\alpha)-1)}\,(p\,/\alpha^3)^{1/4}\,x^{-\alpha}(\log x)^{-3/4}\mathrm{e}^{2\sqrt{p\,\alpha\log x}}
\end{aligned}$$

where

$$K_\pm(\alpha) = \int_0^1 u^{-1}\left((1 \pm u^{1/\alpha})^\alpha - 1\right) du\,.$$

3. Random products of independent regularly varying random variables

In this section we study the tail behavior of the random product

$$\Pi_\nu = \prod_{i=1}^M X_i\,, \tag{3.1}$$

where (X_i) is an iid sequence of non-negative regularly varying random variables independent of the Poisson random variable M with mean ν.[4] We start with the case when the X_i's have a pure Pareto distribution:

$$P(X > x) = x^{-\alpha}\,, \quad x \ge 1\,. \tag{3.2}$$

Notice that

$$X \stackrel{d}{=} U^{-1/\alpha} \stackrel{d}{=} \mathrm{e}^Y\,, \tag{3.3}$$

where U is uniform on $(0,1)$ and Y is exponential with mean $1/\alpha$.

Lemma 3.1. *Assume that X has the Pareto distribution* (3.2) *with parameter $\alpha > 0$ and that M is Poisson distributed with mean $\nu > 0$. Then the product Π_ν in* (3.1) *has tail*

$$P(\Pi_\nu > x) \sim \mathrm{e}^{-\nu}\,(\nu/\alpha^3)^{1/4}\,(2\sqrt{\pi})^{-1}\,x^{-\alpha}\,\mathrm{e}^{2\sqrt{\alpha\,\nu\,\log x}}\,(\log x)^{-3/4}\,, \quad x \to \infty\,. \tag{3.4}$$

Proof. First observe that by virtue of (3.3) for $x > 1$,

$$P(\Pi_\nu > x) = P\left(\sum_{i=1}^M \log X_i > \log x\right) = P(\alpha^{-1}\Gamma_M > \log x)\,, \tag{3.5}$$

where M is independent of the sequence (Γ_i). Thus we will study the tail of a compound Poisson sum Γ_M with iid exponential summands Y_i with mean α^{-1}. It will be convenient to use the saddle-point approximation given in Proposition 4.1 of the Appendix with $y = \log x$. Under the conditions of the lemma, the density f_Y is gamma-like in the sense of (4.1) and f^s is integrable in $\mathbb{R}$ for $s \in (1,2)$. Moreover,

$$\phi_Y(\theta) = (1 - \alpha^{-1}\theta)^{-1}\,.$$

Hence the saddle-point equation $\nu\phi_Y'(\theta) = y$ is given by $\nu\alpha^{-1}(1-\theta/\alpha)^{-2} = y$ and has solution

$$\theta = \theta(y) = \alpha(1 - (y\alpha/\nu)^{-1/2})\,.$$

Calculation with the notation of Proposition 4.1 yields

$$\phi_S(\theta) = \mathrm{e}^{-\nu(1-(y\alpha/\nu)^{1/2})}\,, \quad \sigma_S^2(\theta) = 2(y^3/(\nu\alpha))^{1/2}\,.$$

[4]Note that the distribution of the random variable Π_ν has an atom at zero: $P(\Pi_\nu = 0) = P(M = 0) = \mathrm{e}^{-\nu}$.

Proposition 4.1 now yields the desired approximation as $y = \log x \to \infty$

$$
\begin{aligned}
P(\Pi_\nu > x) &\sim \mathrm{e}^{-\alpha\, y\,(1-(y\alpha/\nu)^{-1/2})} \frac{\mathrm{e}^{-\nu(1-(y\alpha/\nu)^{1/2})}}{\alpha(1-(y\alpha/\nu)^{-1/2})\,\sqrt{2(y^3/(\nu\alpha))^{1/2}}} \frac{1}{\sqrt{2\pi}} \\
&\sim \mathrm{e}^{-\nu} \left(\frac{\nu}{\alpha^3}\right)^{1/4} \frac{1}{2\sqrt{\pi}} \mathrm{e}^{-\alpha\, y + 2\sqrt{\alpha\,\nu\, y}}\, y^{-3/4}\,.
\end{aligned}
$$

This concludes the proof of the lemma. □

4. Appendix

4.1. **Saddle-point approximation to compound Poisson sums.** We have frequently made use of a saddle-point approximation result which follows from Jensen [11], Chapter 7, in particular, Formula (7.1.10) on p. 188 and the calculations in Section 7.2, in particular, Theorem 7.2.4.

Consider the compound Poisson sum $S = \sum_{i=1}^M Y_i$, where (Y_i) is an iid sequence of random variables, independent of the Poisson random variable M with mean $\nu > 0$. Write

$$
\phi_Y(h) = E\mathrm{e}^{hY}\,, \quad \phi_S(h) = E\mathrm{e}^{hS} = \mathrm{e}^{-\nu(1-\phi_Y(h))}\,, \quad \sigma_S^2(h) = \nu\,\phi_Y''(h)\,.
$$

We assume that Y has a *gamma-like density*: if Y has a Lebesgue density f_Y with the property that there exist positive constants x_0, α, β and a slowly varying function ℓ such that

$$
f_Y(y) = x^{\beta-1}\,\ell(x)\,\mathrm{e}^{-\alpha\, x}\,, \quad x \ge x_0\,. \tag{4.1}
$$

Proposition 4.1. *Consider the compound Poisson sum S. Assume the distribution of Y has a gamma-like Lebesgue density f_Y with $\int_{\mathbb{R}} f^s(x)dx < \infty$ for some $s \in (1,2)$ and that $E(Y\mathrm{e}^{\alpha Y}) = \infty$. Then the equation $\nu\phi_Y'(\theta) = y$ has a unique solution $\theta = \theta(y)$ for sufficiently large $y > 0$ and*

$$
P(S > y) \sim \mathrm{e}^{-\theta y} \frac{\phi_S(\theta)}{\theta\sigma_S(\theta)} \frac{1}{\sqrt{2\pi}}\,, \quad y \to \infty\,, \tag{4.2}
$$

Proof. Theorem 7.2.4 in Jensen [11] states that (4.2) holds as $\theta \uparrow \alpha = \sup\{h > 0 : \phi_Y(h) < \infty\}$ provided that Y has a gamma-like density with $\int_0^\infty f_Y^s(x)dx < \infty$ for some $s \in (1,2)$. Notice that $\phi_Y'(h) = E(Y\mathrm{e}^{hY})$ is a continuous function increasing to ∞ for $h \uparrow \alpha$. Therefore the saddle point equation $\nu\phi_Y'(\theta) = y$ has a solution for large $y > 0$. In particular, if $y \uparrow \infty$ then $\theta(y) \uparrow \alpha$. □

4.2. **An auxiliary tail bound.** In view of (3.5) we may wonder which values k of the Poisson random variable M with intensity ν are preponderant in the asymptotic. The answer to this natural question is given in the following lemma.

Lemma 4.2. *Let M be a Poisson random variable with mean ν independent of the sequence of the points (Γ_k) of a unit rate Poisson process. For any positive c the relation*

$$
\begin{aligned}
S_c(x) &= \sum_{|k-\sqrt{\nu x}| \le c(\nu x)^{1/4}} P(M = k)\, P(\Gamma_k > x) \\
&\sim \mathrm{e}^{-\nu}\nu^{1/4}(2\sqrt{\pi})^{-1} x^{-3/4} \mathrm{e}^{-x} \mathrm{e}^{2\sqrt{\nu x}} \int_{-c\sqrt{2}}^{c\sqrt{2}} \varphi(y)\, dy\,.
\end{aligned}
$$

holds as $x \to \infty$, where φ is the standard normal density.

Proof. Here we only sketch the proof since the main results do not rely on this lemma. The classical saddlepoint approximation for sums of iid random variables yields as $x \to \infty$ and uniformly for $k \in \mathcal{K} = \{|k - \sqrt{\nu x}| \le c(\nu x)^{1/4}\}$ (see Jensen [11], Section 2.2)

$$P(\Gamma_k > x) \sim \frac{1}{(k-1)!} \mathrm{e}^{-x}\, x^{k-1}\,.$$

Hence, using Stirling's formula,

$$\begin{aligned} S_c(x) &\sim \mathrm{e}^{-\nu}\mathrm{e}^{-x}\, x^{-1} \sum_{k \in \mathcal{K}} \frac{\nu^k}{(k!)^2}\, k\, x^k \\ &\sim \mathrm{e}^{-\nu}(2\pi)^{-1}\mathrm{e}^{-x}\, x^{-1} \sum_{k \in \mathcal{K}} \left(\frac{\nu \mathrm{e}^2 x}{k^2}\right)^k \\ &\sim \mathrm{e}^{-\nu}(2\pi)^{-1}\mathrm{e}^{-x}\, x^{-1} \int_{\sqrt{\nu x}-c(\nu x)^{1/4}}^{\sqrt{\nu x}+c(\nu x)^{1/4}} \left(\frac{\nu \mathrm{e}^2 x}{y^2}\right)^y dy \\ &= \mathrm{e}^{-\nu}(2\pi)^{-1}\mathrm{e}^{-x}\, x^{-1}\mathrm{e}^{2\sqrt{\nu x}} \int_{-c(\nu x)^{1/4}}^{c(\nu x)^{1/4}} \exp\left\{2\,z - 2\,(\sqrt{\nu x}+z)\log(1+z/\sqrt{\nu x})\right\} dz\,. \end{aligned}$$

A Taylor expansion argument yields

$$\begin{aligned} S_c(x) &\sim \mathrm{e}^{-\nu}(2\pi)^{-1}\mathrm{e}^{-x}\, x^{-1}\mathrm{e}^{2\sqrt{\nu x}} \int_{-c(\nu x)^{1/4}}^{c(\nu x)^{1/4}} \exp\left\{2\,z - 2\,(\sqrt{\nu x}+z)(\frac{z}{\sqrt{\nu x}} - 0.5\frac{z^2}{\nu x})\right\} dz \\ &\sim \mathrm{e}^{-\nu}(2\pi)^{-1}\mathrm{e}^{-x}\, x^{-1}\mathrm{e}^{2\sqrt{\nu x}} \int_{-c(\nu x)^{1/4}}^{c(\nu x)^{1/4}} \mathrm{e}^{-z^2/\sqrt{\nu x}}\, dz \\ &= \mathrm{e}^{-\nu}(2\sqrt{\pi})^{-1}\nu^{1/4}\mathrm{e}^{-x}\, x^{-3/4}\mathrm{e}^{2\sqrt{\nu x}} \int_{-c\sqrt{2}}^{c\sqrt{2}} \varphi(v)\, dv\,. \end{aligned}$$

□

4.3. **More general driving Lévy processes.** The decomposition $\eta_1 = P_{\le 1}P_{>1}$ corresponds to the decomposition of the stable driving process ξ into large and small jumps. From the proofs in Section 2 it is immediate that the asymptotic behavior of the tails of η_1 is determined by $P_{\le 1}$, i.e., by the large jumps, as long as one can ensure that $E(P_{>1}^{\alpha+\epsilon}) < \infty$ for some $\epsilon > 0$.

It is feasible to replace ξ by a more general Lévy process with a regularly varying Lévy measure on $\mathbb{R}$ with index $\alpha > 0$. This means that the left and right tails of the measure are regularly varying with index $-\alpha$ satisfying a tail balance condition. An additional condition on the Lévy measure at zero ensures that $E(P_{>1}^{\alpha+\epsilon}) < \infty$, e.g. if ξ is compound Poisson. Here $P_{\le 1}$ and $P_{>1}$ are the factors in the stochastic exponential η_1 corresponding to the jumps of absolute size smaller or larger than 1, respectively. Regular variation of the Lévy measure implies that the large jumps are of the form $\gamma_i \Gamma_i^{-1/\alpha} L(\Gamma_i^{-1/\alpha})$ for a slowly varying function L and sign variables γ_1, see Rosiński [14]. Then results generalizing Lemma 2.4 may ensure that $P_{\le 1}$, hence η_1, is regularly varying with index α. It is still a major problem to determine the slowly varying function in the tails of

η_1. In this paper we have employed the rather subtle tool of saddlepoint approximation which depends on rather strong assumptions such as gamma-like densities.

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Laboratoire de Statistique et Probabilités, Université Paul Sabatier, 118, Route de Narbonne, 31062 Toulouse Cedex 4, France

E-mail address: Serge.Cohen@math.ups.tlse.fr

Laboratory of Actuarial Mathematics, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen, Denmark

E-mail address: mikosch@math.ku.dk