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ACTIVITY RATES WITH VERY HEAVY TAILS

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Abstract. Consider a data network model in which sources begin to transmit at renewal time points $\{S_n\}$. Transmissions proceed for random durations of time $\{T_n\}$ and transmissions are assumed to proceed at fixed rate unity. We study $M(t)$, the number of active sources at time t, a process we term the *activity rate process*, since $M(t)$ gives the overall input rate into the network at time t. Under a variety of heavy-tailed assumptions on the inter-renewal times and the duration times, we can give results on asymptotic behavior of $M(t)$ and the cumulative input process $A(t) = \int_0^t M(s)\,ds$.

1. The model, notation, preliminary results.

Consider an ordinary renewal process $\{S_n, n \geq 0\}$ such that

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i, \quad n \geq 1,$$

and $\{X_n, n \geq 1\}$ is a sequence of iid non-negative random variables with common distribution F. At time point S_n, an event begins of duration T_n, where we assume $\{T_n, n \geq 0\}$ is a sequence of iid non-negative random variables with common distribution G and $\{T_n\}$ is independent of $\{X_n\}$. The event which was initiated at S_n terminates at $S_n + T_n$. In a data network context, S_n would be the time a user initiates a file download and T_n is the download time. In an insurance context, S_n is the time of a disaster or accident and T_n is the length of time during which all insurance claims from this incident are received so that $S_n + T_n$ is the latest time a claim from the nth accident is received.

A process of interest is

$$M(t) = \sum_{n=1}^{\infty} 1_{[S_n \leq t < S_n + T_n]}, \quad t > 0, \tag{1.1}$$

the *number of active downloads at time t* or the *number of active claims at time t*. If $\{S_k\}$ are the points of a homogeneous Poisson process with intensity λ, the point process $K = \sum_{k=1}^{\infty} \epsilon_{(S_k, T_k)}$ is a Poisson random measure with state space $[0,\infty)^2$ and mean measure $\lambda \mathbb{LEB} \times G$; see Resnick (1992), Proposition 4.4.1 on p. 317. Hence (Resnick (1992), Proposition 4.3.1) $M(t) = K(\{(x,y) : 0 \leq x \leq t < x + y\})$ is a Poisson random variable and asymptotic analysis is relatively easy. It is the aim of this paper to deal with the case when the renewal process $\{S_k\}$ is not a Poisson process. This creates many interesting problems many of which we have solved. We build a general theory about M which paralls and supplements the one for M Poisson.

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In particular, we will consider the asymptotic behavior of $M(t)$ and obtain some novel approximations. We also seek to understand the behavior of the cumulative process

$$A(t) = \int_0^t M(u)\,du, \quad t \geq 0, \tag{1.2}$$

which, in the data networks interpretation corresponds to *cumulative work inputted* provided each transmission initiated at renewal epochs proceeds at unit rate. In particular, we consider the very heavy-tailed cases when

$$\bar{F}(x) = 1 - F(x) \sim x^{-\alpha} L_F(x), \quad \bar{G}(x) = 1 - G(x) \sim x^{-\beta} L_G(x), \quad x \to \infty\,,$$

and $0 \leq \alpha,\ \beta \leq 1$ and some slowly varying functions L_F, L_G. Concerning the relationship of F and G, we assume one of the following:

1. COMPARABLE TAILS: $\beta = \alpha$ and $\bar{F}(x) \sim c\,\bar{G}(x)$, $c > 0$, as $x \to \infty$ so that the distribution tails of X_1 and T_1 are essentially the same. To avoid having to keep writing annoying constants, we assume $c = 1$.
2. G HEAVIER-TAILED:
   - (a) $0 < \beta < \alpha < 1$ or, if $\beta = \alpha$, then $\bar{F}(x)/\bar{G}(x) \to 0$ as $x \to \infty$ so that the distribution tail of X_1 is lighter than the distribution tail of T_1.
   - (b) $0 = \beta < \alpha < 1$ so that the distribution tail of T_1 is slowly varying and thus again heavier than that of X_1.
3. F HEAVIER-TAILED: $\beta > \alpha$ so that the distribution tail of X_1 is heavier than the distribution tail of T_1.

The process M has attracted attention in the data network literature since, under the assumption of unit input rate, it corresponds to *traffic per unit time* which, in several data measurement studies, has been empirically identified as self-similar or possessing long range dependence; see Crovella and Bestavros (1996), Garrett and Willinger (1994), Leland et al. (1994), Park and Willinger (2000). Some standard attempts to provide model based explanations of this empirically observed phenomenon use the infinite source Poisson model in which $\{S_n\}$ are homogeneous Poisson points and $\{T_n\}$ are iid with $\bar{G}$ regularly varying with index $\beta > 1$. This leads to M possessing long range dependence in the sense of covariances slowly decreasing with lag. See for example, the standard argument in Resnick (2003) and Park and Willinger (2000). However, the Poisson based model often does not fit collected data well (Guerin et al. (2003)) and file sizes are sometimes modeled with heavier tails than $\beta > 1$ (Arlitt and Williamson (1996), Resnick and Rootzén (2000)), and it is of interest to consider behavior of models with different assumptions. Hence the present study.

In Resnick and Rootzén (2000), queuing is allowed in the sense that inputs are processed by a server and the contents process is studied under the assumption that $\beta < 1$. We have not attempted to model the processing of offered load in this paper. Some of our composition arguments used later have the flavor of ones employed by Meerschaert and Scheffler (2004), Becker-Kern et al. (2004). However, ours are applied to random measures instead of càdlàg functions as in the latter reference. We finally mention that the methods and techniques of this paper are related to work on Poisson shot noise processes with infinite variance stable limits (see Klüppelberg et al. (2003) and the references therein) and to renewal reward processes with infinite variance stable limits; see Pipiras et al. (2004). The novel approach of this paper is to avoid the Poisson assumption on the renewal process which leads to a variety of rather interesting technical difficulties which we could resolve in some cases.

This paper is organized as follows. In Section 1.1 we give some of the notation used throughout the paper. We continue in Section 1.2 with a mean value analysis of $M(t)$ from which we gain preliminary information about the rate of growth of this process as $t \to \infty$ under different distributional assumption on F and G. In Section 1.3 we study the distributional limits of the renewal counting function of the points $\{S_n\}$ and of its inverse function. In Section 2 we study the case of very heavy-tailed F and G when $\alpha, \beta < 1$. In Section 2.1 we start by studying the asymptotic behavior of $M(t)$ as $t \to \infty$ in the Case (1) of comparable tails. It turns out that $M(t)$ converges in distribution to a random variable which is conditionally Poisson distributed. Section 2.2 is devoted to Case (2) of heavier-tailed G. In this case $[\bar{F}(t)/\bar{G}(t)]M(t)$ converges in distribution to some random variable. In Section 3 we study the case of "lighter-tailed" F in the sense that $\alpha = 1$ or $\mathbb{E}X_1 < \infty$. In Section 3.1 we study the case when $\mathbb{E}X_1 < \infty$ and $\beta \in (0,1)$. In this case, $[t\,\bar{G}(t)]^{-1}M(t)$ converges in probability to a deterministic limit. A similar result holds when $\alpha = 1$ and $\mathbb{E}X_1 = \infty$; see Section 3.2. When both T_1 and X_1 have finite mean it is natural to work with a stationary version of M; see Section 3.3 for such a construction. Section 4 deals with the asymptotic behavior of the cumulative work process A. We understand its limit behavior when $\mathbb{E}X_1 < \infty$ and $\beta \in (1,2)$ (infinite variance stable limits; see Section 4.1), when both T_1 and X_1 have finite variance (Brownian motion limits,; see Section 4.2) and when $0 < \alpha, \beta < 1$ (the limit is an integral with respect to the inverse of an infinite variance stable subordinator; see Section 4.3). We conclude in Section 5 with some unresolved problems.

We present in Table 1 a summary of some of the limiting behavior of $M(t)$.

TABLE 1. Limiting behavior of $M(t)$ as $t \to \infty$.

Conditions	Limit behavior of $M(t)$ as $t \to \infty$
$0 < \alpha < 1$ $\bar{F} \sim \bar{G}$	$M(t) \Rightarrow$ random limit.
$0 \le \beta < \alpha < 1$ or $0 < \alpha = \beta < 1$ and $\bar{F} = o(\bar{G})$	$\frac{\bar{F}(t)}{\bar{G}(t)} M(t) \Rightarrow$ random limit.
$0 < \beta < 1$ $\mathbb{E}(X_1) < \infty$	$\frac{M(t)}{t\bar{G}(t)} \Rightarrow$ constant $\frac{M(t) - \text{random centering}}{\sqrt{t\bar{G}(t)}} \Rightarrow$ Gaussian rv
$0 < \beta \le \alpha = 1$ $\mathbb{E}(X_1) = \infty$	$\frac{M(t)}{t\bar{G}(t)\mu(t)} \Rightarrow$ constant $\mu(t)$= truncated 1st moment
$\mathbb{E}(X_1) < \infty$ $\mathbb{E}(T_1) < \infty$	Stationary version of $M(\cdot)$ exists

1.1. **Basic notation.** In this section we introduce some of the basic notation used throughout the paper.

$$\mu_X = \mathbb{E}X_1\,, \mu_T = \mathbb{E}T_1\,, \sigma_X^2 = \mathrm{Var}(X_1)\,, \sigma_T^2 = \mathrm{Var}(T_1)\,, E = [0,\infty) \times (0,\infty]\,,$$

$\mathbb{C}_K^+(S)$ — the space of continuous functions on S with compact support, equipped with the uniform topology

$\mathbb{D}[0,\infty)$ — the Skorokhod space of real-valued càdlàg functions on $[0,\infty)$ equipped with the J_1-topology

$\mathbb{D}([0,\infty,\mathbb{R}^2))$ — the Skorokhod space of $\mathbb{R}^2$-valued càdlàg functions on $[0,\infty)$ equipped with the J_1-topology

$\mathbb{D}^\uparrow[0,\infty)$ — subspace of $\mathbb{D}[0,\infty)$ containing the non-decreasing functions f such that $f(0) = 0$ and $f(\infty) = \lim_{x\to\infty} f(x) = \infty$

ϵ_x — point mass as x

$f^{\leftarrow}$	the right-continuous inverse of a monotone function f $f^{\leftarrow}(x) = \inf\{y : f(y) > x\}$
$\mathbb{LEB}$	Lebesgue measure
$M_+(S)$	the space of non-negative Radon measures on S
$M_p(E)$	the space of Radon point measures on E
ν_γ	a measure on $(0,\infty]$ given by $\nu_\gamma(x,\infty] = x^{-\gamma}$, $\gamma > 0$, $x > 0$.
PRM(μ)	Poisson random measure on E with mean measure μ.
$\Rightarrow$	convergence in distribution

For information on the space $\mathbb{D}[0,\infty)$ we refer to Billingsley (1968), Resnick (1986), Whitt (2002). For information on point processes, random measures and vague convergence, see Kallenberg (1983), Resnick (1987). There one can also find information about the spaces M_+, M_p.

1.2. **Mean value analysis when $\alpha, \beta < 1$.** The mean value asymptotic behavior of $M(t)$ can be obtained essentially from Karamata's Tauberian theorem. Let

$$U(x) = \sum_{n=0}^{\infty} F^{n*}(x), \quad x > 0\,,$$

be the renewal function for the ordinary renewal sequence $\{S_n\}$. Since $0 < \alpha < 1$ we have Feller (1971), p. 471,

$$U(x) \sim \big(\Gamma(1-\alpha)\,\Gamma(1+\alpha)\,\bar F(x)\big)^{-1} \sim c(\alpha)\,x^\alpha / L_F(x), \quad x \to \infty. \tag{1.3}$$

Therefore it follows that, as $t \to \infty$,

$$\begin{aligned} \mathbb{E}M(t) &= \int_0^t U(dx)\,\bar G(t-x) = \int_0^1 \frac{\bar G(t(1-s))}{\bar G(t)} \frac{U(tds)}{U(t)} \Big(\bar G(t) U(t)\Big) \\ &\sim c(\alpha) \int_0^1 (1-s)^{-\beta} \alpha s^{\alpha-1} ds\, \frac{\bar G(t)}{\bar F(t)} = c'(\alpha) \frac{\bar G(t)}{\bar F(t)}. \end{aligned} \tag{1.4}$$

Thus, in Case (1) of comparable tails, $\mathbb{E}M(t)$ converges to a constant while in Case (2), where G is heavier-tailed, $\mathbb{E}M(t) \to \infty$. In Case (3), $\mathbb{E}M(t) \to 0$ and hence $M(t) \stackrel{L_1}{\to} 0$, so Case (3) may be of lesser interest. It corresponds to the case where renewals are so sparse relative to event durations that at any time there is not likely to be an event in progress. We will not consider this case.

1.3. **Behavior of the renewal counting function when $0 < \alpha < 1$.** Define for $x \geq 0$,

$$N(x) = \sum_{n=0}^{\infty} 1_{[S_n \leq x]} = \inf\{n : S_n > x\}\,.$$

Note that $N(x) = S^{\leftarrow}(x)$, where $S = \{S_{[t]}, t \geq 0\}$. Next, let $\sum_k \epsilon_{(t_k, j_k)}$ be PRM$(\mathbb{LEB} \times \nu_\alpha)$ on E. The process

$$X_\alpha(t) = \sum_{t_k \leq t} j_k, \quad t \geq 0,$$

is α-stable Lévy motion with Lévy measure ν_α; see Samorodnitsky and Taqqu (1994). Finally, define the quantile function of F:

$$b(t) \sim (1/\bar F)^{\leftarrow}(t)\,, \quad t \to \infty\,.$$

When $\alpha > 0$, we can always choose b as continuous and strictly increasing function; see for example, Seneta (1976) and Bingham et al. (1987).

A standard result is that the renewal epochs are asymptotically stable. In fact, if

$$X^{(s)}(t) = \frac{S_{[st]}}{b(s)}, \quad t \geq 0,$$

then in $\mathbb{D}[0,\infty)$ we have as $s \to \infty$, (see, for example, Resnick (1986))

$$X^{(s)} \Rightarrow X_\alpha \,. \tag{1.5}$$

Furthermore, the inverse processes also converge in $\mathbb{D}[0,\infty)$:

$$(X^{(s)})^{\leftarrow} \Rightarrow X_\alpha^{\leftarrow} \,.$$

Unpacking this last result, we get

$$\frac{N(b(s)\cdot)}{s} \Rightarrow X_\alpha^{\leftarrow}(\cdot) \tag{1.6}$$

in $\mathbb{D}[0,\infty)$ or, equivalently, $\bar{F}(s)N(s\cdot) \Rightarrow X_\alpha^{\leftarrow}(\cdot)$ or, equivalently,

$$\frac{1}{s}\sum_{n=0}^{\infty} \epsilon_{\frac{S_n}{b(s)}} \Rightarrow X_\alpha^{\leftarrow} \,,$$

in $M_+[0,\infty)$, where we have used $X_\alpha^{\leftarrow}$ to indicate both the monotone function and the measure. The inverse $X_\alpha^{\leftarrow}$ of the stable subordinator X_α, $\alpha \in (0,1)$, is a well-studied process in the Lévy process literature; see, for example, Bertoin (1996), Section III.2, or Sato (1999), Chapter 9.

2. Activity rates when $\alpha, \beta < 1$

2.1. **Case 1: Comparable tails.** Consider Case (1), where the tails of F and G are asymptotically equivalent. We begin with a result which describes the behavior of the counting function of the points $\{(S_k, T_k), k \geq 0\}$.

Define the mapping $T : \mathbb{D}^{\uparrow}[0,\infty) \times M_+(E) \mapsto M_+(E)$ by

$$T(x,m) = \tilde{m} \,, \tag{2.1}$$

where $\tilde{m}$ is defined by

$$\tilde{m}(f) = \iint f(x(u),v)\, m(du,dv) \,, \quad f \in \mathbb{C}_K^+(E) \,.$$

This means that T replaces the usual time scale of m by one determined by the function x. If m is a point measure with representation $m = \sum_k \epsilon_{(\tau_k, y_k)}$, then

$$T(x,m) = \sum_k \epsilon_{(x(\tau_k), y_k)} \,.$$

Theorem 2.1. *Suppose the Case* (1) *assumptions hold with* $\bar{F}(x) \sim \bar{G}(x)$, *as* $x \to \infty$, $0 < \alpha < 1$, *and let* $N_\infty = \sum_k \epsilon_{(t_k, j_k)}$ *be* PRM($\mathbb{LEB} \times \nu_\alpha$). *Then in* $M_p(E)$ *we have as* $s \to \infty$,

$$N_s^* = \sum_{k=0}^{\infty} \epsilon_{\left(\frac{S_k}{b(s)}, \frac{T_k}{b(s)}\right)} \Rightarrow N_\infty^* = T(X_\alpha, N_\infty) = \sum_k \epsilon_{(X_\alpha(t_k), j_k)}. \tag{2.2}$$

Remark 2.2. The distribution of N_∞^* can be specified by giving its Laplace functional. For $f : E \mapsto [0,\infty)$, we have,

$$\mathbb{E}\Big(e^{-N_\infty^*(f)}\Big) = \mathbb{E}\left(\exp\{-\iint_E \big(1 - e^{-f(X_\alpha(s),y)}\big)\, ds\, \nu_\alpha(dy)\}\right).$$

Proof. Begin with the statement (Resnick (1986, 1987)) that in $M_p(E)$ we have as $s \to \infty$,

$$\sum_{k=0}^{\infty} \epsilon_{(\frac{k}{s}, \frac{T_k}{b(s)})} \Rightarrow N_\infty .$$

Since $\{S_k\}$ is independent of $\{T_k\}$, we then get the joint convergence in $\mathbb{D}[0,\infty) \times M_p(E)$, using (1.5),

$$\Big(\frac{S_{[s\cdot]}}{b(s)}, \sum_{k=0}^{\infty} \epsilon_{(\frac{k}{s}, \frac{T_k}{b(s)})}\Big) \Rightarrow \big(X_\alpha, N_\infty\big) .$$

The function T is a.s. continuous at $\big(X_\alpha, N_\infty\big)$. Hence

$$T\Big(\frac{S[s\cdot]}{b(s)}, \sum_{k=0}^{\infty} \epsilon_{(\frac{k}{s}, \frac{T_k}{b(s)})}\Big) = \sum_{k=0}^{\infty} \epsilon_{(\frac{S_{[sk/s]}}{b(s)}, \frac{T_k}{b(s)})} = \sum_{k=0}^{\infty} \epsilon_{(\frac{S_k}{b(s)}, \frac{T_k}{b(s)})} \Rightarrow T\big(X_\alpha, N_\infty\big) .$$

□

From this result, we get the desired result about M, the number of active sources or events.

Corollary 2.3. *The finite-dimensional distributions of the counting function $M(t)$ defined in* (1.1) *satisfy as $s \to \infty$,*

$$M(s\,t) = \sum_{k=0}^{\infty} 1_{[\frac{S_k}{s} \le t < \frac{S_k + T_k}{s}]} \Rightarrow M_\infty(t) = \sum_k 1_{[X_\alpha(t_k) \le t < X_\alpha(t_k) + j_k]} .$$

Conditionally on $X_\alpha^{\leftarrow}$, the limit $M_\infty(t)$ is Poisson with mean $\Lambda(t) = \int_0^t (t-u)^{-\alpha} dX_\alpha^{\leftarrow}(u)$ and hence the generating function of $M_\infty(t)$ is

$$\mathbb{E}\Big(\tau^{M_\infty(t)}\Big) = \mathbb{E}\exp\{(\tau - 1)\,\Lambda(t)\}, \quad \tau \in (0,1) .$$

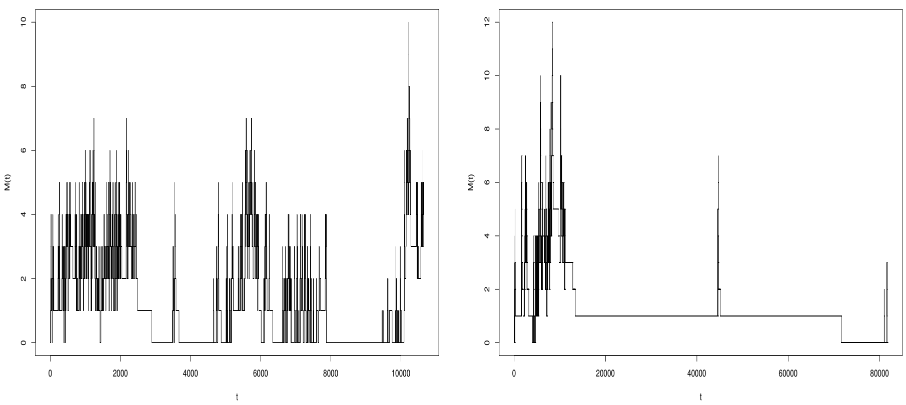

FIGURE 1. A path of the process M for $\alpha = \beta = 0.9$ (left) and $\alpha = \beta = 0.6$ (right).

Proof. Fix $t > 0$. An important point to note is that $\Lambda(t) < \infty$ a.s. To prove this claim we first note that

$$\mathbb{E}X_\alpha^{\leftarrow}(u) = u^\alpha\, \mathbb{E}\big(X_\alpha^{-\alpha}(1)\big) = d_\alpha\, u^\alpha .$$

This results from the self-similar scaling of the Lévy process X_α:

$$\mathbb{E}X_\alpha^{\leftarrow}(u) = \int_0^\infty \mathbb{P}[X_\alpha^{\leftarrow}(u) > x]\, dx = \int_0^\infty \mathbb{P}[u > X_\alpha(x)]\, dx$$

$$= \int_0^\infty \mathbb{P}[u > x^{1/\alpha} X_\alpha(1)]\,dx = u^\alpha\,\mathbb{E}\big(X_\alpha^{-\alpha}(1)\big) = d_\alpha\,u^\alpha.$$

The quantity d_α is finite; see Zolotarev (1986).

We prove $\Lambda(t) < \infty$ a.s. for $t = 1$ as an example of the method. Writing $f(u) = (1-u)^{-\alpha}$, $0 < u < 1$, and observing that $f(0) = 1$, we have

$$\begin{aligned}\int_0^1 f(u)\,dX_\alpha^{\leftarrow}(u) - X_\alpha^{\leftarrow}(1) = \int_0^1 \big(f(u) - f(0)\big)\,dX_\alpha^{\leftarrow}(u) &= \int_0^1\int_0^u f'(s)\,ds\,dX_\alpha^{\leftarrow}(u)\\ &= \int_0^1 \Big(\int_s^1 dX_\alpha^{\leftarrow}(u)\Big)\alpha(1-s)^{-\alpha-1}\,ds\\ &=\alpha\int_0^1 \big(X_\alpha^{\leftarrow}(1) - X_\alpha^{\leftarrow}(s)\big)(1-s)^{-\alpha-1}\,ds.\end{aligned}$$

Taking expectations, we have

$$\mathbb{E}\Big(\int_0^1 f(u)\,dX_\alpha^{\leftarrow}(u)\Big) = d_\alpha + \alpha\,d_\alpha \int_0^1 (1-s^\alpha)(1-s)^{-\alpha-1}\,ds\,.$$

Now, apart from constants, the second term is $\int_0^1\big(1-(1-s)^\alpha\big)\big)s^{-\alpha-1}ds$. The problem for integrability is near 0. But as $s \downarrow 0$, the integrand is asymptotic $\sim \alpha s^{-\alpha}$ which, for $0 < \alpha < 1$, is integrable. This verifies $\Lambda(1) < \infty$ a.s.

Next we prove $M(b(s)t) \Rightarrow M_\infty(t)$ for fixed $t > 0$. As before we choose $t = 1$ in order to demonstrate the method. For positive ϵ, let

$$B_\epsilon = \{(u,v) : u \le 1 < u+v\,, v > \epsilon\}\,,$$

which is relatively compact in E. By virtue of Theorem 2.1, $N_s^*(B_\epsilon) \Rightarrow N_\infty^*(B_\epsilon)$. Also, by monotone convergence and using $\Lambda(1) < \infty$, with probability 1,

$$N_\infty^*(B_\epsilon) \uparrow N_\infty^*(B_0) = M_\infty(1) < \infty$$

From the Converging Together Theorem (Billingsley (1968), Theorem 4.2, p. 25), it suffices to show, for any $\delta > 0$, that

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty} \mathbb{P}[|N_s^*(B_\epsilon) - N_s^*(B_0)| > \delta] = 0\,. \tag{2.3}$$

Observe that

$$N_s^*(B_0) - N_s^*(B_\epsilon) = \sum_k 1_{[S_k \le b(s) < S_k + T_k\,, T_k \le \epsilon b(s)]}\,,$$

By Chebyshev's inequality, it suffices to show that the expectation of this last quantity has a double limit which is zero. We have

$$\begin{aligned}&\sum_k \mathbb{P}[S_k \le b(s) < S_k + T_k\,, T_k \le \epsilon\, b(s)]\\ &= \int_{1-\epsilon}^1 \sum_k F^{k*}(b(s)\,dx)\,\mathbb{P}[1-x < T_k/b(s) \le \epsilon]\\ &= \int_{1-\epsilon}^1 U(b(s)\,dx)\,[\bar G(b(s)\,(1-x)) - \bar G(b(s)\,\epsilon)]\\ &= \int_{1-\epsilon}^1 \frac{\bar G(b(s)(1-x)) - \bar G(b(s)\,\epsilon)}{\bar G(b(s))}\,\frac{U(b(s)\,dx)}{U(b(s))}\,U(b(s))\,\bar G(b(s))\end{aligned}$$

$$\to \quad c(\alpha) \int_{1-\epsilon}^{1} [(1-x)^{-\alpha} - \epsilon^{-\alpha}]\, dx^{\alpha} \quad \text{as } s \to \infty$$

$$\to \quad 0 \quad \text{as } \epsilon \downarrow 0.$$

Thus we proved $M(b(s)\,t) \Rightarrow M_\infty(t)$ for fixed $t > 0$. The convergence of the finite-dimensional distributions follows analogously by an application of Theorem 2.1. Since b can be chosen continuous and strictly increasing, we may rephrase the latter limit relation as $M(s\,t) \Rightarrow M_\infty(t)$. □

Remark 2.4. The above proof rests on a continuous mapping argument applied to the weak convergence relation (2.2). A similar argument ensures the joint convergence

$$(\bar F(s)\, N(s), M(s)) \Rightarrow (X_\alpha^{\leftarrow}(1)\,, M_\infty(1))\ .$$

In particular,

$$\frac{M(s)}{N(s)} \stackrel{d}{=} \bar F(s) \frac{M_\infty(1)}{X_\alpha^{\leftarrow}(1)} (1 + o_{\mathbb{P}}(1))\,.$$

Thus $M(s)/N(s)$ is essentially of the order $\bar F(s) \sim \bar G(s)$. Compare this with the case when $\bar G$ is heavier-tailed than $\bar F$ (Remark 2.8). Then $M(s)/N(s) \sim \bar G(s)$.

2.2. **Case 2: G is heavier-tailed.** In this section we assume the Case (2) conditions $0 \le \beta \le \alpha < 1$ and if $0 < \alpha = \beta$, then $\bar F(t)/\bar G(t) \to 0$, as $t \to \infty$. Recall the definition of the measure ν_γ given by $\nu_\gamma(x, \infty] = x^{-\gamma}$, for $x > 0$, some $\gamma > 0$. For $\gamma = 0$, we interpret this as $\nu_0 = \epsilon_\infty$, i.e., the unit mass at ∞.

As in the previous section we first prove a limit result for the point process generated by the scaled points $(b(s))^{-1}(S_k, T_k)$. Later we use this result in order to derive a distributional limit for $M(s)$ as $s \to \infty$.

Theorem 2.5. *Assume the Case* (2) *conditions. Then in $M_+(E)$ we have*

$$\frac{\bar F(b(s))}{\bar G(b(s))} \sum_{k=0}^{\infty} \epsilon_{\left(\frac{S_k}{b(s)}, \frac{T_k}{b(s)}\right)} \Rightarrow T(X_\alpha\,, \mathbb{LEB} \times \nu_\beta)\,, \tag{2.4}$$

where T was defined in (2.1).

Remark 2.6. Note that the normalization in (2.4) for both S_k and T_k is by the quantile function $b(s) = (1/\bar F)^{\leftarrow}(s)$ for the lighter-tailed distribution. Since this is inappropriate for T_k, it should not be too surprising that the pre-multification by the ratio of the tails (which goes to 0) is necessary.

Proof. Begin by observing that

$$\frac{s\,\bar F(b(s))}{\bar G(b(s))}\, \bar G(b(s)\cdot) \stackrel{v}{\to} \nu_\beta\,,$$

in $M_+(0,\infty]$, where $\stackrel{v}{\to}$ denotes vague convergence in the Borel σ-field of $(0,\infty]$. Hence from Resnick (1987), Example 3.5.7, see also a proof in Resnick (1986), we get

$$\frac{\bar F(b(s))}{\bar G(b(s))} \sum_{k=0}^{[s]} \epsilon_{\frac{T_k}{b(s)}} \Rightarrow \nu_\beta\,.$$

This may be extended as in the proof of Resnick (1987), Proposition 3.21, to show in $M_+(E)$,

$$\frac{\bar F(b(s))}{\bar G(b(s)} \sum_{k=0}^{\infty} \epsilon_{\left(\frac{k}{s}, \frac{T_k}{b(s)}\right)} \Rightarrow \mathbb{LEB} \times \nu_\beta\,.$$

From independence we get the joint convergence in $\mathbb{D}[0,\infty)\times M_+(E)$,

$$\left(\frac{S_{[s\cdot]}}{b(s)}, \frac{\bar{F}(b(s))}{\bar{G}(b(s))}\sum_{k=0}^{\infty}\epsilon_{(\frac{k}{s},\frac{T_k}{b(s)})}\right) \Rightarrow (X_\alpha, \mathbb{LEB}\times\nu_\beta)\,.$$

Now apply the a.s. continuous map T (see (2.1)) to get (2.4). □

From this result, we get the desired result about M, the number of active sources or events.

Corollary 2.7. *The finite-dimensional distributions of the counting function M defined in* (1.1) *satisfy as $s\to\infty$,*

$$\frac{\bar{F}(s)}{\bar{G}(s)}M(s\,t) \Rightarrow \int_0^t (t-u)^{-\beta}\,dX_\alpha^{\leftarrow}(u). \tag{2.5}$$

For any fixed t,

$$\int_0^t (t-u)^{-\beta}\,dX_\alpha^{\leftarrow}(u) \stackrel{d}{=} t^{-\beta\alpha}\int_0^1 (1-u)^{-\beta}\,dX_\alpha^{\leftarrow}(u)\,.$$

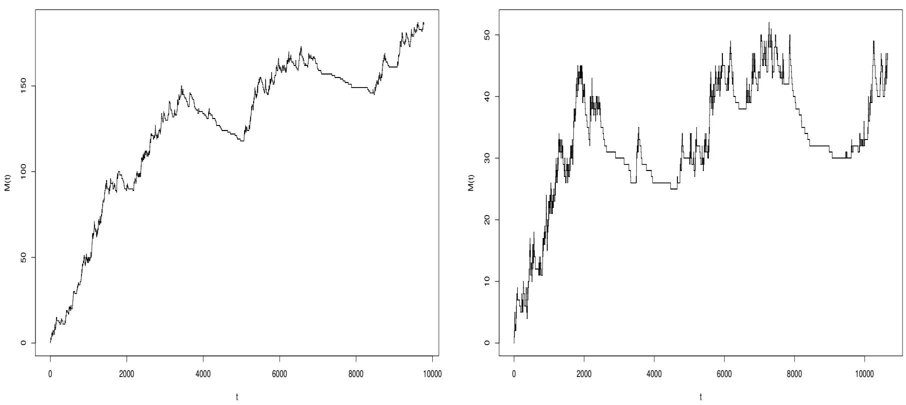

FIGURE 2. A path of the process M for $\alpha=0.9$, $\beta=0.2$ (left) and $\alpha=0.9$, $\beta=0.4$ (right).

Remark 2.8. In particular, for $0=\beta<\alpha<1$, we get

$$\frac{\bar{F}(s)}{\bar{G}(s)}M(s\,t) \Rightarrow X_\alpha^{\leftarrow}(t)\,.$$

Coupled with (1.6) we conclude as $s\to\infty$,

$$\frac{M(s)}{N(s)} \sim \bar{G}(s) \stackrel{\mathbb{P}}{\to} 0\,.$$

Proof. We again consider the case of a fixed $t>0$; the convergence of the finite-dimensional distributions is analogous. We evaluate the convergence in (2.4) on the set $\{(u,v): 0\le u\le t< u+v\}$. After a truncation and Slutsky style argument outlined in (2.3), we get

$$\frac{\bar{F}(b(s))}{\bar{G}(b(s))}M(b(s)\,t) \Rightarrow T(X_\alpha, \mathbb{LEB}\times\nu_\beta)(f)\,, \tag{2.6}$$

where T is the mapping defined in (2.1) and $f(u,v)=1_{[u\le t<u+v]}$. Evaluating the right side, we find

$$T(X_\alpha,\mathbb{LEB}\times\nu_\beta)(f) \;=\; \int\!\!\int f(X_\alpha(v),x)\,dv\,d\nu_\beta(x) = \int_0^{X_\alpha^{\leftarrow}(t)} (t-X_\alpha(v))^{-\beta}\,dv$$

$$= \int_0^t (t-v)^{-\beta}\, dX_\alpha^{\leftarrow}(v)\,,$$

which is the convolution of the measure ν_β and the non-decreasing function $X_\alpha^{\leftarrow}$. The integral also equals

$$t^{-\beta}\int_0^1 (1-v)^{-\beta}\, dX_\alpha^{\leftarrow}(tv) \stackrel{d}{=} t^{-\beta\alpha}\int_0^1 (1-v)^{-\beta}\, dX_\alpha^{\leftarrow}(v)\,.$$

Since b can be chosen continuous and strictly increasing, the $M(b(s)t)$ in (2.6) may be replaced by $M(s\,t)$. This concludes the proof. □

3. Activity rates when $\alpha = 1$ or $\mu_X < \infty$

In this section we collect some results about the activity rates when either μ_X is finite or $\mu_X = \infty$ and $\bar F$ is regularly varying with index -1.

3.1. The case when F has finite mean and $0 < \beta < 1$.

For mean value analysis of $M(t)$, we have from (1.4),

$$\mathbb{E}M(t) = \int_0^t \bar G(t-x)\, U(dx) = \int_0^1 \bar G(t(1-x))\, U(t\,dx)\,.$$

Hence

$$\frac{\mathbb{E}M(s)}{s\,\bar G(s)} \to \int_0^1 (1-x)^{-\beta}\, \mu_X^{-1}\, dx = \mu_X^{-1}(1-\beta)^{-1}\,.$$

This suggests what the correct normalization for $M(t)$ should be.

Proposition 3.1. *Under the assumptions $0 < \mu_X < \infty$ and $\beta \in (0,1)$, the finite-dimensional distributions of M satisfy*

$$\frac{1}{s\,\bar G(s)} M(s\,t) \Rightarrow \mu_X^{-1}\,(1-\beta)^{-1}\, t^{1-\beta}\,, \quad s \to \infty\,.$$

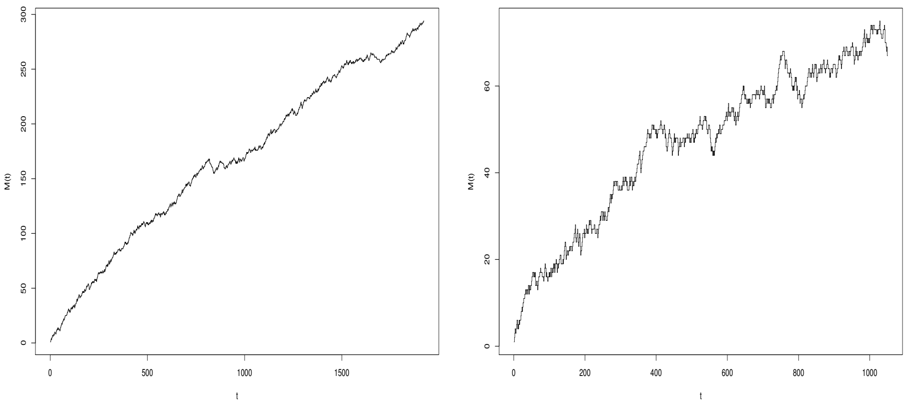

Figure 3. A path of the process M for $\alpha = 2$, $\beta = 0.2$ (left) and $\alpha = 20$, $\beta = 0.5$ (right).

Proof. Since $\beta \in (0,1)$, we have $s\bar G(s) \to \infty$ as $s \to \infty$. Therefore as $s \to \infty$

$$s\,(s\,\bar G(s))^{-1}\, G(s\cdot) \stackrel{v}{\to} \nu_\beta \tag{3.1}$$

in $(0,\infty]$. This is equivalent to (see Resnick (1987), Example 3.5.7, see also proof in Resnick (1986))

$$\frac{1}{s\bar{G}(s)}\sum_{k=0}^{\infty}\epsilon_{\frac{T_k}{s}} \Rightarrow \nu_\beta\,,$$

in $M_+[0,\infty)$, and this can be extended to

$$\frac{1}{s\bar{G}(s)}\sum_{k=0}^{\infty}\epsilon_{(\frac{k}{s},\frac{T_k}{s})} \Rightarrow \mathbb{LEB}\times\nu_\beta\,, \tag{3.2}$$

in $M_+(E)$. The law of large numbers for $\{S_k\}$ together with (3.2) yields as $s\to\infty$,

$$\left(\frac{S_{[s\cdot]}}{s},\frac{1}{s\bar{G}(s)}\sum_{k=0}^{\infty}\epsilon_{(\frac{k}{s},\frac{T_k}{s})}\right) \Rightarrow (\mu_X\cdot\,,\mathbb{LEB}\times\nu_\beta)\,.$$

Therefore, as in earlier sections, for any fixed t, as $s\to\infty$,

$$\frac{M(st)}{s\,\bar{G}(s)} \Rightarrow T(\mu_X\cdot\,,\mathbb{LEB}\times\nu_\beta)(f)\,,$$

where $f(u,v)=1_{[u\le t<u+v]}$. Evaluating the right side, one obtains

$$\int\int 1_{[\mu_X\,v\le t<\mu_X\,v+x]}\,dv\,\nu_\beta(dx)=\int_v^{t/\mu_X}(t-\mu_X\,v)^{-\beta}\,dv=\mu_X^{-1}\,(1-\beta)^{-1}\,t^{1-\beta}\,.$$

Since the limit is deterministic, this implies the convergence of the finite-dimensional distributions in $\mathbb{D}[0,\infty)$. □

This result generalizes equation (2.7) in Resnick and Rootzén (2000). Since the limit is deterministic, Proposition 3.1 should be regarded as the first order behavior of M and suggests there may be second order behavior involving a Gaussian limit as in Theorem 1, p. 760 in Resnick and Rootzén (2000). We have the following result.

Proposition 3.2. *Suppose that $0\le\beta<1$ and $\mu_X<\infty$, and define for $s>0$*

$$W_s(t):=\frac{M(st)-\sum_{k=1}^{N(st)}\bar{G}(st-S_k)}{\sqrt{s\bar{G}(s)}},\quad t\ge 0.$$

Then as $s\to\infty$, the finite dimensional distributions of $W_s(\cdot)$ converge to those of

$$\frac{W_\infty(\cdot)}{\sqrt{\mu_X(1-\beta)}}$$

where $W_\infty(\cdot)$, is a mean-zero Gaussian process with covariance function

$$C(t_1,t_2):=t_2^{1-\beta}-(t_2-t_1)^{1-\beta},\quad 0\le t_1\le t_2.$$

Remark 3.3. The limiting process is self-similar with index $1-\beta$. Except for the case $\beta=0$, W_∞ does not have stationary increments and then it is Brownian motion.

It would be desirable to replace the random centering $\sum_{k=1}^{N(st)}\bar{G}(st-S_k)$ by $\int_0^{st}\bar{G}(st-u)\frac{du}{\mu_X}$ but it is not clear this is in general possible since $N(s)-s/\mu_X$ is of order $\sqrt{s}$ while $\sqrt{s\bar{G}(s)}$ is of order $s^{(1-\beta)/2}$ and $(1-\beta)/2<1/2$.

In the case when S_k are the points of a homogeneous Poisson process the hypothesis of replacing the random centering by the expectation $EM(s\cdot)$ can be made to work by following the lines of the proof in Klüppelberg and Mikosch (1995). (There it is assumed that the shot noise has non-decreasing sample paths which is inessential for the proof in our situation.) In this case, the convergence can be strengthened to a functional CLT (in $(\mathbb{D}[0,\infty),J_1)$) with the limiting process described above.

Proof. We begin by showing one-dimensional convergence and then give the covariance calculation.

Define $\mathcal{S} = \sigma(S_k, k \geq 1)$, for the σ-field generated by the renewal times. Conditionally on $\mathcal{S}$, $M(s)$ is a sum of independent, non-identically distributed Bernoulli random variables

$$M(s) = \sum_{k=1}^{N(s)} 1_{[T_k > s - S_k]}.$$

Thus

$$\frac{\sum_{k=1}^{N(s)} 1_{[T_k > s-S_k]} - \sum_{k=1}^{N(s)} \bar{G}(s - S_k)}{\sqrt{\mathrm{Var}\Big(\sum_{k=1}^{N(s)} 1_{[T_k > s-S_k]} \mid \mathcal{S}\Big)}} \Rightarrow N(0,1),$$

provided the denominator converges to ∞ as $s \to \infty$. To see this note that

$$\begin{aligned}\mathrm{Var}\Big(\sum_{k=1}^{N(s)} 1_{[T_k > s-S_k]} \mid \mathcal{S}\Big) &= \sum_{k=1}^{N(s)} \bar{G}(s - S_k) G(t - S_k) \\ &= \int_0^s \bar{G}(s-u) G(s-u) N(du) \\ &= \int_0^1 \bar{G}(s(1-u)) G(s(1-u)) N(sdu).\end{aligned}$$

Now almost surely, as $s \to \infty$,

$$S_{[s\cdot]}/s \to \mu_X(\cdot),$$

locally uniformly, and therefore also

$$N(s\cdot)/s \to \frac{1}{\mu_X}(\cdot),$$

locally uniformly. Thus it follows that

$$\begin{aligned}\frac{\mathrm{Var}\Big(\sum_{k=1}^{N(s)} 1_{[T_k > s-S_k]} \mid \mathcal{S}\Big)}{s\bar{G}(s)} &= \int_0^1 \frac{\bar{G}(s(1-u))}{\bar{G}(s)} G(s(1-u)) \frac{N(sdu)}{s} \\ &\to \int_0^1 (1-u)^{-\beta} \frac{du}{\mu_X} = \frac{1}{\mu_X(1-\beta)},\end{aligned}$$

if $0 \leq \beta < 1$. Note in this case, that $s\bar{G}(s) \to \infty$.

Thus we conclude that

$$\mathbb{P}[W_s(1) \leq x | \mathcal{S}] \to \mathbb{P}[W_\infty(1)/\sqrt{\mu_X(1-\beta)} \leq x],$$

and taking expectations, we get the same result unconditionally.

For the covariance calculation we again proceed conditionally on $\mathcal{S}$. Suppose $0 \leq t_1 \leq t_2$. Then

$$\mathrm{Cov}(W_s(t_1), W_s(t_2) \mid \mathcal{S}) \quad = \quad \frac{1}{s\bar{G}(s)} \sum_{k=1}^{N(st_1)} \mathrm{Cov}\Big(1_{[T_k > st_1 - S_k]}, 1_{[T_k > st_2 - S_k]} \mid \mathcal{S}\Big)$$

(since the sums for $W_s(t_2)$ involving terms with $N(st_1) < k \leq N(st_2)$ are conditionally independent of terms appearing for $W_s(t_1)$)

$$\begin{aligned} &= \frac{1}{s\bar{G}(s)} \sum_{k=1}^{N(st_1)} \Big(\bar{G}(st_2 - S_k) - \bar{G}(st_1 - S_k)\bar{G}(st_2 - S_k)\Big) \\ &= \int_0^{t_1} \frac{\bar{G}(s(t_2-u))}{\bar{G}(s)} \frac{N(sdu)}{s} - \int_0^{t_1} \frac{\bar{G}(s(t_1-u))}{\bar{G}(s)} \bar{G}(s(t_2-u)) \frac{N(sdu)}{s}\end{aligned}$$

$$\to \int_0^{t_1} (t_2-u)^{-\beta} \frac{du}{\mu_X} = \frac{t_2^{1-\beta} - (t_2-t_1)^{1-\beta}}{\mu_X(1-\beta)}.$$

□

3.2. **The case** $0 < \beta < \alpha = 1$ **with** $\mathbb{E}X_1 = \infty$**.** Then $\int_0^x \bar{F}(u)\,du$ is slowly varying which is the necessary and sufficient condition for relative stability in probability to hold (Feller (1971), p. 236); that is

$$\frac{S_n}{n\mu(n)} \xrightarrow{P} 1, \quad n \to \infty,$$

where

$$\mu(n) = \mathbb{E}\Big(X_1 1_{[X_1 \le b(n)]}\Big).$$

As in (3.1), since $s\bar{G}(s\mu(s)) \to \infty$, this leads to

$$s\left(\frac{1}{s\bar{G}(s\mu(s))}\right) G(s\mu(s)\cdot) \xrightarrow{v} \nu_\beta(\cdot)$$

and therefore we have as $s \to \infty$,

$$\left(\frac{S_{[s\cdot]}}{s\mu(s)}, \frac{1}{\bar{G}(s\mu(s))} \sum_k \epsilon_{\left(\frac{k}{s}, \frac{T_k}{s\,\mu(s)}\right)}\right) \Rightarrow (\cdot, \mathbb{LEB} \times \nu_\beta)\,.$$

Applying composition yields

$$\frac{1}{\bar{G}(s\mu(s))} \sum_k \epsilon_{\left(\frac{S_k}{s\mu(s)}, \frac{T_k}{s\,\mu(s)}\right)} \Rightarrow T\Big(\cdot, \mathbb{LEB} \times \nu_\beta\Big).$$

Finally, we get for $t > 0$,

$$\frac{M(s\mu(s)\,t)}{s\bar{G}(s\mu(s))} \Rightarrow \frac{t^{1-\beta}}{1-\beta}.$$

Since the limit is deterministic the convergence of the finite-dimensional distributions is immediate. This implies the following result which is analogous to Proposition 3.1.

Proposition 3.4. *Under the assumptions* $\mu_X = \infty$ *and* $0 < \beta < \alpha = 1$*, the finite-dimensional distributions of* M *satisfy*

$$\frac{M(s\,t)}{s\bar{G}(s)/\mu(s)} \Rightarrow \frac{t^{1-\beta}}{1-\beta}, \quad s \to \infty\,.$$

3.3. **The case when** F **and** G **have finite mean.** Then we have from the Key Renewal Theorem

$$\mathbb{E}M(t) = \int_0^t \bar{G}(t-x)U(dx) \to \frac{\mu_T}{\mu_X}\,.$$

This suggests that there exists a stationary version of the process M. We make this precise in what follows.

As $s \to \infty$,

$$\sum_{k=0}^{\infty} \epsilon_{S_k - s} \Rightarrow \sum_{k=0}^{\infty} \epsilon_{S_k^{(0)}},$$

in $M_p([0,\infty))$, where $\{S_k^{(0)}, k \ge 0\}$ is the stationary renewal sequence, so that

$$\mathbb{P}[S_0^{(0)} > x] = \frac{1}{\mu_X} \int_x^\infty \bar{F}(u)du,$$

(Resnick (1992)). Since $\{T_k\}$ is independent of $\{S_k\}$ we get

$$\sum_{k=0}^{\infty} \epsilon_{(S_k - s, T_k)} \Rightarrow \sum_{k=0}^{\infty} \epsilon_{(S_k^{(0)}, T_k)},$$

in $M_p([0,\infty)^2)$, where $\{T_k\}$ is independent of $\{S_k^{(0)}\}$. We therefore conclude that as $s \to \infty$, for any $t > 0$

$$\begin{aligned} &\sum_{k=0}^{\infty} \epsilon_{(S_k - s, T_k)}(\{(u,v) : 0 \le u \le t < u+v\}) \\ \Rightarrow & \sum_{k=0}^{\infty} \epsilon_{(S_k^{(0)}, T_k)}(\{(u,v) : 0 \le u \le t < u+v\}) \\ = & \sum_{k=0}^{\infty} 1_{[S_k^{(0)} \le t < S_k^{(0)} + T_k]}. \end{aligned} \tag{3.3}$$

Note that the left side of (3.3) is not all of $M(t+s)$, since from (3.3) we only have

$$\sum_{k=1}^{\infty} 1_{[s \le S_k \le t+s \le S_k + T_k]}.$$

The difference between this and $M(t+s)$ has expectation

$$\int_0^s U(du)\bar{G}(t+s-u) \to \frac{1}{\mu_T}\int_0^\infty \bar{G}(u+t)du = \frac{1}{\mu_T}\int_t^\infty \bar{G}(u)du.$$

However, the way to construct a stationary version of M is clear: start with $\{S_k\}$ a stationary renewal sequence on all of $\mathbb{R}$ and define for $t > 0$

$$M^{(0)}(t) = \sum_k 1_{[S_k \le t < S_k + T_k]}.$$

We observe, additionally, that even when the renewal process is a Poisson process, M is only stationary if one defines the Poisson process on all of $\mathbb{R}$.

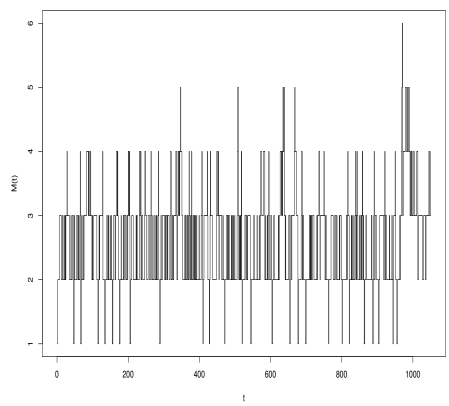

FIGURE 4. A path of the process M for $\alpha = 20$, $\beta = 2$.

4. The cumulative work process

In the Introduction we mentioned that the workload process $A(t)$ is of major interest in the network context. The following decomposition of $A(t)$ will be useful:

$$\begin{aligned}
A(t) &= \int_0^t M(s)\,ds = \sum_{i=1}^{N(t)} \min(T_i, t-S_i) \\
&= \sum_{i=1}^{N(t)} T_i 1_{[S_i+T_i\le t]} + \sum_{i=1}^{N(t)} (t-S_i)\, 1_{[S_i+T_i>t]} \\
&= I_1 + I_2 \qquad (4.1)\\
&= \sum_{i=1}^{N(t)} T_i - \sum_{i=1}^{N(t)} T_i 1_{[S_i+T_i>t]} + \sum_{i=1}^{N(t)} (t-S_i)\, 1_{[S_i+T_i>t]} \\
&= I_{11} - I_{12} + I_2\,. \qquad (4.2)
\end{aligned}$$

4.1. The case $\mu_X < \infty$, $\beta \in (1,2)$.

Define the quantile function of G:

$$\sigma(t) \sim (1/\bar G)^{\leftarrow}(t)\,, \quad t \to \infty\,.$$

We always choose $\sigma(t)$ continuous and strictly increasing.

Theorem 4.1. *Assume $\beta \in (1,2)$. Moreover, assume that the renewal process N is non-arithmetic and that either $\bar F$ is regularly varying with index $-\alpha \in [-2,-1)$ or $\sigma_X^2 < \infty$.*

(1) *Suppose $\bar F$ is regularly varying and either*
   (a) *$\alpha > \beta$ or*
   (b) *$\alpha = \beta$ and $\bar F(x) = o(\bar G(x))$ or*
   (c) *$\sigma_X^2 < \infty$.*

   Set
$$A_s(u) = \sigma(s)^{-1}\Big(A(su) - su\mu_T/\mu_X\Big), \quad u \ge 0.$$
   Then as $s \to \infty$,
$$A_s(\cdot) \Rightarrow \mu_X^{-1/\beta}\, X_\beta(\cdot)\,, \qquad (4.3)$$
   where X_β is a β-stable spectrally positive Lévy motion on $[0,\infty)$.

(2) *If $\bar F$ is regularly varying $\alpha = \beta$ and $\bar F(x) \sim c\,\bar G(x)$, then (4.3) holds, where X_β is β-stable Lévy motion with skewness parameter (4.6).*

(3) *If $\bar F$ is regularly varying and $\alpha < \beta$ or $\alpha = \beta$ and $\bar G(x) = o(\bar F(x))$, then, as $s \to \infty$*
$$(b(s))^{-1}\,[A(\cdot\, s) - s\,(\cdot)\,\mu_T/\mu_X] \Rightarrow \mu_X^{-1/\alpha}\, X_\alpha(\cdot)\,,$$
   where X_α is spectrally negative β-stable Lévy motion.

Here $\Rightarrow$ refers to convergence of the finite-dimensional distributions; it cannot be strengthened to weak convergence in the Skorokhod space $(\mathbb{D}[0,\infty), J_1)$ since A has continuous sample paths and the limiting process has jumps.

Proof. We have for $\gamma \in (0,1)$

$$\begin{aligned}
(\sigma(t))^{-1}\mathbb{E}I_2 &= (\sigma(t))^{-1}\int_0^t (t-x)\,\bar G(t-x)\,U(dx) \\
&\le (\sigma(t))^{-\gamma}\int_0^t (t-x)\,\bar G(t-x)\,(\sigma(t-x))^{-1+\gamma}\,U(dx)
\end{aligned}$$

$$\sim \quad \mu_X^{-1}\,(\sigma(t))^{-\gamma} \int_0^\infty x\,\bar G(x)\,(\sigma(x))^{-1+\gamma}\,dx\,.$$

The right hand integral is finite for small γ. We conclude that $\mathbb{E}I_2 = o(\sigma(t))$.

We have

$$\mathbb{E}I_{12} \quad = \quad \int_0^t \mathbb{E}[T_1 1_{[T_1>t-x]}]\,U(dx)\,.$$

By Karamata's Theorem (see Bingham et al. (1987)),

$$\mathbb{E}[T_1 1_{[T_1>t]}] \sim (\beta-1)^{-1}\,t\,\mathbb{P}[T_1>t]\,.$$

Mohan (1976) proved for a non-arithmetic renewal process N that $U(t)-\mu_X^{-1}t=\widetilde U(t)$ is regularly varying with index $2-\alpha$ if $\bar F$ is regularly varying with index $-\alpha$, $\alpha\in(1,2]$, and $\widetilde U(t)\to c$ for some positive c if $\sigma_X^2<\infty$ (cf. Resnick (1992), p. 243).

Hence, for $\bar F$ regularly varying with index $-\alpha\in(-2,-1)$,

$$\begin{aligned}
\mathbb{E}I_{12} \quad &= \quad \mu_X^{-1}\int_0^t \mathbb{E}[T_1 1_{[T_1>x]}]\,dx + \int_0^1 \frac{\mathbb{E}[T_1 1_{[T_1>t(1-x)]}]}{\mathbb{E}[T_1 1_{[T_1>t]}]}\,\frac{\widetilde U(t\,dx)}{\widetilde U(t)}\,\left(\widetilde U(t)\,\mathbb{E}[T_1 1_{[T_1>t]}]\right)\\
&\sim \quad c(\beta)\,t^2\,\mathbb{P}[T_1>t] + c(\alpha,\beta)\int_0^1 (1-x)^{1-\beta}\,x^{1-\alpha}\,dx\left(\widetilde U(t)\,t\,\mathbb{P}[T_1>t]\right)\\
&= \quad o(\sigma(t))\,.
\end{aligned}$$

Now consider the case when $\sigma_X^2<\infty$ or $\sigma_X^2=\infty$ and $\bar F$ is regularly varying with index -2. Then, as above,

$$\mathbb{E}I_{12} \quad \sim \quad c(\beta)\,t^2\,\mathbb{P}[T_1>t] + \int_0^t \mathbb{E}[T_1 1_{[T_1>t-x]}]\,\widetilde U(dx)\,.$$

We integrate by parts:

$$\begin{aligned}
\int_0^t \mathbb{E}[T_1 1_{[T_1>t-x]}]\,\widetilde U(dx) \quad &= \quad \mathbb{E}[T_1 1_{[T_1>0]}]\widetilde U(t) - \mathbb{E}[T_1 1_{[T_1>t]}]\widetilde U(0) - \int_0^t \widetilde U(x)\,d\mathbb{E}[T_1 1_{[T_1>t-x]}]\\
&= \quad \mathbb{E}[T_1 1_{[T_1>0]}]\widetilde U(t) - \int_0^t \widetilde U(x)\,\mathbb{P}[T_1>t-x]\,dx\,.
\end{aligned}$$

Since $\widetilde U$ is slowly varying and $\mu_T<\infty$ it also follows in this case that $\mathbb{E}I_{12}=o(\sigma(t))$.

Notice that

$$\begin{aligned}
I_{11}-\frac{\mu_T}{\mu_X}\,t \quad &= \quad \sum_{i=1}^{N(t)}(T_i-\mu_T)+\mu_T\;(N(t)-t/\mu_X)\\
&= \quad \sum_{i=1}^{N(t)}(T_i-\mu_T)+\mu_T\left(N(t)-\frac{S_{N(t)}}{\mu_X}\right)+O(X_{N(t)+1})\\
&= \quad \sum_{i=1}^{N(t)}\left(T_i-\frac{\mu_T}{\mu_X}\,X_i\right)+O(X_{N(t)+1})\,.
\end{aligned}$$

From the above decomposition we conclude that

$$(\sigma(t))^{-1}[A(t\cdot)-t\,\cdot\mu_T/\mu_X] = (\sigma(t))^{-1}\sum_{i=1}^{N(t\cdot)}\left(T_i-\frac{\mu_T}{\mu_X}\,X_i\right)+o_P(1)\,,$$

which equation holds for the finite-dimensional distributions.

Since $\bar{G}$ is regularly varying we have

$$\mathbb{P}[T_i - \frac{\mu_T}{\mu_X} X_i > x] \quad \sim \quad \bar{G}(x)\,, \tag{4.4}$$

(cf. Resnick (1986), Lemma 4.2). If $\bar{F}$ is regularly varying with positive index we also have

$$\mathbb{P}[T_i - \frac{\mu_T}{\mu_X} X_i \le -x] \quad \sim \quad \bar{F}(\mu_X x/\mu_T) \sim (\mu_T/\mu_X)^\alpha\, \bar{F}(x)\,.$$

If $\sigma_X^2 < \infty$

$$\mathbb{P}[T_i - \frac{\mu_T}{\mu_X} X_i \le -x] \le \mathbb{P}[-\frac{\mu_T}{\mu_X} X_i \le -x] = o(\bar{G}(x))\,. \tag{4.5}$$

Regular variation of $\bar{G}$ and the conditions (4.4) and (4.5) imply that in $(\mathbb{D}[0,\infty), J_1)$ (see Gikhman and Skorohod (1969), Chapter IX.6)

$$\begin{aligned}
(\sigma(t))^{-1} \sum_{i=1}^{[t\cdot]} \left(T_i - \frac{\mu_T}{\mu_X} X_i\right) &= (\sigma(t))^{-1} \sum_{i=1}^{[t\cdot]} (T_i - \mu_T) + (\sigma(t))^{-1} \sum_{i=1}^{[t\cdot]} \left(\mu_T - \frac{\mu_T}{\mu_X} X_i\right) \\
&= (\sigma(t))^{-1} \sum_{i=1}^{[t\cdot]} (T_i - \mu_T) + o_P(1) \\
&\Rightarrow X_\beta(\cdot)\,,
\end{aligned}$$

where X_β is a spectrally positive β-stable Lévy motion. Notice that (4.5) also holds when $\alpha > \beta$ or $\alpha = \beta$ and $\bar{F}(x) = o(\bar{G}(x))$. Hence the same result applies.

If $\alpha = \beta$ and $\bar{F}(x) \sim c\,\bar{G}(x)$, the corresponding limit theory yields that

$$(\sigma(t))^{-1} \sum_{i=1}^{[t\cdot]} \left(T_i - \frac{\mu_T}{\mu_X} X_i\right) \Rightarrow X_\beta\,,$$

where X_β is a β-stable Lévy motion with skewness parameter

$$1 - 2(1 + c^{-1}(\mu_X/\mu_T)^\alpha)^{-1} \in [-1,1]\,. \tag{4.6}$$

If $\alpha < \beta$ or if $\alpha = \beta$ and $\bar{G}(x) = o(\bar{F}(x))$, then

$$(b(t))^{-1} \sum_{i=1}^{[t\cdot]} \left(T_i - \frac{\mu_T}{\mu_X} X_i\right) \Rightarrow X_\alpha$$

for a spectrally negative α-stable Lévy motion.

Therefore

$$\left(\frac{N(t\cdot)}{t}\,, (b(t))^{-1} \sum_{i=1}^{[t\cdot]} (T_i - \mu_T)\right) \Rightarrow (\mu_X^{-1}\cdot\,, X_\alpha)$$

in $\mathbb{D}([0,\infty), \mathbb{R}^2)$. By a continuous mapping argument we conclude that

$$(b(t))^{-1} \left(A(t\cdot) - \frac{\mu_T}{\mu_X} \cdot t\right) \Rightarrow \mu_X^{-1/\alpha}\, X_\alpha(\cdot)\,,$$

where $\Rightarrow$ refers to the convergence of the finite-dimensional distributions.

The cases when X_β appears in the limit is completely analogous and therefore omitted. □

4.2. **The case when X_1 and T_1 have finite variance.** Under the assumptions $\sigma_T^2 < \infty$ and $\sigma_X^2 < \infty$, the Key Renewal Theorem yields

$$\begin{aligned} \mathbb{E}I_2 &= \int_0^t (t-x)\,\bar G(t-x)\,U(dx) \to \mu_X^{-1}\int_0^\infty x\,\bar G(x)\,dx < \infty\,, \\ \mathbb{E}I_{12} &= \int_0^t \mathbb{E}[T_1 1_{[T_1>t-x]}]\,U(dx) \to \mu_X^{-1}\int_0^\infty \mathbb{E}[T_1 1_{[T_1>x]}]\,dx < \infty\,. \end{aligned}$$

On the other hand, similar arguments as in Section 4.1 show that

$$t^{-1/2}(I_{11} - \frac{\mu_T}{\mu_X}t) \;=\; t^{-1/2}\sum_{i=1}^{N(t)}\Big(T_i - \frac{\mu_T}{\mu_X}X_i\Big) + o_P(1)\,.$$

Following the ideas of the proof on p. 108 in Embrechts et al. (1997), it is now easy to derive the following result:

Proposition 4.2. *Assume $\sigma_T^2 < \infty$ and $\sigma_X^2 < \infty$. Then*

$$t^{-1/2}\left(A(t\cdot) - \frac{\mu_T}{\mu_X}t\cdot\right) \Rightarrow \left([\sigma_T^2 + (\mu_T\sigma_X/\mu_X)^2]\,\mu_X^{-1}\right)^{1/2} B(\cdot)\,,$$

where B is standard Brownian motion and $\Rightarrow$ refers to convergence of the finite-dimensional distributions.

4.3. **The case $0 < \beta, \alpha < 1$.** Observe that

$$\begin{aligned} \frac{\mathbb{E}I_2}{\bar G(t)\,U(t)} &= t\int_0^1 (1-x)\,\frac{\bar G(t(1-x))}{\bar G(t)}\,\frac{U(t\,dx)}{U(t)} \\ &\sim c(\alpha)\,t\,, \\ \frac{\mathbb{E}I_1}{\mathbb{E}[T_1 1_{[T_1\le t]}]\,U(t)} &= \int_0^1 \frac{\mathbb{E}[T_1 1_{[T_1\le t(1-x)]}]}{\mathbb{E}[T_1 1_{[T_1\le t]}]}\,\frac{U(t\,dx)}{U(t)} \sim c(\alpha,\beta)\,. \end{aligned}$$

This means that $\mathbb{E}I_2$ and $\mathbb{E}I_1$ are of the same order $t\bar G(t)U(t) \sim t^{1-\beta+\alpha}L(t)$. The term I_{11} is of order $t^{\alpha/\beta}$ (see Proposition 4.3 below) and hence is either of larger order than $\mathbb{E}I_1$ when $\alpha > \beta$, or of smaller order when $\alpha < \beta$. The analysis of $A(t)$ cannot be based just on I_{11} in this case; one has to understand the interplay between I_1 and I_2.

Since N and (T_i) are independent,

$$(t^{-1}N(b(t)\cdot)\,, (\sigma(t))^{-1}\sum_{i=1}^{[t\cdot]} T_i \Rightarrow (X_\alpha^{\leftarrow}(\cdot), X_\beta(\cdot)) \tag{4.7}$$

in $\mathbb{D}([0,\infty),\mathbb{R}^2)$. Then by a continuous mapping argument

$$(\sigma(t))^{-1}\sum_{i=1}^{N(b(t)\cdot)} T_i \Rightarrow X_\beta(X_\alpha^{\leftarrow}(\cdot))\,.$$

Since $b(t)$ and $\sigma(t)$ can be chosen as continuous functions, we can change time:

$$(\sigma(b^{\leftarrow}(t)))^{-1}\sum_{i=1}^{N(t\cdot)} T_i \Rightarrow X_\beta(X_\alpha^{\leftarrow}(\cdot))\,,$$

in $\mathbb{D}[0,\infty)$. Now observe that $\sigma(b^{\leftarrow}(t)) \sim \sigma(1/\bar F(t))$.

Proposition 4.3. *Assume* $0 < \beta, \alpha < 1$. *Then*

$$(\sigma(1/\bar{F}(t)))^{-1} \sum_{i=1}^{N(t\cdot)} T_i \Rightarrow X_\beta(X_\alpha^{\leftarrow}(\cdot)),$$

where the convergence is in $(\mathbb{D}[0,\infty), J_1)$, X_β *is* β*-stable spectrally positive Lévy motion on* $[0,\infty)$ *and* $X_\alpha^{\leftarrow}$ *is the inverse process to the* α*-stable Lévy motion defined in Section 1.3, and both processes are independent.*

Despite this result, it turns out that $A(t)$ needs a different normalization and we must proceed by relying on Theorem 2.5.

Theorem 4.4. *Suppose the Case* (2) *assumptions hold:* $0 \le \beta \le \alpha < 1$ *and if* $\alpha = \beta$, *then* $\bar{F}(s)/\bar{G}(s) \to 0$, *as* $s \to \infty$. *Then* A *satisfies the relation*

$$\frac{\bar{F}(s)}{s\bar{G}(s)} A(st) \Rightarrow \int_0^t \frac{(t-u)^{1-\beta}}{1-\beta} dX_\alpha^{\leftarrow}(u), \quad t \ge 0, \tag{4.8}$$

in $(\mathbb{D}[0,\infty), J_1)$.

Remark 4.5. The convergence in (4.8) is the result one expects by integrating to the limit in (2.5). It suggests that Corollary 2.7 may hold in the M_1-topology (Whitt (2002)) since integration is continuous in that topology. However, we have not been able to verify this.

Proof. We start by verifiying the convergence of the finite-dimensional distributions and focus on the case of a fixed t. We again decompose $A(t) = I_1 + I_2$ as defined in (4.1). The idea is to express both I_1 and I_2 as functions of the random measure in (2.4).

Fix $\delta > 0$. The map

$$m \mapsto \left(\iint_{\substack{0 \le u \le t, \delta < v \\ u+v \le t}} v\, m(du, dv), \iint_{\substack{0 \le u \le t, \delta < v \\ u+v > t}} (t-u)\, m(du, dv) \right) \tag{4.9}$$

from $M_+(E) \mapsto [0,\infty)^2$ is continuous at measures in

$$\Lambda := \Big\{ m \in M_+(E) : m(\{0 \times [0,\infty)\}) = m(\{(u,v) : u+v = t, v \ge \delta\}) \\ = m([0,\infty) \times \{\delta\}) = m(\{t\} \times [\delta, 0)) = 0 \Big\}.$$

To see this, write, for instance

$$\iint_{\substack{0 \le u \le t, \delta < \delta \\ u+v \le t}} v\, m(du, dv) = \iint_{[0,\infty)^2} 1_{[0,t]}(u)\, v\, 1_{\{u \le t, \delta < v, u+v \le t\}}(u,v)\, m(du, dv) \\ = \iint f(u,v)\, m(du, dv),$$

and proceed as in the proof of the Helly-Bray lemma. An almost identical argument applies to the continuity of the second integral. Referring to Theorem 2.5, note that,

$$\mathbb{P}[T(X_\alpha, \mathbb{LEB} \times \nu_\beta) \in \Lambda^c] = 0.$$

Therefore by continuous mapping, as $s \to \infty$,

$$\begin{aligned} &(I_{1,\delta}(t), I_{2,\delta}(t)) \\ = \;& \frac{\bar{F}(b(s))}{\bar{G}(b(s))} \left(\sum_{k=1}^{N(b(s)t} \frac{T_k}{b(s)} 1_{[\frac{T_k}{b(s)} \le t - \frac{S_k}{b(s)}, \frac{T_k}{b(s)} \ge \delta]}, \sum_{k=1}^{N(b(s)t} \big(t - \frac{S_k}{b(s)}\big) 1_{[\frac{T_k}{b(s)} > t - \frac{S_k}{b(s)}, \frac{T_k}{b(s)} \ge \delta]} \right) \\ \Rightarrow \;& (I_{1,\delta}^{(\infty)}(t), I_{2,\delta}^{(\infty)}(t)) \end{aligned}$$

$$= \left(\iint_{\substack{u\le t, v\ge \delta\\ u+v\le t}} vT(X_\alpha, \mathbb{LEB}\times \nu_\beta)(du,dv), \iint_{\substack{u\le t, v\ge \delta\\ u+v> t}} (t-u)T(X_\alpha, \mathbb{LEB}\times \nu_\beta)(du,dv)\right).$$

As $\delta \downarrow 0$,

$$(I^{(\infty)}_{1,\delta}(t), I^{(\infty)}_{2,\delta}(t)) \Rightarrow \left(\int_0^t \frac{\beta}{1-\beta}(t-u)^{1-\beta} dX_\alpha^{\leftarrow}(u), \int_0^t (t-u)^{1-\beta} dX_\alpha^{\leftarrow}(u)\right).$$

Note the sum of the last two terms is

$$\int_0^t \frac{1}{1-\beta}(t-u)^{1-\beta} dX_\alpha^{\leftarrow}(u),$$

as claimed in the statement (4.8).

So it remains to show for any $\eta > 0$,

$$\lim_{\delta\downarrow 0}\limsup_{s\to\infty} \mathbb{P}[|I_{j,\delta}(t) - I_j| > \eta] = 0, \quad j=1,2. \tag{4.10}$$

For $j=1$ the probability is

$$\begin{aligned}
P\Big[&\frac{\bar F(b(s))}{\bar G(b(s))}\sum_{k=1}^{N(b(s)t}\frac{T_k}{b(s)}1_{[\frac{T_k}{b(s)}\le t-\frac{S_k}{b(s)}, \frac{T_k}{b(s)}\le \delta]} > \eta\Big]\\
&\le \eta^{-1}\frac{\bar F(b(s))}{\bar G(b(s))}\mathbb{E}\left(\sum_{k=1}^{N(b(s)t}\frac{T_k}{b(s)}1_{[\frac{T_k}{b(s)}\le t-\frac{S_k}{b(s)}, \frac{T_k}{b(s)}\le \delta]}\right) \qquad \text{(Chebyshev)}\\
&= \eta^{-1}\frac{\bar F(b(s))}{\bar G(b(s))}\int_0^{b(s)t}\mathbb{E}\Big(\frac{T_1}{b(s)}1_{[T_1\le b(s)t-u, T_1\le b(s)\delta]}\Big)U(du)\\
&= \eta^{-1}\int_0^t \frac{\mathbb{E}\Big(T_1 1_{[T_1\le ((t-y)\wedge \delta) b(s)]}\Big)}{b(s)\bar G(b(s))}\bar F(b(s))U(b(s)dy)\\
&= \int_{t-\delta}^t \frac{\mathbb{E}\Big(T_1 1_{[T_1\le ((t-y) b(s)]}\Big)}{b(s)\bar G(b(s))}\bar F(b(s))U(b(s)dy)\\
&\qquad + \eta^{-1}\int_0^{t-\delta} \frac{\mathbb{E}\Big(T_1 1_{[T_1\le \delta b(s)]}\Big)}{b(s)\bar G(b(s))}\bar F(b(s))U(b(s)dy)\\
&= A + B\,.
\end{aligned}$$

Now for A we have the bound (apart from the factor η^{-1}),

$$A \le \frac{\mathbb{E}\Big(T_1 1_{[T_1\le \delta b(s)]}\Big)}{b(s)\bar G(b(s))}\Big(\bar F(b(s))U(b(s)t) - \bar F(b(s))U(b(s)(t-\delta))\Big)$$

and as $s\to\infty$. This is asymptotic to

$$\sim c_1\delta^{1-\beta}\Big(c_2 t^\alpha - c_2(t-\delta)^\alpha\Big) \to 0, \quad \delta\downarrow 0\,.$$

For B we have

$$B \sim c\delta^{1-\beta}\bar F(b(s))U(b(s)(1-\delta)) \sim c\delta^{1-\beta}(1-\delta)^\alpha \to 0, \quad (\delta\downarrow 0)\,.$$

For $j=2$ in (4.10), we have for the probability

$$P\Big[\frac{\bar F(b(s))}{\bar G(b(s))}\sum_{k=1}^{N(b(s)t)}\big(t-\frac{S_k}{b(s)}\big)1_{[\frac{T_k}{b(s)}> t-\frac{S_k}{b(s)}, \frac{T_k}{b(s)}\le \delta]} > \eta\Big]$$

$$\leq \eta^{-1}\frac{\bar F(b(s))}{\bar G(b(s))}\mathbb{E}\left((t-\frac{S_k}{b(s)})1_{[\frac{T_k}{b(s)}>t-\frac{S_k}{b(s)},\frac{T_k}{b(s)}\leq\delta]}>\eta\right),$$

Letting $\mathcal{S}=\sigma(S_k\,,k\geq 1)$, we get by iterating expectations

$$\begin{aligned}
&=\eta^{-1}\frac{\bar F(b(s))}{\bar G(b(s))}EE\left(\sum_{k=1}^{N(b(s)t)}(t-\frac{S_k}{b(s)})1_{[\frac{T_k}{b(s)}>t-\frac{S_k}{b(s)},\frac{T_k}{b(s)}\leq\delta]}>\eta\mid\mathcal{S}\right)\\
&=\eta^{-1}\frac{\bar F(b(s))}{\bar G(b(s))}E\left(\sum_{k=1}^{N(b(s)t)}(t-\frac{S_k}{b(s)})P[b(s)t-S_k<T_k\leq b(s)\delta\mid\mathcal{S}]\right)\\
&=\eta^{-1}\frac{\bar F(b(s))}{\bar G(b(s))}E\left(\sum_{k=1}^{N(b(s)t)}(t-\frac{S_k}{b(s)})\Big(\bar G(b(s)t-S_k)-\bar G(b(s)\delta)\Big)_+\right)\\
&=\eta^{-1}\frac{\bar F(b(s))}{\bar G(b(s))}\int_0^t(t-u)\Big(\bar G(b(s)(t-u))-\bar G(b(s)\delta)\Big)_+U(b(s)du)\\
&\leq\eta^{-1}\int_{t-\delta}^t(t-u)\frac{\bar G(b(s)(t-u))}{\bar G(b(s))}\bar F(b(s))U(b(s)du)\\
&\sim c\int_{t-\delta}^t(t-u)^{1-\beta}du^\alpha\qquad(s\to\infty)\\
&\to 0\qquad(\delta\downarrow 0).
\end{aligned}$$

This proves convergence of the one-dimensional distributions in Theorem 4.4. The convergence of the finite dimensional distributions is straightforward: The multivariate analog of the map in (4.9) is also almost surely continuous and once this is noted, it is clear how to proceed.

The tightness of the converging processes in $(\mathbb{D}[0,\infty),J_1))$ follows from the convergence of the finite-dimensional distributions together with the observation that the sample paths of A and of the limiting process are monotone and continuous; see Jacod and Shiryaev (1987), Theorem VI.3.37. □

5. Unresolved problems

Several questions remain unanswered.

5.1. The case $\mu_X<\infty$ and $\beta\in(0,1)$.

An analysis similar to what was performed at the beginning of Subsection 4.3, shows that $\mathbb{E}I_1$ and $\mathbb{E}I_2$ are of the same order and of lower order than I_{11}; see below. Hence I_{11} does not help here.

By the independence of N and (T_i),

$$\left(t^{-1}N(t\cdot)\,,(\sigma(t))^{-1}\sum_{i=1}^{[t\cdot]}T_i\right)\Rightarrow\left(\mu_X^{-1}\cdot\,,X_\beta\right),$$

in $\mathbb{D}([0,\infty),\mathbb{R}^2)$, where X_β is spectrally positive β-stable Lévy motion. By a continuous mapping argument,

$$(\sigma(t))^{-1}\sum_{i=1}^{N(t\cdot)}T_i\Rightarrow\mu_X^{-1/\beta}X_\beta(\cdot)\,,$$

in $(\mathbb{D}[0,\infty),J_1)$. A similar argument as for Proposition 4.3 finally gives the following result:

Proposition 5.1. *Assume $\beta \in (0,1)$ and $\mu_X < \infty$. Then*

$$(\sigma(t))^{-1} \sum_{i=1}^{N(t\cdot)} T_i \Rightarrow \mu_X^{-1/\beta} X_\beta(\cdot)$$

in $(\mathbb{D}[0,\infty), J_1)$, where X_β is spectrally positive β-stable Lévy motion on $[0,\infty)$.

Referring to Proposition 3.2, we would expect a Gaussian limit for $A(t)$ in this case.

5.2. **Other problems.** Here is a list of problems whose resolution is unsatisfactory:

(1) The Gaussian limit in Proposition 3.2 is only obtained after a random centering. It can be replaced by the expected value if $\{S_k\}$ constitutes a Poisson process. When can the random centering be replaced by a non-random centering?
(2) The Gaussian approximation in Proposition 3.2 is only in the sense of convergence of finite-dimensional distributions. We suspect that the convergence can be considerably strengthened allowing integration to the limit which would resolve the asymptotic behavior of $A(t)$.
(3) We expect that the mode of convergence in Corollary 2.7 can be strengthened. If so, this would provide a convenient way to obtain Theorem 4.4.
(4) Connections to data networks rarely occur according to a Poisson process, and it is unlikely they occur according to a renewal process (Guerin et al. (2003)). What more general class of connection models would be tractable?
(5) Transmissions do not occur at unit rate as assumed here and more general models are needed.

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