MATHEMATICS OF OPERATIONS RESEARCH
Vol. 32, No. 4, November 2007, pp. 890–919
ISSN 0364-765X | EISSN 1526-5471 | 07 | 3204 | 0890

DOI 10.1287/moor.1070.0267
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Scaling Limits for Cumulative Input Processes

Thomas Mikosch
Laboratory of Actuarial Mathematics, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen, Denmark,
mikosch@math.ku.dk

Gennady Samorodnitsky
School of Operations Research and Information Engineering, Cornell University, Ithaca, New York 14853,
gennady@orie.cornell.edu

We study different scaling behavior of very general telecommunications cumulative input processes. The activities of a telecommunication system are described by a marked-point process $((T_n, Z_n))_{n\in\mathbb{Z}}$, where T_n is the arrival time of a packet brought to the system or the starting time of the activity of an individual source, and the mark Z_n is the amount of work brought to the system at time T_n. This model includes the popular ON/OFF process and the infinite-source Poisson model. In addition to the latter models, one can flexibly model dependence of the interarrival times $T_n - T_{n-1}$, clustering behavior due to the arrival of an impulse generating a flow of activities, but also dependence between the arrival process (T_n) and the marks (Z_n). Similarly to the ON/OFF and infinite-source Poisson model, we can derive a multitude of scaling limits for the input process of one source or for the superposition of an increasing number of such sources. The memory in the input process depends on a variety of factors, such as the tails of the interarrival times or the tails of the distribution of activities initiated at an arrival T_n, or the number of activities starting at T_n. It turns out that, as in standard results on the scaling behavior of cumulative input processes in telecommunications, fractional Brownian motion or infinite-variance Lévy stable motion can occur in the scaling limit. However, the fractional Brownian motion is a much more robust limit than the stable motion, and many other limits may occur as well.

Key words: telecommunications; cumulative input process; marked-point process; stable Lévy motion; fractional Brownian motion; scaling limits; long-range dependence
MSC2000 subject classification: Primary: 60K25; secondary: 60K30
OR/MS subject classification: Primary: queues, networks; secondary: probability, stochastic model application
History: Received April 11, 2006; revised September 6, 2006.

1. Introduction and the basic model. Recent analysis of broadband measurements of teletraffic shows that the data exhibit the following characteristic properties: heavy tails, self-similarity, and long-range dependence (LRD). A standard model for explaining these empirical facts is the ON/OFF model. In it, traffic is generated by a large number of independent ON/OFF sources (such as workstations in a big computer space). An ON/OFF source transmits data at a constant rate to a server if it is ON and remains silent if it is OFF. Every individual ON/OFF source generates an ON/OFF process consisting of independent alternating ON and OFF periods. The ON periods are iid and so are the lengths of the OFF periods. Moreover, the ON and the OFF periods for each source are independent. Teletraffic is then generated by the superposition of a large number of these iid ON/OFF sources. Support for this model in the form of statistical analysis of Ethernet local area network traffic of individual sources was provided in Willinger et al. [38]. One of the conclusions of this study was that the lengths of the ON and the OFF periods are heavy tailed and in fact Pareto-like with tail index α between one and two. Further evidence on infinite-variance distributions in teletraffic is given in Crovella and Bestavros [5], Crovella et al. [6], Leland et al. [24], Hernandes-Campos et al. [16], and Hernandes-Campos et al. [17], which present evidence of infinite-variance Pareto-like tails in file lengths, transfer times, and idle times in the World Wide Web traffic.

One of the immediate consequences of the assumption of Pareto-like tails with tail index α between one and two is that a stationary version of the ON/OFF process of an individual source exhibits LRD in the sense that its covariance function stays positive and is not integrable; see Heath et al. [15] for a mathematical proof. This mathematical fact explains LRD at the individual source level, but not at the level of teletraffic. In the ON/OFF model, teletraffic is considered as the superposition of iid individual ON/OFF processes, and its cumulative input is the integrated superposition of the ON/OFF processes.

This cumulative input process has been the object of intensive research over the past 15 years. In particular, limit theory for the scaled and centered cumulative input process has been employed to prove some fundamental results about teletraffic; see Leland et al. [24], Taqqu et al. [37], Mikosch et al. [28], and Gaigalas and Kaj [12]. One of the aims of this line of research was to show that the LRD of the individual sources (which is due to the Pareto-like ON/OFF times) can be inherited by the limit process of the cumulative input. However, depending on the number of superimposed ON/OFF processes, one can get quite different limit processes. Mikosch et al. [28] show that if the number of ON/OFF sources increases "fast" with time, one gets fractional Brownian motion

with Hurst coefficient $H \in (0.5, 1)$, i.e., a Gaussian process with LRD in its increments. If this number grows "slowly" one gets an α-stable Lévy motion as weak limit of the scaled cumulative input process. This limit is a process with stationary and independent α-stable increments. In this model, the limit process inherits the heavy tails from the individual ON/OFF sources. However, its increments have no dependence at all, let alone LRD. Gaigalas and Kaj [12] show in the case of "intermediate" growth that the limit is neither α-stable Lévy motion nor fractional Brownian motion.

In this paper we consider a stationary marked-point process (MPP)

$$((T_n, Z_n))_{n\in\mathbb{Z}}, \tag{1}$$

where we interpret $\cdots \le T_{-1} \le T_0 \le 0 \le T_1 \le T_2 \le \cdots$ as the arrival times of a packet brought to the system or as the starting time for the activity of one source, and $Z_n \ge 0$ is the amount of work brought to the system at time T_n. For example, in the popular ON/OFF model, the arrival times T_n correspond to the beginning of an ON period and Z_n is the length of the period initiated at time T_n. We assume that the point process (T_n) is simple and has a finite intensity λ. In general, we do not assume independence between the arrival process (T_n) and the mark process (Z_n).

The *number of active sources* at time t is given by the process

$$M(t) = \sum_{n\in\mathbb{Z}} I\{T_n \le t < T_n + Z_n\}, \quad t \ge 0. \tag{2}$$

The number of sources arriving in the interval $(s, t]$ is described by

$$N(s, t] = \sum_{n\in\mathbb{Z}} I\{s < T_n \le t\}, \quad s < t,$$

and we write $N(t) = N(0, t]$, $t \ge 0$, for the corresponding counting process. The amount of work brought into the system in the interval $[0, t]$ is given by the *input process*

$$A(t) = \int_0^t M(y)\,dy = \sum_{n\in\mathbb{Z}} [Z_n \wedge (t - T_n)_+ - Z_n \wedge (-T_n)_+], \quad t \ge 0. \tag{3}$$

Assuming that the marks Z_n have, under the Palm distribution, a finite mean, we will show in §2 that the process A is well defined in the sense that it is finite for every $t \ge 0$ and that it has stationary increments. In fact, $A(t)$ will have a finite mean, and so by the stationarity, $EA(t) = \mu t$ for all $t > 0$. Here, $\mu > 0$ is a constant whose meaning is the expected amount of work arriving in a time interval of unit length.

A common way of viewing the behavior of a communication system is to assume that the cumulative input to such a system is provided by the superposition of a large number of iid individual input processes. Each one of the input processes generates work in the system according to the model (3). Furthermore, one also speeds up time by a large factor (i.e., adopts the bird's-eye point of view of the system). Then the limiting behavior of the deviation of the cumulative input process from the average is of interest when both the number of input processes and the time scale increase.

To fix notation, let A_i, $i = 1, 2, \ldots$, be iid copies of the input process A in (3). With n input processes, $(\sum_{i=1}^n A_i(t))_{t\ge 0}$ is the *cumulative input process*, which is sometimes also called *total accumulated work*. The deviation of the cumulative input from its mean at the time scale $T > 0$ is the process

$$D_{n,T}(t) = \sum_{i=1}^n (A_i(tT) - \mu tT), \quad t \ge 0. \tag{4}$$

We are interested in the limits of the suitably normalized sequence of the processes $D_{n,T}$ as $n, T \to \infty$. Such limits quite clearly depend on the relative speed at which the number of input processes n and the timescale T grow. For two special cases of (3), the ON/OFF model and the infinite-source Poisson model (see §3), it has been established in Mikosch et al. [28] that if the number of input processes grows relatively slowly, under a proper normalization the sequence $(D_{n,T})$ converges in distribution (in the M_1 topology) to a stable Lévy motion. If, on the other hand, the number of input processes grows relatively fast, a properly normalized process $D_{n,T}$ converges in distribution (in the J_1 topology) to a fractional Brownian motion.

A natural question arises: To what extent is it in general true that, under a slow growth condition for the number of input processes, the system "looks like a stable motion," and that, under a fast growth condition for the number of input processes, the system "looks like a fractional Brownian motion?" We address the fast growth situation in §4. We will see that fractional Brownian, indeed, arises frequently in such situations, the key condition being regular variation of $\mathrm{Var}(A(\cdot))$ of the input process in (3).

Throughout, we will consider the following examples from §3:

- the ON/OFF model with regularly varying ON/OFF times, see §3.1;
- a model with iid marks Z_n, independent of (T_n) (including the infinite-source Poisson model), see §3.2;
- a renewal Poisson cluster process, see §3.2.

We will use these examples for the illustration of the theory. In particular, we will study which components of these processes cause regular variation of $\mathrm{Var}(A(\cdot))$.

The purpose of §5 is to investigate the slow growth situation of the number of input processes. Here we identify various situations when we obtain a stable Lévy motion in the limit, but, surprisingly, fractional Brownian motion can appear as well. This is of course impossible for the ON/OFF or the infinite-source Poisson models. Furthermore, various unfamiliar limit processes will occur as well.

In what follows, we generalize and extend some of the standard models of telecommunications, including the popular ON/OFF and infinite-source Poisson models, in different ways. We allow for dependence between (T_n) and (Z_n), but also for dependence of the interarrival times $T_n - T_{n-1}$. Moreover, the models explain the occurrence of LRD and self-similarity of the input process A and superpositions of iid copies of A by heavy-tailed components in the structure of the processes, such as regular variation of the interarrival times, the marks Z_n, or the number of activities started at the points T_n.

2. Basic properties of the input process A. In this section we study some of the basic properties of the process A, such as stationarity of its increments and its first- and second-moment structures. In what follows, we will frequently make use of the Palm distribution of the stationary MPP $((T_n, Z_n))$ defined in (1). Our main reference will be Baccelli and Brémaud [2]. We mention that under the Palm distribution, we have that $T_0 = 0$ is a point of the process N with probability 1.

For convenience we list here some of the *standard notation* used throughout.

C	Any positive constant, possibly different from line to line or formula to formula.
λ	Intensity of the stationary MPP $((T_n, Z_n))$.
γ	Intensity measure of the stationary MPP.
γ_2	Covariance measure of the stationary MPP.
$\hat{\gamma}^*$	Reduced covariance measure.
m_2	Second-moment measure of the stationary MPP.
P_0	Palm distribution of the mark process (Z_n).
E_0, Var_0	Expectation and variance with respect to P_0.
$\bar{F}$	Tail $1 - F$ of the distribution function F.
F_X	Distribution function of X.
$f \in \mathrm{RV}_p$	f is regularly varying with index p.
$A, B, C, \ldots$	The generic element of a stationary sequence (A_i), (B_i), (C_i), etc.
$\stackrel{d}{=}$	Equality of the finite-dimensional distributions.

2.1. Stationarity of the increments.

Lemma 2.1. *The input process A defined in (3) of the stationary MPP $((T_n, Z_n))$ in (1) is well defined and has stationary increments, provided that under the Palm distribution P_0 the stationary marks Z_n have a finite first moment.*

Proof. We start by showing that the process has stationary increments, assuming for the moment that it is well defined. Denote by (θ_h) the group of left shifts of the MPP $((T_n, Z_n))$; see Baccelli and Brémaud [2, p. 5]. The stationarity of the point process implies that for any $h \geq 0$,

$$\begin{aligned}
(A(t+h) - A(h))_{t\geq 0} &= \left(\sum_{n\in\mathbb{Z}} [Z_n \wedge (t+h-T_n)_+ - Z_n \wedge (h-T_n)_+]\right)_{t\geq 0} \\
&= \theta_h\left(\sum_{n\in\mathbb{Z}} [Z_n \wedge (t-T_n)_+ - Z_n \wedge (-T_n)_+]\right)_{t\geq 0} \\
&\stackrel{d}{=} \left(\sum_{n\in\mathbb{Z}} [Z_n \wedge (t-T_n)_+ - Z_n \wedge (-T_n)_+]\right)_{t\geq 0} \\
&= (A(t))_{t\geq 0}.
\end{aligned}$$

This proves the stationarity of the increments of the process A.

Thus, it suffices to show that $A(t)$ is finite with probability one for every $t \geq 0$. To this end, we show that $EA(t) < \infty$ for every $t \geq 0$. Because each term in the sum (3) defining $A(t)$ does not exceed t, and because the

expected number of arrivals in $[0, t]$ is finite, it suffices to verify that

$$I = E\left[\sum_{n<0} I\{T_n + Z_n > 0\}\right] < \infty. \tag{5}$$

Recall that γ is the intensity measure of the stationary MPP. Then, by stationarity,

$$\begin{aligned} I &= \sum_{m=0}^{\infty} E\left[\sum_{n<0} I\{T_n \in [-m-1, -m), T_n + Z_n > 0\}\right] \\ &\le \sum_{m=0}^{\infty} E\left[\sum_{n<0} I\{T_n \in [-m-1, -m), Z_n > m\}\right] \\ &= \sum_{m=0}^{\infty} E\left[\sum_{n\in\mathbb{Z}} I\{T_n \in [0, 1), Z_n > m\}\right] \\ &= \sum_{m=0}^{\infty} \gamma([0, 1) \times (m, \infty)). \end{aligned} \tag{6}$$

Then for $x \ge 0$,

$$P_0(Z_0 > x) = \lambda^{-1} \sum_{n\in\mathbb{Z}} P(T_n \in [0, 1), Z_n > x) = \lambda^{-1}\gamma([0, 1) \times (x, \infty)).$$

Because we assume that the marks have a finite first moment under P_0, we may conclude from the latter identity and (6) that $I < \infty$. This concludes the proof. □

The assumption $E_0 Z_0 < \infty$ in Lemma 2.1 guaranteed that $EA(t) < \infty$ for all $t \ge 0$, implying $A(t) < \infty$ a.s. Without this assumption, the conclusion does not remain valid in general, as the following example shows. Assume that (T_n) is a homogeneous Poisson process with intensity λ and Z_n are iid marks independent of the Poisson process. Then the number of sources that arrive by time zero and remain open by time 1 (say) is Poisson with mean $\lambda \int_0^\infty P_0(Z_0 > x + 1)\, dx$, which is finite if and only if $E_0 Z_0 < \infty$.

2.2. First- and second-moment structure. It is easy to describe the first moment of $A(t)$ in terms of the intensity measure of the marked-point process.

Lemma 2.2. *Assume under the Palm distribution P_0 that the stationary marks Z_n have a finite first moment. Then $EA(t) = t\gamma(\{(s, u)\colon s \le 0 < s + u\})$ is finite as well.*

Proof. The fact that $EA(t) < \infty$ for each t follows from the argument in Lemma 2.1. By stationarity,

$$\begin{aligned} EA(t) &= tEM(0) = tE\left[\sum_{n\in\mathbb{Z}} I\{T_n \le 0, T_n + Z_n > 0\}\right] \\ &= t\gamma(\{(s, u)\colon s \le 0 < s + u\}), \end{aligned}$$

as required. □

It is not easy to relate the moment properties of the stationary point process N and of the Palm distribution of the marks to the existence of the second moment of the input A. However, in applications it is usually straightforward to check the finiteness of the second moment of M. Next, we obtain an expression for the second moment and the variance of $A(t)$ in terms of the moment measures of the MPP. This will also provide us with conditions for finite second moments.

The natural language here is that of the *second-moment measure* (see Daley and Vere-Jones [7, §6.4] and certain other related measures. Assume, therefore, that the stationary MPP $((T_n, Z_n))$ has a second-moment measure m_2. That is, for every measurable $f\colon \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$,

$$E\left(\sum_{n\in\mathbb{Z}} f(T_n, Z_n)\right)^2 = \int_{\mathbb{R}^2\times\mathbb{R}_+^2} f(t_1, z_1) f(t_2, z_2) m_2(dt_1, dt_2, dz_1, dz_2); \tag{7}$$

and both sides of this equality are finite at the same time. Applying (7) to the representation (3) of $A(t)$, we obtain

$$E[(A(t))^2] = \int_{[0,t]^2} h(x, y)\, dx\, dy, \tag{8}$$

where

$$h(x,y)=\int_{\mathbb{R}^2\times\mathbb{R}_+^2} I\{s_1\le x<s_1+u_1, s_2\le y<s_2+u_2\}\, m_2(ds_1,ds_2,du_1,du_2).$$

In particular, the second moment of $A(t)$ is finite if and only if the integral on the right-hand side of (8) is finite. In this case, a convenient expression for the variance of $A(t)$ can be derived by using the *covariance measure* of the MPP (see Karr [22]) defined by

$$\gamma_2(ds_1,ds_2,du_1,du_2)=m_2(ds_1,ds_2,du_1,du_2)-\gamma(ds_1,du_1)\gamma(ds_2,du_2).$$

It follows from (8) and Lemma 2.2 that

$$\mathrm{Var}(A(t))=E[(A(t))^2]-(EA(t))^2=\int_{[0,t]^2} g(x,y)\,dx\,dy,$$

where

$$g(x,y)=\int_{\mathbb{R}^2\times\mathbb{R}_+^2} I\{s_1\le x<s_1+u_1, s_2\le y<s_2+u_2\}\,\gamma_2(ds_1,ds_2,du_1,du_2). \tag{9}$$

By the stationarity of the MPP, the covariance measure is invariant under the transformation $T_a\colon \mathbb{R}^2\times\mathbb{R}_+^2\to\mathbb{R}^2\times\mathbb{R}_+^2$ defined by $T_a(s_1,s_2,u_1,u_2)=(s_1+a,s_2+a,u_1,u_2)$, any $a\in\mathbb{R}$. Therefore, the function g in (9) depends only on the difference $|y-x|$. We denote (abusing the notation in the usual way) the resulting function of one variable also by g, i.e.,

$$g(x)=\int_{\mathbb{R}^2\times\mathbb{R}_+^2} I\{s_1\le 0<s_1+u_1, s_2\le x<s_2+u_2\}\,\gamma_2(ds_1,ds_2,du_1,du_2). \tag{10}$$

We now summarize our findings in the following proposition.

Proposition 2.1. *The variance of the input $A(t)$ is given by*

$$\mathrm{Var}(A(t))=2\int_0^t (t-x)g(x)\,dx. \tag{11}$$

The following is an immediate consequence of Karamata's theorem (see Bingham et al. [4, §1.6]).

Corollary 2.1. *Assume that $g\in \mathrm{RV}_\beta$ for some $\beta\le 0$. If $\beta\in[-1,0]$, then $\mathrm{Var}(A(\cdot))\in\mathrm{RV}_{2+\beta}$. In fact, if $\beta\in(-1,0]$, then*

$$\mathrm{Var}(A(t))\sim\frac{2}{(1+\beta)(2+\beta)}t^2 g(t),\quad t\to\infty.$$

If $\beta<-1$, then $\mathrm{Var}(A(t))\sim Ct$ for some finite constant C, which is positive unless $\int_0^\infty g(x)\,dx=0$.

3. Examples.

3.1. The ON/OFF process. This is perhaps the most popular model for teletraffic. Consider a single ON/OFF source such as a workstation as described in Heath et al. [15], Leland et al. [24], Taqqu et al. [37], Pipiras and Taqqu [31], Levy and Taqqu [25], Mikosch et al. [28], Stegeman [35], Gaigalas and Kaj [12], and Pipiras et al. [32]. During an ON period, the source generates traffic at a constant rate one, for example, one byte per time unit. (However, Pipiras and Taqqu [31] and Pipiras et al. [32] also allow for random rates (rewards).) During an OFF period, the source remains silent and the input rate is zero. Let (Z_i) and (Y_i) be independent sequences of iid nonnegative random variables representing the lengths of ON periods and OFF periods, respectively. Here Z_i and Y_i with positive (negative) index represent ON/OFF periods happening after (before) time 0. Then,

$$W_i=Z_i+Y_i,\quad i\in\mathbb{Z},$$

are the interarrival times under the Palm measure of a stationary ON/OFF process, which we will denote by (T_n); see Heath et al. [15] for an explicit construction. The nth mark is simply Z_{n+1}, the length of the next ON period. Notice that the renewal process (T_n) and the mark process (Z_n) are dependent. Obviously, $M(t)=0$ or 1 in the ON/OFF model.

Assuming that $\bar{F}_{\mathrm{ON}}\in\mathrm{RV}_{-\alpha_{\mathrm{ON}}}$ for some $\alpha_{\mathrm{ON}}\in(1,2)$, the tail of Y is lighter than the one of Z, and W has a spread-out distribution, Heath et al. [15] showed that

$$g(t)\sim\frac{\mu_{\mathrm{OFF}}^2}{(\alpha_{\mathrm{ON}}-1)\tilde{\mu}^3}t\bar{F}_{\mathrm{ON}}(t),\quad t\to\infty, \tag{12}$$

where $\tilde{\mu} = \mu_{\mathrm{ON}} + \mu_{\mathrm{OFF}}$, μ_{ON}, μ_{OFF} are the expectations of W, Z, Y, respectively, and F_{ON} is the distribution Z. By Corollary 2.1 we see that for the ON/OFF model,

$$\mathrm{Var}(A(t)) \sim \frac{2}{(\alpha_{\mathrm{ON}}-1)\alpha_{\mathrm{ON}}(\alpha_{\mathrm{ON}}+1)} \frac{\mu_{\mathrm{OFF}}^2}{\tilde{\mu}^3} t^3 \bar{F}_{\mathrm{ON}}(t), \quad t \to \infty.$$

3.2. Marks independent of the point process. In this model the sequence of marks Z_n and the sequence (T_n) are independent, and (Z_n) constitutes a nonnegative stationary process. By (2.2.4) in Baccelli and Brémaud [2], the intensity measure of a stationary MPP whose marks are independent of the point process is given by

$$\gamma = \lambda \,\mathrm{Leb} \times F,$$

where Leb is Lebesgue measure on $\mathbb{R}$, and F is the law of the marks. By Lemma 2.2,

$$EA(t) = \lambda t E Z.$$

A common particular case occurs when the marks form an iid sequence. In that case, we can use Proposition 6.4.IV in Daley and Vere-Jones [7] to see that we have $m_2(ds_1, ds_2, du_1, du_2) = m_2^{\mathrm{g}}(ds_1, ds_2)F(du_1)F(du_2)$ off the diagonal $s_1 = s_2$, where m_2^{g} is the second-moment measure of the unmarked (*ground*) stationary point process (T_n). On the diagonal $s_1 = s_2$, we have

$$m_2\left(\{s_1 = s_2 \in A\} \times B_1 \times B_2\right) = \lambda \,\mathrm{Leb}(A) F\left(B_1 \cap B_2\right).$$

Therefore,

$$\gamma_2(ds_1, ds_2, du_1, du_2) = \gamma_2^{\mathrm{g}}(ds_1, ds_2)F(du_1)F(du_2) + \gamma^{\mathrm{d}}(ds_1, ds_2, du_1, du_2),$$

where γ_2^{g} is the covariance measure of the ground stationary point process (T_n), and γ^{d} is the *diagonal* (signed) measure defined by

$$\gamma^{\mathrm{d}}\left(A_1 \times A_2 \times B_1 \times B_2\right) = \lambda \,\mathrm{Leb}\left(A_1 \cap A_2\right)\left[F\left(B_1 \cap B_2\right) - F(B_1)F(B_2)\right]$$

for any Borel sets A_i, B_i, $i = 1, 2$. Therefore, by (10),

$$g(t) = \lambda\left[E[(Z-t)_+] - E[Z_1 \wedge (Z_2 - t)_+]\right] + \int_{\mathbb{R}^2} I\left\{s_1 \le 0, s_2 \le t\right\} \bar{F}_Z(-s_1)\bar{F}_Z(t-s_2)\, \gamma_2^{\mathrm{g}}(ds_1, ds_2), \tag{13}$$

where Z_1 and Z_2 are two independent mark variables.

In a further particular case where the ground point process N is a homogeneous Poisson process, we have $\gamma_2^{\mathrm{g}}(A \times B) = \lambda \,\mathrm{Leb}(A \cap B)$ for any Borel sets A and B (see Karr [22]), and so (13) reduces to

$$g(t) = \lambda E(Z-t)_+.$$

In particular, if $\bar{F}_Z \in \mathrm{RV}_{-\alpha}$ for some $\alpha > 1$, we obtain from Karamata's theorem

$$g(t) = \lambda \int_t^\infty \bar{F}_Z(x)\, dx \sim \frac{\lambda}{\alpha - 1} t \bar{F}_Z(t),$$

and by Corollary 2.1 we see that

$$\mathrm{Var}(A(t)) \sim \frac{2\lambda}{(\alpha-1)\alpha(\alpha+1)} t^3 \bar{F}_Z(t), \quad t \to \infty.$$

This model has attracted a lot of attention under the name of *infinite-source Poisson model* in the literature on teletraffic; see Konstantopoulos and Lin [23], Mikosch et al. [28], Guerin et al. [13], Maulik et al. [27], and Maulik and Resnick [26].

Another model with marks independent of the point process and forming an iid sequence, more general than the infinite-source Poisson model, occurs when the ground process is a *cluster Poisson process*:

- Clusters arrive according to a homogeneous Poisson process $\widetilde{N}$ with rate λ_0 and points Γ_i whose points are enumerated such that $\cdots < \Gamma_{-1} < 0 < \Gamma_1 < \Gamma_2 < \cdots$.
- At each cluster center Γ_n an independent copy of a *finite* point process N_c starts. Here

$$\lambda = \lambda_0 E N_c[0, \infty),$$

and the last expectation is assumed to be finite.

If the process N_c is a randomly stopped renewal process, we obtain a model that we will here call a *renewal Poisson cluster process*. This model was studied in Faÿ et al. [11], see also Hohn and Veitch [18] and Hohn et al. [19] for some empirical studies. It can be explicitly constructed as follows. The ground process N is a

point process with the points

$$Y_{jk} = \Gamma_j + \sum_{i=1}^{k} X_{ji} = \Gamma_j + S_{jk}, \quad j \in \mathbb{Z}, \quad 0 \le k \le K_j, \tag{14}$$

where (Γ_j) is as above, X_{ji} are iid nonnegative random variables, and K_j are iid integer-valued random variables with a finite mean. We assume that (Γ_j), (K_j), and (X_{ji}) are mutually independent. Notice that N is a stationary point process due to the stationarity of the underlying Poisson process, and its intensity is related to the Poisson intensity via $\lambda = \lambda_0(1+EK)$. Recall that, unless mentioned otherwise, the marks are assumed to be iid, whether within a cluster or not.

A possible interpretation of this model goes as follows: A packet arrives at time Γ_j, initiating various activities (such as opening and closing windows or splitting the arriving packet into a number K_j+1 of smaller pieces, represented by X_{ji}) at times $Y_{j0}, Y_{j1}, \ldots, Y_{jK_j}$.

Write (X_n) for an iid sequence with the same distribution as X, define the random walk

$$S_0 = 0, \quad S_n = X_1 + \cdots + X_n, \quad n \ge 1$$

and a measure on $\mathbb{R}_+^2$ by

$$U^*(A \times B) = E\left[\sum_{n_1=0}^{K} I\{S_{n_1} \in A\} \sum_{n_2=0}^{K} I\{S_{n_2} \in B\}\right]. \tag{15}$$

Then the the covariance measure of the ground point process N is given by

$$\gamma_2^{\mathrm{g}}(A) = \lambda_0 \int_{\mathbb{R}} U^*(A+\mathbf{s})\, ds,$$

where A is any Borel set in $\mathbb{R}^2$, and $\mathbf{s}$ is a vector in $\mathbb{R}^2$ with both components equal to s.

The behavior of the function g in (13) is seen to be dependent on the interplay between the impact of the ground process and the marks. We refer the reader to Faÿ et al. [11] for the details on the above statements and on more information on the behavior of the ground process and its possible limits.

4. Limiting behavior of the cumulative input process: Fast growth condition. An extreme fast growth condition corresponds, of course, to the situation when we take the limit of a properly normalized sequence of the processes $D_{n,T}$ in (4) as $n \to \infty$ for a fixed timescale T, and then let the timescale $T \to \infty$. We will work under the assumption that $\mathrm{Var}(A(t)) < \infty$ for every $t \ge 0$. Then it follows by the stationary increments and the multivariate central limit theorem that as $n \to \infty$,

$$S_{n,T}(t) = n^{-1/2} D_{n,T}(t) \xrightarrow{\text{fidi}} G(tT), \quad t \ge 0, \tag{16}$$

where $\xrightarrow{\text{fidi}}$ refers to the convergence of the finite-dimensional distributions and G is a mean zero Gaussian process with stationary increments and incremental variance

$$\mathrm{Var}(G(t+h) - G(t)) = \mathrm{Var}(G(t)) = \mathrm{Var}(A(t)), \quad t \ge 0, \quad h \ge 0.$$

Now assume that the function $g \in \mathrm{RV}_\beta$ for some $\beta \in [-1, 0]$. From Corollary 2.1 we know that $\mathrm{Var}(A(\cdot)) \in \mathrm{RV}_{2+\beta}$. Write for $T > 0$,

$$G_T(t) = [\mathrm{Var}(A(T))]^{-1/2} G(tT), \quad t \ge 0.$$

It is immediate that the finite-dimensional distributions of G_T converge to those of fractional Brownian motion.

Theorem 4.1. *Assume that $g \in \mathrm{RV}_\beta$ for some $\beta \in [-1, 0]$. Then, as $T \to \infty$,*

$$(G_T(t))_{t \ge 0} \xrightarrow{\text{fidi}} (B_H(t))_{t \ge 0}, \tag{17}$$

where B_H denotes fractional Brownian motion with covariance structure

$$\mathrm{cov}(B_H(t), B_H(s)) = 0.5(t^{2H} + s^{2H} - |t-s|^{2H}), \quad s, t \ge 0,$$

and $H = 1 + \beta/2$ is the corresponding Hurst coefficient.

If $g \in \mathrm{RV}_\beta$ for some $\beta < -1$ and $\int_0^\infty g(x)dx \ne 0$, then (17) holds with $H = 0.5$, i.e., B_H is Brownian motion.

For an extensive discussion of fractional Brownian motion and its properties, we refer to Samorodnitsky and Taqqu [34, Ch. 7]. Notice that a Hurst coefficient $H < 0.5$ is excluded in this theorem. The relatively rare cases when such Hurst coefficients occur correspond to the situation when $\int_0^\infty g(x)dx = 0$.

Example 4.1. For the ON/OFF model, under the assumptions in §3, it follows from (12) that $g \in \mathrm{RV}_\beta$ with $\beta = 1-\alpha$. Hence, (17) holds with $H = (3-\alpha)/2$. This result was proved in the ON/OFF case in the celebrated papers (Leland et al. [24], Taqqu et al. [37]).

We now proceed to investigate how fast the number n of input processes should grow relative to the timescale to preserve the convergence to fractional Brownian motion appearing in Theorem 4.1. The following language will be used. In a system with n input processes we will let the timescale be λ_n. Fast growth for the number of input processes translates, then, into sufficiently slow growth for the scale λ_n.

The next result gives sufficient conditions on the rate of growth of λ_n such that a properly normalized sequence of processes $(D_{n,T})$ in (4) converges to the same fractional Brownian motion limit as in the extreme fast growth situation of Theorem 4.1. Furthermore, it turns out that in many cases the limit exists in the sense of weak convergence in $\mathbb{C}[0,\infty)$.

For future use, we denote

$$\tilde{S}_n(t) = [n\mathrm{Var}(A(\lambda_n))]^{-1/2} D_{n,\lambda_n}(t).$$

Theorem 4.2. *Assume $g \in \mathrm{RV}_\beta$ for some $\beta \le 0$ and that the stationary number of open sources in (2) satisfies $E[|M(0)|^{2+\delta}] < \infty$ for some $\delta > 0$.*

(i) *If $\beta \in (-1,0]$ or $\beta = -1$ and $\int_0^\infty g(x)\,dx = \infty$ and for some $\delta' < \delta$*

$$\lambda_n = o(n^{1/(|\beta|(1+2/\delta'))}), \tag{18}$$

then with $H = 1+\beta/2$

$$(\tilde{S}_n(t))_{t\ge 0} \xrightarrow{\text{fidi}} (B_H(t))_{t\ge 0}. \tag{19}$$

Moreover, if (18) holds for some $\delta' < \min(2,\delta)$, then (19) can be extended to convergence in $\mathbb{C}[0,\infty)$.

(ii) *Suppose that $\beta < -1$ or $\beta = -1$ and $\int_0^\infty |g(x)|\,dx < \infty$. If $\int_0^\infty g(x)\,dx \ne 0$, and*

$$\lambda_n = o(n^{1/(1+2/\delta)}), \tag{20}$$

then (19) holds with $H = 0.5$. The convergence can be strengthened to hold in $\mathbb{C}[0,\infty)$ if (20) holds for $0 < \delta \le 2$ or if $\delta > 2$ and

$$\lambda_n = O(\sqrt{n}). \tag{21}$$

Proof. An application of the Hölder inequality and stationarity of the process M yields

$$\begin{aligned} E[|A(t)-\mu t|^{2+\delta}] &\le t^{1+\delta}\int_0^t E[|M(y)-\mu|^{2+\delta}]\,dy \\ &= t^{2+\delta} E[|M(0)-\mu|^{2+\delta}]. \end{aligned} \tag{22}$$

Using this estimate, Corollary 2.1, the growth conditions (18) and (20) and the Potter bounds (see, e.g., Resnick [33]), we see that the Lyapunov condition

$$\lim_{n\to\infty} n^{-\delta/2} \frac{E|A(\lambda_n)-\mu\lambda_n|^{2+\delta}}{(\mathrm{Var}(A(\lambda_n)))^{1+\delta/2}} = 0, \tag{23}$$

is satisfied. Hence, according to classical central limit theory (e.g., Petrov [30]), $\tilde{S}_n(t)$ satisfies the central limit theorem for every $t \ge 0$.

The proof of the convergence of the finite-dimensional distributions to a Gaussian limit G is completely analogous by employing the Cramér-Wold device. By Corollary 2.1, $\mathrm{Var}(A(\cdot))$ is regularly varying. This immediately implies that the one-dimensional marginal distributions of $\tilde{S}_n$ converge to those of B_H. Because $\tilde{S}_n$, and hence the limiting process, have stationary increments, the covariance structure of the limiting process and hence its finite-dimensional distributions, are completely determined by its one-dimensional marginal distributions. We obtain, thus, convergence of the finite-dimensional distributions in (19).

For functional convergence it suffices to show convergence in $\mathbb{C}[0,r]$ for every $r > 1$. We restrict ourselves to prove tightness in $\mathbb{C}[0,1]$ in order to show the method. We will check that for some $\rho > 1$ and $C > 0$,

$$E[|\tilde{S}_n(t)-\tilde{S}_n(s)|^{2+\delta}] \le C|t-s|^\rho \tag{24}$$

for all $s,t \in [0,1]$; see, e.g., Theorem 12.3 in Billingsley [3]. Because the processes $\tilde{S}_n$ have stationary increments, it is enough to prove (24) for $s = 0$ and $t \in [0,1]$.

Let $J = \lceil \log(2+\delta)/\log 2\rceil - 1$. For $i = 1,\ldots,n$ let $B_{0,i}(t) = A_i(t) - \mu t$ and for $j = 1,\ldots,J$, $B_{j,i}(t) = [B_{j-1,i}(t)]^2 - E([B_{j-1,i}(t)]^2)$, $t \ge 0$. It is clear from (22) and the definition of J that these are well-defined zero mean processes. We now repeatedly use the Burkholder-Davis-Gundy inequality (see, e.g., Nualart [29, p. 236])

and the bound $(a+b)^p \le 2^{p-1}(a^p+b^p)$ for $a, b \ge 0$ and $p \ge 1$ to obtain

$$\begin{aligned}
E\left[\left|\sum_{i=1}^{n}(A_i(\lambda_n t)-\mu\lambda_n t)\right|^{2+\delta}\right] &\le CE\left[\left|\sum_{i=1}^{n}[B_{0,i}(\lambda_n t)]^2\right|^{(2+\delta)/2}\right] \\
&\le C(nE([B_{0,1}(\lambda_n t)]^2))^{(2+\delta)/2} + CE\left[\left|\sum_{i=1}^{n}B_{1,i}(\lambda_n t)\right|^{(2+\delta)/2}\right] \\
&\le \sum_{j=0}^{J-1} C\left(nE([B_{j,1}(\lambda_n t)]^2)\right)^{(2+\delta)/2^{j+1}} + CE\left[\left|\sum_{i=1}^{n}B_{J,i}(\lambda_n t)\right|^{(2+\delta)/2^J}\right].
\end{aligned}$$

A simple inductive argument shows that for all $j = 0, \ldots, J-1$,

$$E\left([B_{j,1}(t)]^2\right) \le CE\left([B_{0,1}(t)]^{2^{j+1}}\right). \tag{25}$$

If $J \ge 2$ (which is equivalent to $\delta > 2$), then for $j = 1, \ldots, J-1$ we have, using (25) and Hölder's inequality as in (22),

$$\begin{aligned}
[n\mathrm{Var}(A(\lambda_n))]^{-(1+\delta/2)}\left(nE\left([B_{j,1}(\lambda_n t)]^2\right)\right)^{(2+\delta)/2^{j+1}} &\le [n\mathrm{Var}(A(\lambda_n))]^{-(1+\delta/2)}C\left(n(\lambda_n t)^{2^{j+1}}\right)^{(2+\delta)/2^{j+1}} \\
&\le Ct^{2+\delta}\left(\frac{\lambda_n^2}{\mathrm{Var}(A(\lambda_n))}\right)^{1+\delta/2}\frac{n^{(2+\delta)/2^{j+1}}}{n^{(2+\delta)/2}} \\
&\le Ct^{2+\delta}\left(\frac{\lambda_n^2}{\mathrm{Var}(A(\lambda_n))}\right)^{1+\delta/2} n^{-(2+\delta)/4}
\end{aligned}$$

(because $j \ge 1$). Under the stronger assumptions the n-dependent coefficient is a bounded function of n and, hence,

$$[n\mathrm{Var}(A(\lambda_n))]^{-(1+\delta/2)}\left(nE\left(B_{j,1}(\lambda_n t)^2\right)\right)^{(2+\delta)/2^{j+1}} \le Ct^{2+\delta} \tag{26}$$

for $j = 1, \ldots, J-1$.

Furthermore, using the Potter bounds (if $\beta \in [-1, 0]$) and either (18) or (20), we see that there is $u_0 > 0$ such that, given $\epsilon > 0$ small enough, for all n large enough, and $t \in [0, 1]$ such that $\lambda_n t > u_0$, we have

$$[n\mathrm{Var}(A(\lambda_n))]^{-(1+\delta/2)}\left(nE\left([B_{0,1}(\lambda_n t)]^2\right)\right)^{(2+\delta)/2} \le Ct^{2H(1+\delta/2)-\epsilon}.$$

Note that $2H(1+\delta/2)-\epsilon > 1$ if ϵ is small enough. For such $\epsilon > 0$, now consider the case $\lambda_n t \le u_0$. It follows from (11) that $\mathrm{Var}(A(t)) \le Ct^2$ for all $t > 0$ and that, under our assumptions, $\mathrm{Var}(A(t)) \ge Ct$ for all $t > 0$ large enough (to see that the latter claim is true in the case $\beta = -1$, write (11) in the form $\mathrm{Var}(A(t)) = 2\int_0^t(\int_0^y g(x)\,dx)\,dy$ and use the assumption that the integral of g diverges to infinity). Therefore, for all n large enough and t such that $\lambda_n t \le u_0$, we have

$$[n\mathrm{Var}(A(\lambda_n))]^{-(1+\delta/2)}\left(nE\left([B_{0,1}(\lambda_n t)]^2\right)\right)^{(2+\delta)/2} \le C\left(\lambda_n t^2\right)^{1+\delta/2} \le Ct^{1+\delta/2}.$$

We conclude that there is $\rho > 1$ such that for all n large enough and $t \in [0, 1]$,

$$[n\mathrm{Var}(A(\lambda_n))]^{-(1+\delta/2)}\left(nE\left([B_{0,1}(\lambda_n t)]^2\right)\right)^{(2+\delta)/2} \le Ct^{\rho}. \tag{27}$$

Finally, once again using the Burkholder-Davis-Gundy inequality, the definition of J, and the triangle inequality, we see that

$$\begin{aligned}
E\left[\left|\sum_{i=1}^{n}B_{J,i}(\lambda_n t)\right|\right]^{(2+\delta)/2^J} &\le CnE\left|B_{J,1}(\lambda_n t)\right|^{(2+\delta)/2^J} \\
&\le CnE\left|B_{J-1,1}(\lambda_n t)\right|^{(2+\delta)/2^{J-1}} \le \cdots \le CnE\left|B_{0,1}(\lambda_n t)\right|^{(2+\delta)},
\end{aligned}$$

where in the last steps we used Jensen's inequality. Using the bound in (22), we see that

$$[n\mathrm{Var}(A(\lambda_n))]^{-(1+\delta/2)}E\left|\sum_{i=1}^{n}B_{J,i}(\lambda_n t)\right|^{(2+\delta)/2^J} \le Ct^{2+\delta}n^{-\delta/2}\left(\frac{\lambda_n^2}{\mathrm{Var}(A(\lambda_n))}\right)^{1+\delta/2}.$$

Because under our assumptions the n-dependent coefficient is a bounded function of n, we conclude that

$$[n\mathrm{Var}(A(\lambda_n))]^{-(1+\delta/2)}E\left|\sum_{i=1}^{n}B_{J,i}(\lambda_n t)\right|^{(2+\delta)/2^J} \le Ct^{2+\delta} \tag{28}$$

for all n and $t \in [0, 1]$.

The inequality (24) now follows from (26)–(28), and so we have established the weak convergence. □

4.1. Gaussian limits in the ON/OFF model. Recall the ON/OFF model from §3.1. For $1 < \alpha < 2$, the relation (12) established in Heath et al. [15] shows that $g \in \mathrm{RV}_\beta$ for $\beta = 1 - \alpha$. In this case, we have $E[|M(0)|^{2+\delta}] < \infty$ for all $\delta > 0$. Theorem 4.2 applies with limit B_H, $H = (3-\alpha)/2$. Condition (18) on the growth of (λ_n) turns into $\lambda_n = o(n^{1/(\alpha-1+\varepsilon)})$ for any $\varepsilon > 0$. It was shown in Mikosch et al. [28] that this condition can be weakened to $\lambda_n^2/\mathrm{Var}(A(\lambda_n)) = o(n)$.

4.2. Gaussian limits in the model with iid marks independent of the point process. Recall the model with iid marks independent of the point process, introduced in §3.2. To apply Theorem 4.2, we need to verify that $E[|M(0)|^{2+\delta}] < \infty$ for some $\delta > 0$ and that the function g is regularly varying. The following result gives sufficient conditions for that; it is convenient to state it in terms of the *reduced covariance measure* $\hat{\gamma}^*$ of the unmarked point process. Recall that $\hat{\gamma}^*$ is a signed measure on $\mathbb{R}$ such that for a measurable function h

$$\int_{\mathbb{R}^2} h(s_1, s_2)\, \gamma_2^{\mathrm{g}}(ds_1, ds_2) = \int_{\mathbb{R}} du \int_{\mathbb{R}} h(u, u+s)\, \hat{\gamma}^*(ds),$$

provided the integrals are well defined, see Karr [22].

Proposition 4.1. *Assume $\bar{F}_Z \in \mathrm{RV}_{-\alpha}$ for some $\alpha > 1$ and $E[(N(0,1])^{2+\delta}] < \infty$ for some $\delta > 0$. Then $E[|M(0)|^{2+\delta}] < \infty$.*

Moreover, assume that

$$g_h(t) = \int_{\mathbb{R}} h(t-s)\, \hat{\gamma}^*(ds) = o(t\bar{F}_Z(t)) \quad \text{as } t \to \infty, \tag{29}$$

with $h(t) = \int_0^\infty \bar{F}_Z(u)\bar{F}_Z(u+|t|)du$, $t \in \mathbb{R}$. Then $g \in \mathrm{RV}_\beta$ for $\beta = 1-\alpha$.

If, on the other hand,

$$g_h \in \mathrm{RV}_\theta \quad \text{for some } \theta \in (0, \alpha-1), \tag{30}$$

then $g \in \mathrm{RV}_\beta$ with $\beta = -\theta$.

In both cases, the convergence to fractional Brownian motion in Theorem 4.2 holds.

We observe that the dominated convergence theorem implies

$$h(t) \sim EZ\bar{F}_Z(t) \quad \text{as } |t| \to \infty. \tag{31}$$

Proof. Observe that in the decomposition (13) of g, for the first term, by Karamata's theorem, $\lambda E(Z-t)_+ \sim \lambda(\alpha-1)^{-1}t\bar{F}_Z(t)$. For the second term,

$$E[Z_1 \wedge (Z_2 - t)_+] \le EZ_1 I\{Z_2 > t\} = EZ\bar{F}_Z(t) = o(t\bar{F}_Z(t)).$$

Finally, the third term in (13) can be rewritten with the reduced covariance measure $\hat{\gamma}^*$ as follows

$$\begin{aligned}
&\int_{\mathbb{R}} du \int_{\mathbb{R}} \hat{\gamma}^*(ds) I\{u \le 0, u+s \le t\}\, \bar{F}_Z(-u)\bar{F}_Z(t-(u+s)) \\
&\quad = \int_{\mathbb{R}} \hat{\gamma}^*(ds) \int_{-\infty}^{0\wedge(t-s)} \bar{F}_Z(-u)\bar{F}_Z(t-(u+s))\, du \\
&\quad = \int_{\mathbb{R}} \hat{\gamma}^*(ds) \int_0^\infty \bar{F}_Z(u)\bar{F}_Z(u+|s-t|)\, du = g_h(t).
\end{aligned}$$

Now the proposition is a consequence of the following lemma. □

Lemma 4.1. *Assume $0 < EZ < \infty$ and let $\delta \ge 1$. Then $E[|M(0)|^\delta] < \infty$ if and only if $E[(N(0,1])^\delta] < \infty$.*

Proof. The necessity of the condition $E[(N(0,1])^\delta] < \infty$ is obvious. The key to the proof of sufficiency is the observation that the random variables

$$B_m = \sum_{n\in\mathbb{Z}} I\{T_n \in (-m-1, -m], Z_n > m\}, \quad m = 0, 1, 2, \ldots,$$

are independent binomially $(D_m, \bar{F}_Z(m))$ distributed, conditionally on (D_m), where $D_m = N(-m-1, -m]$ and F_Z is the distribution of the iid marks Z_n. Notice that (D_m) is a stationary process. Now observe that

$$M(0) \le \sum_{n\in\mathbb{Z}} \sum_{m=0}^\infty I\{T_n \in (-m-1, -m], Z_n > m\} = \sum_{m=0}^\infty B_m.$$

Assume $\delta \in [k, k+1]$ for some integer $k \geq 1$. Then, by Hölder's inequality, conditionally on (D_l),

$$\begin{aligned} E([M(0)]^\delta) &\leq E\left(\sum_{m=0}^{\infty} B_m\right)^\delta \\ &= E\left(\sum_{m_1=0}^{\infty} \cdots \sum_{m_{k+1}=0}^{\infty} B_{m_1} \cdots B_{m_{k+1}}\right)^{\delta/(k+1)} \\ &\leq E\left(\sum_{m_1=0}^{\infty} \cdots \sum_{m_{k+1}=0}^{\infty} E(B_{m_1} \cdots B_{m_{k+1}} \mid (D_l))\right)^{\delta/(k+1)}. \end{aligned}$$

There is a finite number of possibilities such that the subscripts $m_1, \ldots, m_{k+1}$ coincide. Therefore, it is enough to prove that for every $j \geq 1$ and $n_1 \geq 1, \ldots, n_j \geq 1$, $n_1 + \cdots + n_j = k+1$,

$$E\left(\sum_{m_1=0}^{\infty} E(B_{m_1}^{n_1} \mid D_{m_1}) \cdots \sum_{m_j=0}^{\infty} E(B_{m_j}^{n_j} \mid D_{m_j})\right)^{\delta/(k+1)} < \infty. \tag{32}$$

A straightforward induction argument shows that if X is a binomial random variable with parameters n and p, then for every $d \geq 1$ there is a finite constant C_d such that $EX^d \leq C_d[np + (np)^d]$. Therefore, (32) will follow once we check that for all $d_1 \geq 1, \ldots, d_j \geq 1$, $d_1 + \cdots + d_j \leq k+1$,

$$E\left(\sum_{m_1=0}^{\infty} D_{m_1}^{d_1} [\bar{F}_Z(m_1)]^{d_1} \cdots \sum_{m_j=0}^{\infty} D_{m_j}^{d_j} [\bar{F}_Z(m_j)]^{d_j}\right)^{\delta/(k+1)} < \infty.$$

To this end, note that

$$E\left(\sum_{m_1=0}^{\infty} D_{m_1}^{d_1} [\bar{F}_Z(m_1)]^{d_1} \cdots \sum_{m_j=0}^{\infty} D_{m_j}^{d_j} [\bar{F}_Z(m_j)]^{d_j}\right)^{\delta/(d_1+\cdots+d_j)} \leq \prod_{i=1}^{j} \left(E\left(\sum_{m_i=0}^{\infty} D_{m_i}^{d_i} [\bar{F}_Z(m_i)]^{d_i}\right)^{\delta/d_i}\right)^{d_i/(d_1+\cdots+d_j)},$$

and so we only need to check that each term in the product is finite. Suppose first that $\delta/d_i \geq 1$. Write

$$p = \sum_{m=0}^{\infty} [\bar{F}_Z(m)]^{d_i},$$

and notice that $p < \infty$. Then, by Lyapunov's inequality and by stationarity of (D_m),

$$\begin{aligned} E\left(\sum_{m_i=0}^{\infty} D_{m_i}^{d_i} [\bar{F}_Z(m_i)]^{d_i}\right)^{\delta/d_i} &= p^{\delta/d_i} E\left(\sum_{m_i=0}^{\infty} D_{m_i}^{d_i} [\bar{F}_Z(m_i)]^{d_i}/p\right)^{\delta/d_i} \\ &\leq p^{\delta/d_i} \sum_{m_i=0}^{\infty} E D_{m_i}^{\delta} [\bar{F}_Z(m_i)]^{d_i}/p \\ &= p^{\delta/d_i} E(N(0,1])^\delta < \infty, \end{aligned}$$

as required. The case $\delta/d_i < 1$ is possible only when $j = 1$ and $d_1 = k+1$. In this case,

$$\begin{aligned} E\left(\sum_{m_1=0}^{\infty} D_{m_1}^{k+1} [\bar{F}_Z(m_1)]^{k+1}\right)^{\delta/k+1} &\leq E \sum_{m_1=0}^{\infty} D_{m_1}^{\delta} [\bar{F}_Z(m_1)]^\delta \\ &= E[(N(0,1])^\delta] \sum_{m_1=0}^{\infty} \bar{F}_Z(m_1)^\delta < \infty \end{aligned}$$

as well. This proves the statement. □

4.3. Gaussian limits in the Poisson cluster model. The simplest case is that of the infinite-source Poisson model of §3.2. Because for a rate λ Poisson process $\hat{\gamma}^* = \lambda\delta_{\mathbf{0}}$ (see Karr [22]), we see that $g_h = h$, and so (29) holds. Therefore, Theorem 4.2 applies with limit B_H, $H = (3-\alpha)/2$ if $1 < \alpha \leq 2$ and $H = 1/2$ if $\alpha > 2$. (It is easy to check that here $\int_0^\infty g(x)\,dx > 0$.)

As in the ON/OFF case, condition (18) on the growth of (λ_n) becomes $\lambda_n = o(n^{1/(\alpha-1+\varepsilon)})$ for any $\epsilon > 0$, and it is known that this can be relaxed to $\lambda_n^2/\mathrm{Var}(A(\lambda_n)) = o(n)$, see Mikosch et al. [28].

For the general Poisson cluster model of §3.2, the scaling limits depend, mostly, on the relation between the tails of the marks and cluster sizes. However, the tails of the interarrival times within each cluster also play a role.

We start with the case where the tails of the marks are heavy relative to those of the cluster sizes.

Proposition 4.2. (i) *Suppose that* $\bar{F}_Z \in \mathrm{RV}_{-\alpha}$ *for some* $1 < \alpha < 2$, *that* $EK^\theta < \infty$ *for some* $\theta > \max(\alpha, 3-\alpha)$, *and* $EX < \infty$. *Then for any sequence* (λ_n) *satisfying* $\lambda_n = o(n^{1/(\alpha-1+\varepsilon)})$ *for some* $\epsilon > 0$, *the convergence* (19) *to a fractional Brownian motion* B_H *holds with* $H = (3-\alpha)/2$.

(ii) *Suppose that* $\bar{F}_Z \in \mathrm{RV}_{-\alpha}$ *for some* $\alpha \geq 2$. *Assume that* $EK^2 < \infty$ *and that*

$$P(S_L > t) = o(t\bar{F}_Z(t)) \quad \text{as } t \to \infty, \tag{33}$$

where L *is a random variable independent of* (X_n) *with distribution*

$$P(L = k) = \theta_k/\Theta = E(K-k+1)_+/\Theta, \quad k \geq 1, \quad \Theta = \sum_{l=1}^{\infty} \theta_l. \tag{34}$$

Then for any sequence (λ_n) *satisfying* $\lambda_n = o(n^{1/(1+\varepsilon)})$ *for some* $\epsilon > 0$, *the convergence* (19) *to a Brownian motion* $(H = 1/2)$ *holds* (*assuming that* $\int_0^\infty g(x)\,dx \neq 0$ *if* $\int_0^\infty |g(x)|\,dx < \infty$).

The quantity Θ is finite because $EK^2 < \infty$ in part (2) of the proposition, and so L is a well-defined random variable. Bounds on the tail of S_L are readily available in many standard cases; see, for example, Embrechts et al. [10, Theorem A3.20] or Faÿ et al. [11].

Proof. (1) We use Proposition 4.1. Observe first that $E[(N(0,1])^{2+\delta}] < \infty$ for all $\delta > 0$; see Faÿ et al. [11]. Next we study the function g_h. It is straightforward to check that for the Poisson cluster model the reduced covariance measure is given by

$$\hat{\gamma}^* = \lambda E\left[\sum_{n_1=0}^{K}\sum_{n_2=0}^{K} \delta_{S_{n_2}-S_{n_1}}\right],$$

where, as usual, δ_x is a point mass at x. Therefore, the function g_h in (29) can be written, after some algebra, in the form

$$\begin{aligned}
g_h(t) &= \lambda E\left[\sum_{n_1=0}^{K}\sum_{n_2=0}^{K} h(t+S_{n_2}-S_{n_1})\right] \\
&= \lambda(EK+1)h(t) + \lambda E\left[\sum_{n_1=0}^{K}\sum_{n_2=n_1+1}^{K} h(t+S_{n_2}-S_{n_1})\right] + \lambda E\left[\sum_{n_2=0}^{K}\sum_{n_1=n_2+1}^{K} h(t+S_{n_2}-S_{n_1})\right] \\
&= \lambda(EK+1)h(t) + \lambda E\left[\sum_{k=1}^{K}(K-k+1)\int_0^\infty \bar{F}_Z(x)P(Z > x+t+S_k)\,dx\right] \\
&\quad + \lambda E\left[\sum_{k=1}^{K}(K-k+1)\int_0^\infty \bar{F}_Z(x)P(S_k \leq x+t \leq S_k+Z)\,dx\right] \\
&= \lambda(EK+1)h(t) + \lambda g_2(t) + \lambda g_3(t), \quad t \geq 0.
\end{aligned} \tag{35}$$

We start by estimating the function g_2. Write

$$I_k(t) = \int_0^\infty \bar{F}_Z(x)P(Z > x+t+S_k)\,dx.$$

Then

$$\begin{aligned}
g_2(t) &= E\left[I\{K \leq t\}\sum_{k=1}^{K}(K-k+1)I_k(t)\right] + E\left[I\{K > t\}\sum_{k=1}^{K}(K-k+1)I_k(t)\right] \\
&= g_{21}(t) + g_{22}(t).
\end{aligned} \tag{36}$$

Because $I_k(t) \leq h(t)$ for all $k \geq 1$ and $t > 0$, we have for $\theta < 2$

$$g_{21}(t) \leq E[K^2 I\{K \leq t\}]h(t) = E[K^\theta(K^{2-\theta}I\{K \leq t\})]h(t) \leq EK^\theta t^{2-\theta}h(t),$$

and for $\theta \geq 2$ we have $g_{21}(t) \leq EK^2 h(t)$. Because $\theta > 1$, we see by (31) that

$$g_{21}(t) = o(t\bar{F}_Z(t)) \quad \text{as } t \to \infty. \tag{37}$$

Further,

$$g_{22}(t) \leq E[KI\{K > t\}] \sum_{k=1}^{\infty} I_k(t) \leq EK^\theta t^{1-\theta} t h(t) + EK^\theta t^{1-\theta} \sum_{k>t} I_k(t). \tag{38}$$

We already know that the first term on the right-hand side of (38) is of a smaller order than $t\bar{F}_Z(t)$. For the second term we need a different bound on $I_k(t)$. First of all, the fact that $X_1 > 0$ a.s. implies (via an exponential Markov inequality) that there is $a > 0$ and $0 < \rho < 1$ such that $P(S_k < ka) \leq \rho^k$ for all k large enough. Therefore, for all k large enough,

$$I_k(t) \leq \rho^k h(t) + \bar{F}_Z(ak).$$

Because by Karamata's theorem for some constant $c > 0$,

$$t^{1-\theta} \sum_{k>t} \bar{F}_Z(ak) \sim ct^{2-\theta} \bar{F}_Z(at) = o(t\bar{F}_Z(t)) \quad \text{as } t \to \infty,$$

we conclude that $g_{22}(t) = o(t\bar{F}_Z(t))$, and then also by (36) and (37),

$$g_2(t) = o(t\bar{F}_Z(t)) \quad \text{as } t \to \infty. \tag{39}$$

Next we estimate the function g_3 in (35). We start with the case $1 < \alpha < 2$. Write

$$g_3(t) = \sum_{k=1}^{\infty} \theta_k \int_0^\infty \bar{F}_Z(x) P(S_k \leq x + t \leq S_k + Z)\, dx, \tag{40}$$

where (θ_k) is defined in (34). Under the assumption $EK^\theta < \infty$ for $\theta > 1$, we see that for some $C > 0$, $\theta_k \leq Ck^{-(\theta-1)}$ for all $k \geq 1$. Therefore,

$$\begin{aligned} g_3(t) &\leq C \int_0^\infty \bar{F}_Z(x)\, dx \int_0^\infty \left[\sum_{k=1}^{\infty} k^{-(\theta-1)} P(x + t - y < S_k \leq x + t) \right] F_Z(dy) \\ &= C \int_0^\infty \bar{F}_Z(x)\, dx \int_0^\infty [U(x+t) - U((x+t-y)_+)] F_Z(dy), \end{aligned} \tag{41}$$

where for $x \geq 0$

$$U(x) = \sum_{k=1}^{\infty} k^{-(\theta-1)} P(S_k \leq x).$$

We may assume, without loss of generality, that $\theta < 2$. It follows from Theorem 2 in Alsmeyer [1] that

$$U(x) - U(x-1) \leq Cx^{-(\theta-1)} \tag{42}$$

for all x large enough (because only an upper bound is required, the assumption of nonarithmetic distribution in Alsmeyer [1] is not needed). Write the right-hand side of (41) as

$$\begin{aligned} &C \int_0^\infty \bar{F}_Z(x)\, dx \int_0^{t/2} [U(x+t) - U(x+t-y)] F_Z(dy) \\ &\quad + C \int_0^\infty \bar{F}_Z(x)\, dx \int_{t/2}^\infty [U(x+t) - U((x+t-y)_+)] F_Z(dy) = a(t) + b(t). \end{aligned}$$

We have by (42)

$$\begin{aligned} a(t) &\leq C \int_0^\infty \bar{F}_Z(x)\, dx \int_0^{t/2} [U(x+t) - U(x+t-\lceil y \rceil)] F_Z(dy) \\ &= C \int_0^\infty \bar{F}_Z(x)\, dx \int_0^{t/2} \left[\sum_{j=1}^{\lceil y \rceil} (U(x+t-(j-1)) - U(x+t-j)) \right] F_Z(dy) \\ &\leq C \int_0^\infty \bar{F}_Z(x)\, dx \int_0^{t/2} \left[\sum_{j=1}^{\lceil y \rceil} ((x+t-j)^{-(\theta-1)}) \right] F_Z(dy) \\ &\leq CEZ \int_0^\infty (x+t)^{-(\theta-1)} \bar{F}_Z(x)\, dx \\ &\leq C(EZ)^2 t^{-(\theta-1)} = o(t\bar{F}_Z(t)) \quad \text{as } t \to \infty \end{aligned} \tag{43}$$

because $\theta > \alpha$. Furthermore, by Alsmeyer [1],

$$\begin{aligned} b(t) &\le \bar{F}_Z(t/2)\int_0^\infty \bar{F}_Z(x)U(x+t)\,dx \\ &\le C\bar{F}_Z(t/2)\int_0^\infty \bar{F}_Z(x)(x+t)^{2-\theta}\,dx \\ &\le Ct^{2-\theta}P(Z>t) = o(t\bar{F}_Z(t)) \quad \text{as } t\to\infty \end{aligned} \tag{44}$$

because $\theta > 3-\alpha$. It follows from (41), (43), and (44) that

$$g_3(t) = o(t\bar{F}_Z(t)) \quad \text{as } t\to\infty. \tag{45}$$

Now the statement (29) follows from (35), (31), (39), and (45). This proves the statement of the proposition in the case $1<\alpha<2$.

(2) For this part we only need to prove (45). We have by (40)

$$g_3(t) = C\int_0^\infty \bar{F}_Z(x)P(S_L \le x+t \le S_L+Z)\,dx,$$

and so it is enough to check that $P(S_L \le t \le S_L+Z) = o(t\bar{F}_Z(t))$ as $t\to\infty$. This clearly follows if

$$\int_0^t P(t-z<S_L\le t)F_Z(dz) = o(t\bar{F}_Z(t)) \quad \text{as } t\to\infty.$$

This, however, is an immediate consequence of (33), and so the proof of the proposition is complete. □

More common in real-life teletraffic data is the situation when the cluster size K is heavy tailed. We give a limit theorem in one such situation, when the tails of K dominate those of the marks. Such a model was studied in Faÿ et al. [11] and applied to real-life and simulated data. In this case the scaling limit is determined by the tail of K, as the following result shows.

Proposition 4.3. *Assume that $\bar{F}_K \in \mathrm{RV}_{-\alpha}$ for some $\alpha\in(1,2)$, and that $\bar{F}_Z(t) = o(\bar{F}_K(t))$ as $t\to\infty$. Assume, further, that X has a nonarithmetic distribution and $EX<\infty$. Then for any sequence (λ_n) satisfying $\lambda_n = o(n^{1/(\alpha-1+\varepsilon)})$ for some $\epsilon>0$, the convergence (19) to a fractional Brownian motion holds with $H=(3-\alpha)/2$.*

Proof. Here we will directly use Theorem 4.2. We still have $E[|M(0)|^\delta]<\infty$ for all $\delta>0$, so we only need to check the regular variation of the function g in (9). In fact, we will prove that

$$g(t) \sim \frac{\lambda}{\alpha-1}(EX)^{\alpha-2}(EZ)^2 t\bar{F}_K(t) \tag{46}$$

as $t\to\infty$. For the first term on the right-hand side in (13) we have

$$E[(Z-t)_+] = \int_t^\infty P(Z>x)\,dx = o(1)\int_t^\infty P(K>x)\,dx = o(tP(K>t)). \tag{47}$$

Further, $E[Z_1\wedge(Z_2-t)_+] \le E[(Z-t)_+] = o(tP(K>t))$. For the third term on the right-hand side in (13), equal to $g_h(t)$, we use the decomposition in (35). Note that by (31), $h(t) = o(P(K>t)) = o(tP(K>t))$. The same argument as in the proof of Proposition 4.2 shows that

$$g_2(t) = o(tP(K>t)). \tag{48}$$

To estimate the function g_3 we use (40). Write

$$\begin{aligned} g_3(t) &= \int_0^\infty \bar{F}_Z(x)\,dx\int_{y\le t/2}[U_\Theta(x+t)-U_\Theta((x+t-y)_+)]F_Z(dy) \\ &\quad + \int_0^\infty \bar{F}_Z(x)\,dx\int_{y>t/2}[U_\Theta(x+t)-U_\Theta((x+t-y)_+)]F_Z(dy) \\ &= g_{3m}(t)+g_{3r}(t), \end{aligned}$$

where

$$U_\Theta(x) = \sum_{k=1}^\infty \theta_k P(S_k\le x).$$

By Karamata's theorem $\theta_k \sim (\alpha-1)^{-1}kP(K>k)$ as $k\to\infty$, and applying Theorem 2 in Alsmeyer [1], we see that for every $h>0$

$$U_\Theta(t+h)-U_\Theta(t) \sim \frac{h}{(\alpha-1)(EX)^{2-\alpha}}tP(K>t) \tag{49}$$

as $t\to\infty$. In particular, for all t large enough, for every $x>0$ and $y\le t/2$,

$$\begin{aligned}
U_\Theta(x+t)-U_\Theta((x+t-y)_+) &\le U_\Theta(x+t)-U_\Theta(x+t-\lceil y\rceil)\\
&= \sum_{j=1}^{\lceil y\rceil}(U_\Theta(x+t-j+1)-U_\Theta(x+t-j))\\
&\le C\sum_{j=1}^{\lceil y\rceil}(x+t-j)P(K>x+t-j)\\
&\le C\lceil y\rceil \sup_{z\le t/2} zP(K>z) \le C\lceil y\rceil(t/2)P(K>t/2) \le C\lceil y\rceil tP(K>t).
\end{aligned}$$

Therefore, by the dominated convergence theorem and (49)

$$\begin{aligned}
\lim_{t\to\infty}\frac{g_{3m}(t)}{tP(K>t)} &= \int_0^\infty \bar{F}_Z(x)\,dx\int_0^\infty \lim_{t\to\infty}\frac{U_\Theta(x+t)-U_\Theta(x+t-y)}{tP(K>t)}F_Z(dy)\\
&= (\alpha-1)^{-1}(EX)^{\alpha-2}(EZ)^2.
\end{aligned} \tag{50}$$

Further, by (49)

$$\begin{aligned}
g_{3r}(t) &\le P(Z>t/2)\int_0^\infty \bar{F}_Z(x)U_\Theta(x+t)\,dx\\
&\le CP(Z>t/2)\left(\int_{x\le t}+\int_{x>t}\right)(x+t)P(K>x+t)P(Z>x)\,dx\\
&= C[g_{3r1}(t)+g_{3r2}(t)].
\end{aligned}$$

Now,

$$g_{3r1}(t)\le CP(Z>t/2)tP(K>t)=o(tP(K>t)),$$

and by Karamata's theorem,

$$\begin{aligned}
g_{3r2}(t) &\le CP(Z>t/2)\int_t^\infty xP(K>x)P(Z>x)\,dx\\
&\le CP(Z>t/2)\int_t^\infty x(P(K>x))^2\,dx\\
&\sim CP(Z>t/2)t^2(P(K>t))^2=o(tP(K>t)).
\end{aligned}$$

Therefore,

$$g_{3r}(t)=o(tP(K>t)) \tag{51}$$

as $t\to\infty$. Now (46) follows from (47)–(48), (50), and (51). This completes the proof of the proposition. □

5. Limiting behavior of the cumulative input process: Slow growth condition. The extreme slow growth condition corresponds to the situation when we take the limit of a properly normalized sequence of processes $(D_{n,T})$ in (4) as we speed up time with $T\to\infty$ for a fixed number n of sources. Under certain assumptions this limit will exist. In the literature it is almost invariably a stable Lévy motion, i.e., a process with independent and stationary infinite-variance increments. Results of this type were obtained in Leland et al. [24], Taqqu et al. [37], Willinger et al. [38], and Konstantopoulos [23] for the ON/OFF model and further extended (also to superpositions of iid copies of the input A) in Mikosch et al. [28] and Pipiras et al. [32] for the ON/OFF and the infinite-source Poisson models. We will see below that the limit may be much more general than stable Lévy motion. In most "reasonable" cases this limit will be either a Gaussian process, a stable process, or a process in the domain of attraction of such a process. In that case, taking a subsequent limit on the number n of sources will lead, after appropriate normalization, to the corresponding Gaussian or stable limit.

As one would expect, this last limit persists if both the number of the input processes and the timescale grow at the same time, as long as the timescale grows fast enough relative to the number of sources. As in §4, in a system with n input processes we will let the timescale be equal to λ_n.

To see the causes of the asymptotic behavior of $A(T)$ (for a single-input process), the following decomposition is very useful.

$$\begin{aligned} A(t) &= \sum_{n=1}^{N(t)} Z_n + \sum_{n<0} (T_n + Z_n)_+ \wedge t - \sum_{n=1}^{\infty} I\{T_n \le t\}(T_n + Z_n - t)_+ \\ &= \sum_{n=1}^{N(t)} Z_n + I_1(t) - I_2(t). \end{aligned} \tag{52}$$

As a first consequence, we obtain the following result.

Proposition 5.1. *If the stationary marks Z_m have a finite first-moment $E_0 Z$ under the Palm distribution P_0, then*

$$[(A(t) - \mu t)]_{t\ge 0} \stackrel{d}{=} \left(\sum_{m=1}^{N(t)} Z_m - E\left[\sum_{m=1}^{N(t)} Z_m\right]\right)_{t\ge 0} + O_P(1) \tag{53}$$

$$\stackrel{d}{=} \left(\sum_{m=1}^{N(t)} (Z_m - E_0(Z)) + E_0 Z[N(t) - \lambda t]\right)_{t\ge 0} + O_P(1) \tag{54}$$

$$\stackrel{d}{=} \left(\sum_{m=1}^{N(t)} (Z_m - \lambda E_0 Z(T_m - T_{m-1}))\right)_{t\ge 0} + O_P(1), \tag{55}$$

where $O_P(1)$ refers to a collection of random variables whose laws form (under the stationary measure P) a tight family. In particular, if $a(T)$ is any positive function satisfying $a(T) \to \infty$, then the asymptotic behavior of $[(a(T))^{-1}(A(tT) - \mu tT)]_{t\ge 0}$ is determined by the first terms on the right-hand side of the various expressions above.

The proof is given below. The three expressions above emphasize different important features of the input process that may affect limiting behavior. Thus, (54) makes it clear that the departures of the input process from its mean may be due to the departures of cumulative sums of the marks from their mean, and to the departure of the input process from its mean. On the other hand, the main piece in the expression (55) is a random sum of a sequence $G_m = Z_m - \lambda E_0 Z(T_m - T_{m-1})$, $m \in \mathbb{Z}$. Note that the sequence (G_m) is stationary under the Palm measure; see Remark 3.2.2 in Baccelli and Brémaud [2]. This makes our situation similar to that of stopped random walks, and allows one to use similar ideas, see Gut [14] for a general treatment.

Proof. Observe that $I_1(t) \le I^*(0)$ and $I_2(t) \le I^*(t)$ where,

$$I^*(t) = \sum_{m\in\mathbb{Z}} I\{T_m \le t\}(T_m + Z_m - t)_+$$

is the total amount of work in the sessions arriving by time t that are not finished by that time. We proved in (5) that the number of such sessions is finite with probability one for every t, provided that the stationary marks Z_m have a finite first moment under the Palm distribution. Then $(I^*(t))_{t\in\mathbb{R}}$ constitutes a finite stationary process, and so

$$(A(t) - \mu t)_{t\ge 0} \stackrel{d}{=} \left(\sum_{m=0}^{N(t)} Z_m - \mu t\right)_{t\ge 0} + O_P(1). \tag{56}$$

We know that $I_1(t) \le tM(0)$. Thus, $EI_1(t) < \infty$ for $t \ge 0$ because $EM(0) < \infty$ by (5). We will now show that $EI_1(t) = EI_2(t)$. Note that by the shift invariance of the intensity measure γ of the MPP,

$$\begin{aligned} EI_1(t) &= E\left[\sum_{m\in\mathbb{Z}} I\{T_m \le 0\}[(T_m + Z_m)_+ \wedge t]\right] \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}_+} I\{x \le 0\}[(x+z)_+ \wedge t]\gamma(dx, dz) \\ &= \sum_{k=0}^{\infty} \int_{\mathbb{R}}\int_{\mathbb{R}_+} I\{x \in (-t(k+1), -tk]\}[(x+z)_+ \wedge t]\gamma(dx, dz) \end{aligned}$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}_+} I\{x \in (0,t]\} \sum_{k=0}^{\infty} [(x-(k+1)t+z)_+ \wedge t] \gamma(dx, dz)$$
$$= \int_{\mathbb{R}} \int_{\mathbb{R}_+} I\{x \in (0,t]\} (x+z-t)_+ \gamma(dx, dz)$$
$$= E\left[\sum_{n=-\infty}^{\infty} I\{T_n \in (0,t]\}(T_n + Z_n - t)_+\right] = EI_2(t).$$

Here we used the fact that for any $a \geq 0$, $\sum_{k=1}^{\infty}(a-kt)_+ \wedge t = (a-t)_+$. This, together with (56), proves (55).

To prove (54), first observe that by the Campbell-Little-Mecke formula (see (3.3.3) in Baccelli and Brémaud [2]), $E[\sum_{n=1}^{N(t)} Z_n] = \lambda t E_0 Z$, and so

$$\sum_{m=1}^{N(t)} Z_m - E\left[\sum_{m=1}^{N(t)} Z_m\right] = \sum_{m=1}^{N(t)} (Z_m - E_0 Z) + E_0 Z[N(t) - \lambda t].$$

Finally, to show (55), we write instead

$$\sum_{m=1}^{N(t)} Z_m - E\left[\sum_{m=1}^{N(t)} Z_m\right] = \sum_{n=1}^{N(t)} (Z_n - \lambda E_0 Z(T_n - T_{n-1})) - \lambda T_0 + \lambda E_0 Z[T_{N(t)} - t],$$

and note that under the measure P of the MPP, the distribution of $T_{N(t)} - t$ does not depend on t (and is given in (4.2.4b) of Baccelli and Brémaud [2]). This completes the proof. □

One application of Proposition 5.1 is as follows.

Proposition 5.2. *Assume the following conditions hold*:

(i) *The stationary marks Z_m have a finite first moment under the Palm measure P_0.*

(ii) *There exists a function $a(T)$ with $a(T) \to \infty$ as $T \to \infty$ such that under the measure $\tilde{P}$ given by $d\tilde{P}/dP_0(\omega) = \lambda T_1(\omega)$,*

$$\left((a(T))^{-1} \sum_{m=1}^{[tT]} G_m\right)_{t \geq 0} \xrightarrow{\text{fidi}} (V(t))_{t \geq 0} \tag{57}$$

for some nondegenerate at zero stochastic process V.

(iii) *N is ergodic.*

(iv) *An Anscombe condition (see Gut [14]) of the following type holds: For every $x > 0$,*

$$\lim_{\varepsilon \downarrow 0} \limsup_{T \to \infty} P\left(\max_{0 \leq k \leq \varepsilon T} \left|\sum_{m=1}^{k} G_m\right| > x a(T)\right) = 0. \tag{58}$$

Then the function $a \in \mathrm{RV}_\alpha$ for some $\alpha > 0$ and

$$[(a(T))^{-1}(A(tT) - \mu t T)]_{t \geq 0} \xrightarrow{\text{fidi}} (\lambda^\alpha V(t))_{t \geq 0}, \tag{59}$$

under the law P of the stationary MPP.

Proof. We note, first of all, that by the inversion formula (4.1.2b) in Baccelli and Brémaud [2], the sequence of marks (G_m) has the same finite-dimensional distributions under the law $\tilde{P}$ as under the measure P of the stationary MPP. We conclude that (57) holds under P as well.

Next, the regular variation of the function $a(T)$ is a consequence of the Lamperti theorem (see, e.g., Theorem 2.1.1 in Embrechts and Maejima [8]). Because N is ergodic, we have $N(T)/T \xrightarrow{\text{a.s.}} \lambda$, and hence by regular variation of $a(T)$, $a(N(T))/a(T) \xrightarrow{\text{a.s.}} \lambda^\alpha$. Using a standard argument (see Embrechts [10, Lemma 2.5.8 and the proof of Theorem 2.5.9 on p. 102]; see also Gut [14]) based on the Anscombe condition (58), we conclude that

$$\left[(a(T))^{-1} \sum_{m=1}^{N(tT)} G_m\right]_{t \geq 0} \xrightarrow{\text{fidi}} (\lambda^\alpha V(t))_{t \geq 0}$$

under the measure P of the MPP. An application of (55) finishes the argument. □

The following statement is a version of Proposition 5.2 that uses (54) instead of (55). It describes the situation when the limit is caused by the variability of the marks, and is proved in the same way as Proposition 5.2.

Proposition 5.3. *Assume the following conditions hold*:

(i) *The stationary marks Z_m have a finite first moment under the Palm measure P_0.*

(ii) *There exists a function a with $a(T) \to \infty$ as $T \to \infty$ such that under the measure $\tilde{P}$ given by $d\tilde{P}/dP_0(\omega) = \lambda T_1(\omega)$,*

$$\left((a(T))^{-1} \sum_{m=1}^{[tT]} (Z_m - E_0(Z))\right)_{t\geq 0} \xrightarrow{\text{fidi}} (V(t))_{t\geq 0} \tag{60}$$

for some nondegenerate at zero stochastic process V.

(iii) *N is ergodic, and*

$$(a(T))^{-1}(N(T) - \lambda T) \to 0 \quad \textit{in probability as } T \to \infty.$$

(iv) *An Anscombe condition for (Z_m) holds: For every $x > 0$,*

$$\lim_{\varepsilon \downarrow 0} \limsup_{T\to\infty} P\left(\max_{0\leq k\leq \varepsilon T} \left|\sum_{m=1}^{k} (Z_m - E_0(Z))\right| > xa(T)\right) = 0. \tag{61}$$

Then the function $a \in \mathrm{RV}_\alpha$ for some $\alpha > 0$, and (59) holds under the law P of the stationary MPP.

Remark 5.1. If relation (57) can be strengthened to convergence in distribution in the Skorokhod space $\mathbb{D}[0,\infty)$ endowed with the J_1-topology (see, e.g., Billingsley [3] or Jacod and Shiryaev [20]), then the ergodicity of N and (57), as well as regular variation of a, imply that

$$\left(\frac{N(tT)}{T}, \frac{a(N(T))}{a(T)}, (a(T))^{-1} \sum_{n=1}^{[tT]} G_n\right)_{t\geq 0} \xrightarrow{d} (\lambda t, \lambda^\alpha, V(t))_{t\geq 0},$$

where $\xrightarrow{d}$ denotes convergence in distribution in $\mathbb{D}[0,\infty) \times \mathbb{R} \times \mathbb{D}[0,\infty)$. Then the continuous mapping theorem implies that (59) holds in the sense of convergence in distribution in $\mathbb{D}[0,\infty)$, provided the "small terms" in Proposition 5.1 remain appropriately "small" in the J_1-topology. A similar remark applies to Proposition 5.3.

Remark 5.2. If (60) holds under the Palm probability P_0 of the stationary sequence (Z_n) and T_1 is under P_0 independent of $(Z_1, Z_2, \dots)$, then (60) also holds under $\tilde{P}$. This class includes marks independent of the point process, the stationary ON/OFF process, and, more generally, any MPP with unpredictable marks; see Definition 6.4.III in Daley and Vere-Jones [7].

Relation (60) under P_0 also implies (60) under $\tilde{P}$ if a cross-mixing condition of the following type holds. For every $B \in \sigma(Z_m,\ m \in \mathbb{Z})$ and any Borel set A,

$$P_0(\{T_1 \in A\} \cap \theta_m(B)) \to P_0(T_1 \in A)P_0(B)$$

as $m \to \infty$, where (θ_m) is the group of left shifts of the MPP $((T_m, Z_m))$; see Baccelli and Brémaud [2, p. 7]. This is the case if the sequence $((T_n - T_{n-1}, Z_n))_{n\in\mathbb{Z}}$ is mixing under the law P_0.

Remark 5.3. The Anscombe conditions (58) or (61) are usually verified by an application of maximal inequalities such as Kolmogorov's (in the iid case) or Doob's (in the martingale difference case). Alternatively, (58) or (61) can be verified if the partial-sum process of the marks is tight in the Skorokhod space $\mathbb{D}[0,\infty)$ equipped with some topology making suprema over compact intervals continuous functionals.

The most important message of Propositions 5.2 and 5.3 is that, in a very general situation, a scaled single-input process has the same limit as a scaled partial-sum process of the stationary marks Z_m or modified marks G_m. There exists a large variety of scaling limits for a stationary sequence. The limit could be Gaussian (Brownian motion or fractional Brownian motion), one of many kinds of self-similar stationary increments stable processes, or processes that are neither Gaussian nor stable, a well-known example being the Rosenblatt process in Taqqu [36]. In all known nontrivial cases the limiting process has finite-dimensional distributions that are in the domain of attraction of a Gaussian or stable law. If one then passes to the limit as the number of sources grows, the result will provide a large variety of possible Gaussian or stable limits. This should be compared to Theorem 4.1 above, which guarantees that under the extreme fast growth condition and some fairly weak assumptions, the limit always is a fractional Brownian motion. In this sense, the fractional Brownian limit under fast growth conditions is robust under the departures from the ON/OFF process or the infinite source Poisson model of Mikosch et al. [28], whereas the stable Lévy motion limit under slow growth conditions is not similarly robust.

The next result is yet another version of Proposition 5.2. It also uses (54), but this time we look at a situation when the limit is caused by the variability in the underlying point process. Once again, the proof is the same as that of Proposition 5.2.

Proposition 5.4. *Assume the following conditions hold*:

(i) *The stationary marks Z_m have a finite first moment under the Palm measure P_0.*

(ii) *N is ergodic, and there exists a function $a(T)$ with $a(T) \to \infty$ as $T \to \infty$ such that*

$$((a(T))^{-1}(N(tT) - \lambda tT))_{t\geq 0} \xrightarrow{\text{fidi}} (V(t))_{t\geq 0} \tag{62}$$

for some nondegenerate at zero stochastic process V.

(iii) *The following relation holds in probability as $T \to \infty$,*

$$(a(T))^{-1} \sum_{m=1}^{[T]} (Z_m - E_0 Z) \to 0.$$

Then the function $a \in \mathrm{RV}_\alpha$ for some $\alpha > 0$ and

$$[(a(T))^{-1}(A(tT) - \mu tT)]_{t\geq 0} \xrightarrow{\text{fidi}} ((E_0 Z)V(t))_{t\geq 0} \tag{63}$$

under the law P of the stationary MPP.

We now address the question of the relationship between the number n of input processes and the time scale λ_n required to preserve the same limit as in the extreme slow growth case. As in the case of the latter, we will look separately at the situations when the limit is caused by the variability of the marks, and at the situations when the limit is caused by the variability in the underlying point process.

We will start with the former situation. A large number of possibilities exist. We have chosen to concentrate on a particular situation, when the marks form an iid sequence under the Palm measure (not necessarily independent of the point process N).

The following theorem is, then, one possible counterpart of Theorem 4.2 in the slow growth case. It sheds light on what determines the minimal rate at which the timescale λ_n should grow.

Theorem 5.1. *Assume that the marks Z_m form, under the Palm measure, a sequence of iid random variables with a finite first moment. Assume that this sequence is unpredictable with respect to the underlying point process and satisfies (60) with respect to the Palm measure. Assume also that the underlying point process is ergodic and that the following conditions hold.*

(i) *For $t > 0$ and iid copies N_i of N,*

$$(a(n\lambda_n))^{-1} \sum_{i=1}^{n} (N_i(\lambda_n t) - \lambda\lambda_n t) \xrightarrow{P} 0, \quad n \to \infty. \tag{64}$$

(ii)

$$(a(n\lambda_n))^{-1} \sum_{i=1}^{n} I_i^*(0) \xrightarrow{P} 0, \quad n \to \infty, \tag{65}$$

where $I_i^*(0)$ is, for the ith input process, the total amount of work in the sessions arriving by time 0 that are not finished by that time.

Denote

$$\tilde{S}_n(t) = (a(n\lambda_n))^{-1} D_{n,\lambda_n}(t) = (a(n\lambda_n))^{-1} \sum_{i=1}^{n} (A_i(\lambda_n t) - \mu\lambda_n t), \quad t \geq 0.$$

Then the process $(V(t))_{t\geq 0}$ in (60) is an α-stable Lévy motion for some $0 < \alpha \leq 2$, the function $a \in \mathrm{RV}_\alpha$, and

$$(\tilde{S}_n(t))_{t\geq 0} \xrightarrow{\text{fidi}} (\lambda^\alpha V(t))_{t\geq 0}, \quad n \to \infty. \tag{66}$$

Proof. Because α-stable and Gaussian laws are the only weak possible limits of normalized and shifted sums of iid random variables, the process $(V(t))_{t\geq 0}$ in (60) is automatically an α-stable Lévy motion for some $0 < \alpha < 2$ or a Brownian motion, and the fact that the function $a(T)$ is regularly varying with exponent α follows, once again, from the Lamperti theorem. We write

$$\begin{aligned}\tilde{S}_n(t) = {} & (a(n\lambda_n))^{-1} \sum_{i=1}^{n} \sum_{m=1}^{N_i(t)} (Z_m^{(i)} - E_0(Z)) + (a(n\lambda_n))^{-1} E_0(Z) \sum_{i=1}^{n} (N_i(\lambda_n t) - \lambda\lambda_n t) \\ & + (a(n\lambda_n))^{-1} \sum_{i=1}^{n} (I_1^{(i)}(t) - I_2^{(i)}(t))\end{aligned} \tag{67}$$

(cf. the decomposition (52)), where the superscript denotes to which input process a particular variables belongs.

The fact that the sequence of the marks is iid, and (60), imply that

$$\left((a(n\lambda_n))^{-1}\sum_{i=1}^{n}\sum_{m=1}^{[\lambda_n t]}(Z_m^{(i)}-E_0(Z))\right)_{t\geq 0}\xrightarrow{\text{fidi}}(V(t))_{t\geq 0}$$

under the Palm measure. Because the sequence of the marks is unpredictable, this relation also holds under the measure $\tilde{P}$ and then also under the stationary measure P (see Remarks 5.2, 5.3, and the proof of Proposition 5.2). Moreover, convergence to Lévy motion for sums of iid random variables also holds in the Skorokhod topology. Therefore, the appropriate Anscombe condition holds (see Remark 5.1), and we conclude, as before, that also

$$\left((a(n\lambda_n))^{-1}\sum_{i=1}^{n}\sum_{m=1}^{N_i(t)}(Z_m^{(i)}-E_0(Z))\right)_{t\geq 0}\xrightarrow{\text{fidi}}(\lambda^{\alpha}V(t))_{t\geq 0} \tag{68}$$

under the stationary measure P. Because the last two terms on the right-hand side of (67) go to zero in probability by (64), (65), and stationarity of each $(I_i^*(t))$, the claim of the theorem follows. □

We now look at the situation when the limit is caused by the variability in the underlying point process, and study the relationship between the number n of input processes and the timescale λ_n required to preserve the same limit as in the extreme slow growth case. The following theorem is another possible counterpart of Theorem 4.2 in the slow growth case. It also sheds light on what determines the minimal rate at which the timescale λ_n should grow.

THEOREM 5.2. *Assume that the marks have a finite first moment under the Palm distribution P_0. Assume further that for some sequence $b_n\uparrow\infty$*

(i)

$$\left(b_n^{-1}\sum_{i=1}^{n}(N_i(\lambda_n t)-\lambda\lambda_n t)\right)_{t\geq 0}\xrightarrow{\text{fidi}}(V(t))_{t\geq 0} \tag{69}$$

for iid copies N_i of N and some nondegenerate at zero stochastic process V.

(ii)

$$b_n^{-1}\sum_{i=1}^{n}\sum_{m=1}^{[\lambda_n]}(Z_m^{(i)}-E_0(Z))\to 0 \quad \textit{in } \tilde{P}\textit{-probability as } n\to\infty, \tag{70}$$

where $d\tilde{P}/dP_0(\omega)=\lambda T_1(\omega)$, and $(Z_m^{(i)})_{m\in\mathbb{Z}}$ for $i=1,2,\ldots$ are iid copies of $(Z_m)_{m\in\mathbb{Z}}$.

(iii) *The following version of the assumption (65) is satisfied:*

$$b_n^{-1}\sum_{i=1}^{n}I_i^*(0)\xrightarrow{P}0,\quad n\to\infty. \tag{71}$$

Then, with $\tilde{S}_n(t)=b_n^{-1}D_{n,\lambda_n}(t)$ for $n\geq 1$ and $t\geq 0$, we have

$$(\tilde{S}_n(t))_{t\geq 0}\xrightarrow{\text{fidi}}(E_0(Z)V(t))_{t\geq 0},\quad n\to\infty. \tag{72}$$

The proof follows from the decomposition (67) in the same way as in Theorem 5.1.

5.1. A renewal Poisson cluster process: The case where the variability of the marks is dominating. In the case where the variability of the marks dominates that of the underlying Poisson process, many different limits are possible; see the discussion after Remark 5.3. In this section we will only consider the situation of Theorem 5.1, and apply it to the renewal Poisson cluster process of §3.2.

Once again, the simplest case is that of the infinite-source Poisson model. Assuming that the marks are regularly varying with index $1<\alpha<2$, we see that (60) holds with respect to the Palm measure, with $a(t)=\bar{F}_Z^{\leftarrow}(1/t)$ for $t>1$, where we are using the generalized inverse of $\bar{F}_Z$; see, e.g., Resnick [33]. Because the function $a(t)$ is regularly varying with exponent $1/\alpha>1/2$, it is simply seen by computing the second moment that the condition (64) holds for all rates λ_n. Furthermore, the random variable $I^*(0)$ satisfies

$$P(I^*(0)>x)\sim\lambda\int_x^{\infty}\bar{F}_Z(u)\,dy\sim\frac{\lambda}{\alpha-1}x\bar{F}_Z(x).$$

Therefore, the condition (65) is equivalent to $nP(I^*(0)>a(n\lambda_n))\to 0$, which is the same as

$$\lim_{n\to\infty}a(n\lambda_n)/\lambda_n\to 0\quad\text{as } n\to\infty, \tag{73}$$

and under this condition Theorem 5.1 gives us convergence to an α-stable Lévy motion. This is the slow growth condition of Mikosch et al. [28].

This conclusion is a particular case of the following result describing one situation when Lévy stable limits are obtained for a renewal Poisson cluster input process.

Proposition 5.5. *Suppose $\bar{F}_Z \in \mathrm{RV}_{-\alpha}$ for some $1 < \alpha < 2$, and assume that both*

$$EK^\theta < \infty \qquad \text{and} \qquad EX^\theta < \infty \quad \text{for some } \theta > 3 - 2/\alpha. \tag{74}$$

Let $a(t) = \bar{F}_Z^{\leftarrow}(1/t)$. Then, for any sequence (λ_n) satisfying (73), the cumulative input process $(\tilde{S}_n)$ converges to an α-stable Lévy process, i.e., (66) holds, where $V(1)$ has the $S_\alpha(\sigma, 1, 0)$ distribution, with $\sigma^{-\alpha} = C_\alpha$ being the stable tail exponent, see Samorodnitsky and Taqqu [34].

Proof. We will check the assumptions of Theorem 5.1. As before, (60) holds with respect to the Palm measure by the independence and regular variation of the marks. To check (64), we use the notation $N^A(B)$ for the number of active sources in B initiated in A, and estimate the variance. Note that for every $t > 0$,

$$\mathrm{Var} N^{[0,t]}([0, t]) = \lambda_0 t E(N_c[0, tU])^2 = \lambda_0 t E[K^2 \wedge (N_r[0, tU])^2]$$

(see the notation of §3.2), where N_r is the (nonstopped) renewal process (potentially) generated by each cluster, and U is an independent standard uniform random variable. The obvious stochastic domination of $N_r[0, t]$ by a negative binomial random variable shows that there is $b \le 1$ and $C > 0$ such that

$$E[(N_r[0, t])^2 I\{N_r[0, t] > bt\}] \le C\mathrm{e}^{-t/C}.$$

Therefore, using (74) we obtain (changing, if necessary, the constant C)

$$\mathrm{Var}(N^{[0,t]}([0, t])) \le \lambda_0 t[C\mathrm{e}^{-t/C} + b^2 E(K^2 \wedge t^2)] \le C t^{1+(2-\theta)_+}. \tag{75}$$

Furthermore, $N^{(-\infty,0)}([0, t])$ is an infinitely divisible random variable with Lévy measure given by

$$\mu(B) = \int_0^\infty P(N_c[x, x+t] \in B)\, dx$$

for any Borel set B, and so

$$\begin{aligned}\mathrm{Var}(N^{(-\infty,0)}([0, t])) &= \int_0^\infty E(N_c[x, x+t])^2\, dx \\ &\le tE(N_c[0, 2t])^2 + E(N_r[0, t])^2 \int_t^\infty P\left(\sum_{j=1}^K X_j > x\right) dx.\end{aligned} \tag{76}$$

We have already checked that the first term on the right-hand side of (76) is bounded by $Ct^{1+(2-\theta)_+}$. Furthermore, the assumption (74) implies that $E(\sum_{j=1}^K X_j)^\theta < \infty$. The stochastic domination of $N_r[0, t]$ by a negative binomial random variable now shows that

$$\mathrm{Var}(N^{(-\infty,0)}([0, t])) \le Ct^{1+(2-\theta)_+} + Ct^2 t^{-(\theta-1)} E\left(\sum_{j=1}^K X_j\right)^\theta \le Ct^{1+(2-\theta)_+} \tag{77}$$

with, perhaps, changing constants C. Using (75) and (77), we see that for $t > 0$

$$\mathrm{Var}\left((a(n\lambda_n))^{-1} \sum_{i=1}^n (N_i(\lambda_n t) - \lambda\lambda_n t)\right) \sim \frac{n\mathrm{Var}N(\lambda_n t)}{(n\lambda_n)^2} \le C\frac{n\lambda_n^{1+(2-\theta)_+}}{(n\lambda_n)^2} \to 0, \quad n \to \infty,$$

by the fact that a is regularly varying with exponent $1/\alpha$ and the lower bound on θ given in (74). Therefore, (64) follows (without any restrictions on the sequence (λ_n)).

Next we check condition (65). Notice that $I^*(0)$ is an infinitely divisible random variable with Lévy measure given by $\mu(B) = \int_0^\infty P(A^{(c)}(x) \in B)\, dx$ for any Borel set B, where $A^{(c)}(x)$ is the total amount of work in the

sessions belonging to a single cluster, initiated at zero, that starts by the time $x > 0$ but does not finish by that time. Therefore, for $z > 0$,

$$\begin{aligned}\mu(z,\infty) &\le \int_0^z P(A^{(c)}(x) > z)\,dx + \int_z^\infty P\left(\sum_{j=1}^K X_j + \max\bigl(Z_0, Z_1, \ldots, Z_K\bigr) > x\right) dx \\ &= R_1(z) + R_2(z). \end{aligned} \tag{78}$$

Notice that

$$R_1(z) \le zP\left(\sum_{j=0}^K Z_j > z\right) \sim EKzP(Z > z)$$

as $z \to \infty$; see, e.g., Proposition 4.1 in Faÿ et al. [11]. Furthermore, the fact that $E(\sum_{j=1}^K X_j)^\theta < \infty$ implies that $P(\sum_{j=1}^K X_j > z) = o(\bar{F}_Z(z))$. Because we also have $P(\max(Z_0, Z_1, \ldots, Z_K) > z) = O(\bar{F}_Z(z))$, we see that

$$P\left(\sum_{j=1}^K X_j + \max\bigl(Z_0, Z_1, \ldots, Z_K\bigr) > z\right) = O(P(Z > z)),$$

and for large z,

$$R_2(z) \le C\int_z^\infty P(Z > x)\,dx \sim \frac{C}{\alpha - 1} zP(Z > z), \quad z \to \infty.$$

By (78) we then have for large z, $\mu(z,\infty) \le CzP(Z > z)$, where C is a finite constant. A stochastic domination argument and the fact that, if the Lévy measure of an infinitely divisible random variable has a subexponential tail, then the distributional tail of the random variable is asymptotically equivalent to the tail of the Lévy measure (see Embrechts et al. [9]), show that for large z $P(I^*(0) > z) \le Cz\bar{F}_Z(z)$. Therefore, as in the case of the infinite-source Poisson model, we conclude that (65) holds if (73) does. This completes the proof. □

5.2. A Poisson cluster process: The case when the variability of the underlying point process is dominating. Surprisingly, even in the case when the variability of the underlying point process dominates that of the marks, many different limits are possible. We will consider the situation of Theorem 5.2, and we will apply it to Poisson cluster processes (not only Poisson cluster renewal processes) of §3.2.

Specifically, we will assume that the cluster point process N_c is a general stopped point process

$$N_c[0,t] = N_0[0,t] \wedge (K+1), \quad t \ge 0, \tag{79}$$

where N_0 is a point process that has a point at the origin, independent of a nonnegative integer-valued random variable K.

We will assume in this section that the tail $\bar{F}_K \in \mathrm{RV}_{-\alpha}$ for some $\alpha \in (1,2)$. It turns out that the limiting behavior of the input process is largely determined by the relation between the tail index α and the rate of asymptotic growth of the arrival times of the point process N_0. The first result here exhibits a situation when the latter rate of growth is relatively slow, and the input process has a stable Lévy process in the limit.

Proposition 5.6. *Assume N_c satisfies (79), where K is integer-valued with tail $\bar{F}_K \in \mathrm{RV}_{-\alpha}$ for some $\alpha \in (1,2)$. Moreover, assume that the arrival times of N_0, $0 = T_0^{(0)} \le T_1^{(0)} \le T_2^{(0)} \le \cdots$, satisfy the relation*

$$ET_n^{(0)} \le Cn^{\alpha-\varepsilon}, \quad n \ge 1 \tag{80}$$

for some $C > 0$ and $\varepsilon \in (0, \alpha - 1]$. Assume that the marks Z_m form, under the Palm measure, a sequence of iid random variables independent of the underlying point process, and such that

$$P_0(|Z| > z) = o(P(K > z)), \quad z \to \infty. \tag{81}$$

Let $a(T) \uparrow \infty$ be such that $P(K > a(T)) \sim T^{-1}$ as $T \to \infty$. Then for any sequence $\lambda_n \to \infty$

$$\lim_{n\to\infty} (a(n\lambda_n))^{\alpha-\epsilon}/\lambda_n \to 0, \tag{82}$$

the cumulative input process $(\tilde{S}_n)$ satisfies (72) with $b_n = a(n\lambda_n)$, where the limit process V is an α-stable Lévy motion and $V(1)$ has the same distribution as in Proposition 5.5.

Assumption (80) is clearly satisfied for renewal processes with a finite first moment if one chooses $\epsilon = \alpha - 1$. Notice that, in this case, the slow growth condition (82) coincides with the slow growth condition (73) of the previous section.

The proof of Proposition 5.6 is given in §5.2.1.

If assumption (80) fails, the limit can be different from a Lévy stable motion, and, in fact, one can have a fractional Brownian limit under slow growth conditions as well! Specifically, suppose that for some $\beta > \alpha$ that nth arrival time $T_n^{(0)}$ of the point process N_0 is, roughly speaking, of the order n^β. We will see that in certain circumstances one can expect in the limit a fractional Brownian motion with

$$H = (\beta + 2 - \alpha)/(2\beta). \tag{83}$$

More precisely, assume that there is a function $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ that is regularly varying at infinity with exponent $\beta > \alpha$ such that

$$(T_{[yz]}^{(0)}/h(y))_{z\geq 0} \xrightarrow{\text{fidi}} (T_*(z))_{z\geq 0} \tag{84}$$

to some right-continuous process $(T_*(z))$. We have two basic examples in mind.

Example 5.1. Assume that $T_n^{(0)}$ is a deterministic sequence, given by

$$T_n^{(0)} = bn^\beta, \quad n = 0, 1, 2, \ldots, \quad \text{for some } b > 0. \tag{85}$$

In this case $h(u) = u^\beta$ and (84) holds with $T_*(z) = bz^\beta$, $z \geq 0$.

Example 5.2. Assume that $(T_n^{(0)})$ form an infinite mean renewal process with $X_n = T_n^{(0)} - T_{n-1}^{(0)}$ for $n \geq 1$ being independent random variables with $\bar{F}_X \in \mathrm{RV}_{-1/\beta}$. In this case, one can take $h(u) = (1 - F_X)^{\leftarrow}(1/u)$ for $u > 1$, and (84) holds with $(T_*(z))$ being a strictly $1/\beta$-stable subordinator.

Under slow growth conditions the suitable normalization is given by

$$b_n = (n\lambda_n (h^{\leftarrow}(\lambda_n))^2 P(K > h^{\leftarrow}(\lambda_n)))^{1/2}, \tag{86}$$

and, under certain assumptions, the limiting process V in (72) will be a fractional Brownian motion $V = B_H$ with H given by (83) and variance

$$\begin{aligned}\mathrm{Var}(B_H(1)) &= \frac{2-\alpha}{2+\beta-\alpha}\int_0^\infty y^{-(2+\beta-\alpha)/\beta} P(T_*(1) \leq y)\, dy \\ &\quad + \int_0^\infty E\left[\frac{4-\alpha}{2-\alpha}(I(w+1))^{2-\alpha} + \frac{4-2\alpha}{\alpha-1}(I(w+1))^{1-\alpha} I(w) - \frac{\alpha^2-3\alpha+4}{(2-\alpha)(\alpha-1)}(I(w))^2\right] dw. \\ &= \sigma_1^2 + \sigma_2^2.\end{aligned} \tag{87}$$

Here $(I(w))$ is the first hitting time process of $(T_*(z))$:

$$I(w) = \inf\{z \geq 0\colon T_*(z) > w\}, \quad w \geq 0. \tag{88}$$

In Proposition 5.7 we establish the above convergence in the setup of Example 5.1. The situation of Example 5.2 will be considered elsewhere.

Proposition 5.7. *Assume N_c satisfies (79), where K is integer valued and $\bar{F}_K \in \mathrm{RV}_{-\alpha}$ for some $\alpha \in (1, 2)$. Suppose that the arrival times of N_0 satisfy (85) of Example 5.1. Assume that the marks Z_m form, under the Palm measure, a sequence of iid random variables, independent of the underlying point process, and such that*

$$E_0|Z|^\gamma < \infty \quad \textit{for some } \gamma > \max\left(\frac{2\beta}{2+\beta-\alpha}, \frac{\beta+1}{\beta-1}\right). \tag{89}$$

Choose any sequence $\lambda_n \uparrow \infty$ and (b_n) from (86) such that

$$\lim_{n\to\infty} (n\lambda_n)^{2/\min(\gamma,2)}/b_n^2 = 0, \tag{90}$$

$$\lim_{n\to\infty} nb_n P(K > b_n) = 0, \tag{91}$$

$$\lim_{n\to\infty} n^{2/\min(2,\,\gamma-(\beta+1)/(\beta-1))}/b_n^2 = 0. \tag{92}$$

Then the cumulative input process $(\tilde{S}_n)$ satisfies (72) with b_n given by (86), and the limit process V is fractional Brownian motion B_H with H given by (83) and $\mathrm{Var}(B_H(1))$ given by (87).

It is interesting to observe that the limiting fractional Brownian motion satisfies $0.5 < H < \alpha^{-1}$ with $H \to \alpha^{-1}$ as $\beta \downarrow \alpha$ and $H \to 0.5$ as $\beta \uparrow \infty$.

Note that (90) does not impose any constraints on the sequence (λ_n) in the case $\gamma \geq 2$, whereas neither (90) nor (92) impose any constraints on the sequence (λ_n) in the case $\gamma \geq 2 + (\beta+1)/(\beta-1)$. Furthermore, a sufficient condition for (91) is $n^{(1+\alpha')/(\alpha'-1)+1} = o(b_n^2)$ for some $1 < \alpha' < \alpha$.

The proof of Proposition 5.7 is given in §5.2.2.

5.2.1. Proof of Proposition 5.6. We will verify the assumptions of Theorem 5.2. We start by checking the convergence assumption (69). Observe that we can write

$$\begin{aligned}\sum_{i=1}^{n}(N_i(\lambda_n t) - \lambda\lambda_n t) &= \sum_{i=1}^{n}(N_i^{(0,\lambda_n t]}(\lambda_n t) - EN^{(0,\lambda_n t]}(\lambda_n t)) + \sum_{i=1}^{n}(N_i^{(-\infty,0]}(\lambda_n t) - EN^{(-\infty,0]}(\lambda_n t)) \\ &= a(n\lambda_n)[S_n^+(t) + S_n^-(t)], \end{aligned} \tag{93}$$

where for any Borel sets A and B, $N^A(B)$ is the number of active sources in B initiated in A, and the subscript i refers, as usual, to a particular input process. For convenience, we also write here $N_i^{(0,\lambda_n t]}(\lambda_n t) = N_i^{(0,\lambda_n t]}(0, \lambda_n t]$. We will show that, for every $t > 0$,

$$S_n^+(t) \xrightarrow{d} V(t) \tag{94}$$

$$S_n^-(t) \xrightarrow{P} 0. \tag{95}$$

By the stationarity of the increments and by the fact that a Poisson random measure is independently scattered, this will imply the convergence to Lévy motion stated in the proposition; cf. the proof of Proposition 3.5 in Faÿ et al. [11] for a similar situation.

Notice that both $S_n^+(t)$ and $S_n^-(t)$ are infinitely divisible random variables whose characteristic functions can be written in the form

$$E\exp\{i\theta S_n^{\pm}(t)\} = \exp\left\{\int_0^\infty (e^{i\theta x} - 1 - i\theta x)\nu_n^{\pm}(dx)\right\},$$

where $\nu_n^{\pm}$ are the corresponding Lévy measures, given by

$$\nu_n^{\pm} = n(P_1 \times \mathrm{Leb}) \circ T_{\pm}^{-1}. \tag{96}$$

Here $(\Omega_1, \mathcal{F}_1, P_1)$ is the probability space on which a cluster point process $(N_c(u), u \geq 0)$ is defined, and the map T_+: $\Omega_1 \times \mathbb{R} \to [0,\infty)$ is given by $T_+(\omega_1, u) = N_c(0,u]/a(n\lambda_n)$, and T_-: $\Omega_1 \times [0, \lambda_n t] \to [0,\infty)$ by $T_-(\omega_1, u) = N_c(u, u+\lambda_n t]/a(n\lambda_n)$. For notational simplicity in the calculations below, we drop the subscript (i.e., write P, E instead of P_1, E_1, etc.).

For the proofs of (94) and (95) we will use standard results for the weak convergence of infinitely divisible distributions; see, e.g., Theorem 15.14 in Kallenberg [21]. The necessary and sufficient conditions for convergence are formulated in the next lemma, after which we proceed to verify its assumptions. We will exploit the following notation:

$$I_1(n) = n\int_0^\infty E(1 \wedge T_-^2)\,du,$$

$$I_2(n) = n\int_0^\infty E(T_- I\{T_- > x\})\,du,$$

$$I_3(n) = n\int_0^{\lambda_n t} P(T_+ > x)\,du,$$

$$I_4(n) = n\int_0^{\lambda_n t} E(T_+^2 I\{T_+ \leq \epsilon\})\,du,$$

$$I_5(n) = n\int_0^{\lambda_n t} E(T_+ I\{T_+ > y\})\,du.$$

Lemma 5.1. *If* $\lim_{n\to\infty} I_1(n) = \lim_{x\to\infty}\limsup_{n\to\infty} I_2(n) = 0$, *then* (95) *holds. If* $\lim_{n\to\infty} I_3(n) = tx^{-\alpha}$ *for all* $x > 0$, $\lim_{\epsilon\to 0}\limsup_{n\to\infty} I_4(n) = \lim_{y\to\infty}\limsup_{n\to\infty} I_5(n) = 0$, *then* (94) *holds.*

Verification of the assumptions of Lemma 5.1. We have

$$\begin{aligned} I_1(n) &\le n\int_0^\infty E(1\wedge((a(n\lambda_n))^{-1}N_c(u,\infty))^2)\,du \\ &= n\int_0^\infty P(N_c(u,\infty)>a(n\lambda_n))\,du + \frac{n}{(a(n\lambda_n))^2}\int_0^\infty E[(N_c(u,\infty))^2 I\{N_c(u,\infty)\le a(n\lambda_n)\}]\,du \\ &= I_{11}(n)+I_{12}(n). \end{aligned} \tag{97}$$

We have

$$\begin{aligned} I_{11}(n) &\le nE[T_K^{(0)} I\{K>a(n\lambda_n)\}] \le CnE[K^{\alpha-\epsilon} I\{K>a(n\lambda_n)\}] \\ &\sim Cn(a(n\lambda_n))^{\alpha-\epsilon}P(K>a(n\lambda_n)) \sim C(a(n\lambda_n))^{\alpha-\epsilon}/\lambda_n \to 0 \quad \text{as } n\to\infty, \end{aligned} \tag{98}$$

where we have used (80) and (82). We remind the reader that C is a generic finite positive constant, not always the same as the one in (80). Further,

$$\begin{aligned} I_{12}(n) &= \frac{n}{(a(n\lambda_n))^2}E\left[I\{K\le a(n\lambda_n)\}\int_0^{T_K^{(0)}}(N_c[u,\infty))^2\,du\right] \\ &\quad + \frac{n}{(a(n\lambda_n))^2}E\left[I\{K>a(n\lambda_n)\}\int_0^{T_K^{(0)}}(N_c[u,\infty))^2 I\{N_c(u,\infty)\le a(n\lambda_n)\}\,du\right] \\ &= I_{121}(n)+I_{122}(n). \end{aligned} \tag{99}$$

Observe that, again using (80) and (82),

$$\begin{aligned} I_{121}(n) &\le \frac{n}{(a(n\lambda_n))^2}E[K^2T_K^{(0)}I\{K\le a(n\lambda_n)\}] \\ &\le C\frac{n}{(a(n\lambda_n))^2}E[K^{2+\alpha-\varepsilon}I\{K\le a(n\lambda_n)\}] \\ &\sim C\frac{n}{(a(n\lambda_n))^2}(a(n\lambda_n))^{2+\alpha-\epsilon}P(K>a(n\lambda_n)) \\ &\sim C(a(n\lambda_n))^{\alpha-\epsilon}/\lambda_n \to 0 \quad \text{as } n\to\infty. \end{aligned} \tag{100}$$

Furthermore,

$$I_{122}(n) \le nE[I\{K>a(n\lambda_n)\}T_K^0] \to 0 \quad \text{as } n\to\infty \tag{101}$$

as in (98). Now $\lim_{n\to\infty} I_1 = 0$ follows from (97)–(101).

Observe that for $a>0$, and a point mass at a point $b>0$, $\int_0^\infty \delta_b(u,u+a]\,du \le a$. Because N_c has K points, we have for $x>0$,

$$\begin{aligned} I_2(n) &\le \frac{n}{a(n\lambda_n)}E\left[I\{K>xa(n\lambda_n)\}\int_0^\infty N_c(u,u+\lambda_n t]\,du\right] \\ &\le \frac{n}{a(n\lambda_n)}E[I\{K>xa(n\lambda_n)\}\lambda_n tK] \\ &\sim C[n\lambda_n/a(n\lambda_n)][(xa(n\lambda_n))P(K>xa(n\lambda_n))] \\ &\to Cx^{-(\alpha-1)}, \quad n\to\infty, \end{aligned}$$

by regular variation of $\bar{F}_K$. Now $\lim_{x\to\infty}\limsup_{n\to\infty} I_2 = 0$ follows because $\alpha>1$, and so (95) is established.

We now switch to proving (94). We have for $I_3(n)$ the upper bound

$$\limsup_{n\to\infty} I_3(n) \le \limsup_{n\to\infty} n(\lambda_n t)P(K>xa(n\lambda_n)) = \limsup_{n\to\infty} n(\lambda_n t)x^{-\alpha}(n\lambda_n)^{-1} = tx^{-\alpha}.$$

On the other hand, using (80) and the condition (82), we have the lower bound

$$\begin{aligned} I_3(n) &= nE(I\{K>xa(n\lambda_n)\}(\lambda_n t - T^{(0)}_{[xa(n\lambda_n)]})_+) \\ &\ge nE(I\{K>xa(n\lambda_n)\}(\lambda_n t - ET^{(0)}_{[xa(n\lambda_n)]})) \\ &\ge nE(I\{K>xa(n\lambda_n)\}(\lambda_n t - C(xa(n\lambda_n))^{\alpha-\varepsilon})) \\ &= (1-o(1))nP(K>xa(n\lambda_n))(\lambda_n t) \sim tx^{-\alpha}, \quad n\to\infty. \end{aligned}$$

We conclude that $\lim_{n\to\infty} I_3(n) = tx^{-\alpha}$ holds for fixed $t,x>0$.

For $I_4(n)$ we have for any $\varepsilon, \delta > 0$ and for large n by virtue of Karamata's theorem

$$\begin{aligned} I_4(n) &\le \frac{n}{(a(n\lambda_n))^2}(\lambda_n t)E(K^2 I\{K \le \delta a(n\lambda_n)\}) \\ &\quad + \frac{n}{(a(n\lambda_n))^2}\int_0^{\lambda_n t} E((N_c(0,u])^2 I\{K > \delta a(n\lambda_n), N_c(0,u] \le \varepsilon a(n\lambda_n)\})\,du \\ &\le C\delta^2(n\lambda_n)P(K > \delta a(n\lambda_n)) + \frac{n}{(a(n\lambda_n))^2}(\lambda_n t)\varepsilon^2(a(n\lambda_n))^2 P(K > \delta a(n\lambda_n)) \\ &\le C(n\lambda_n)P(K > \delta a(n\lambda_n))(\delta^2+\varepsilon^2) \sim C\delta^{-\alpha}(\delta^2+\varepsilon^2), \quad n \to \infty. \end{aligned}$$

Hence,

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} I_4(n) \le C\delta^{2-\alpha} \to 0 \quad \text{as } \delta \to 0.$$

Finally, for $I_5(n)$ we have

$$I_5(n) \le \frac{n}{a(n\lambda_n)}(\lambda_n t)E(KI\{K > ya(n\lambda_n)\}),$$

and then $\lim_{y\to\infty}\limsup_{n\to\infty} I_5(n) = 0$ follows in the same way as the corresponding statement for $I_2(n)$ above. And so we have established (94).

We have now verified condition (69) of Theorem 5.2. The second assumption of that theorem, (70), follows directly from (81). It remains to check assumption (71). We use a decomposition somewhat different from (78). Notice that for $x > 0$,

$$P(A^{(c)}(x) > z) \le P\left(\sum_{j=0}^K Z_j > z, T_K^{(0)} > x/2\right) + P(\max(Z_0, Z_1, \ldots, Z_K) > x/2).$$

Therefore,

$$\begin{aligned} \mu(z,\infty) &\le zP\left(\sum_{j=0}^K Z_j > z\right) + P\left(\sum_{j=0}^K Z_j > z\right)\int_z^\infty P(T_K^{(0)} > x/2)\,dx + \int_z^\infty P(\max(Z_0, Z_1, \ldots, Z_K) > x/2)\,dx \\ &= R_1(z) + R_2(z) + R_3(z). \end{aligned}$$

Therefore, as in the proof of Proposition 5.5, the assumption (65) will follow once we show that

$$R_i(z) \le CzP(K > z) \quad \text{for } i = 1, 2, 3 \text{ and } z \text{ large enough.} \tag{102}$$

Using Proposition 4.3 in Faÿ et al. [11], we see that (102) holds for $i = 1, 3$. Because

$$\int_z^\infty P(T_K^{(0)} > x/2)\,dx \le 2ET_K^{(0)} \le CEK^{\alpha-\epsilon} < \infty,$$

we can once again use Proposition 4.3 in Faÿ et al. [11] to see that (102) holds for $i = 2$. This proves (65) and therefore completes the proof of the proposition. □

5.2.2. Proof of Proposition 5.7. Once again, we verify the assumptions of Theorem 5.2, and we start by checking the convergence assumption (69). For this, we will establish that for every $t > 0$,

$$b_n^{-1}\sum_{i=1}^n (N_i(\lambda_n t) - \lambda\lambda_n t) \xrightarrow{d} B_H(t). \tag{103}$$

Then, stationarity of the N_is implies that for any $t_1 < t_2$

$$b_n^{-1}\sum_{i=1}^n (N_i(\lambda_n t_1, \lambda_n t_2) - \lambda\lambda_n(t_2 - t_1)) \xrightarrow{d} B_H(t_2) - B_H(t_1). \tag{104}$$

The latter relation implies tightness of the family of random variables $b_n^{-1}\sum_{i=1}^n (N_i(\lambda_n t_j) - \lambda\lambda_n t_j)$, $j = 1, \ldots, k$, for any choice of $0 \le t_1 < t_2 < \cdots < t_k$ and $k \ge 1$. Any of the laws of the above family is infinitely divisible and

so are their weak limits. Because the marginal laws of any such weak limit point are Gaussian, the weak limit points of the above family are Gaussian as well. Relation (104) determines the covariance structure of the weak limits, which coincides with the one of B_H. This will prove (69).

We proceed, therefore, to show (103). To this end we will again use decomposition (93), but the normalization $a(n\lambda_n)$ will be replaced by b_n, and we also use the symbols $S_n^{\pm}$ abusing notation.

By the obvious independence it is then enough to show that

$$S_n^+(t) \xrightarrow{d} t^H N(0,\sigma_1^2) \qquad \text{and} \qquad S_n^-(t) \xrightarrow{d} t^H N(0,\sigma_2^2), \tag{105}$$

where σ_i^2, $i=1,2$, are defined in (87). We will check (105) for S_n^+; the proof for S_n^- is similar. Again, Theorem 15.14 in Kallenberg [21] gives necessary and sufficient conditions for this convergence in terms of Lévy measures. Using the Lévy measure description given in (96) with the corresponding modification of the normalizing sequence, one needs to prove

$$\lim_{n\to\infty}\int_0^{\lambda_n t} E\left[\left(\frac{N_c(0,u]}{b_n}\right)^2 I\{N_c(0,u]\le \epsilon b_n\}\right] du = t^{2H}\sigma_1^2, \quad \epsilon>0, \tag{106}$$

$$\lim_{n\to\infty}\int_0^{\lambda_n t} E\left[\frac{N_c(0,u]}{b_n} I\{N_c(0,u]> y b_n\}\right] du = 0, \quad y>0. \tag{107}$$

Notice that by (86), $\lambda_n/h(b_n)\to 0$ and so, for large n, the integral on the left-hand side of (107) vanishes. We now concentrate on (107), in which we set $\epsilon=1$, the general case being analogous. Denote by $J(n)$ the expression under the limit on the left-hand side of (107). We have

$$\begin{aligned} J(n) &= \frac{n}{b_n^2} E\left(I\{K\le b_n\}\int_0^{\lambda_n t}(N_c(0,u])^2\,du\right) + \frac{n}{b_n^2}E\left(I\{K>b_n\}\int_0^{(\lambda_n t)\wedge T^{(0)}_{[b_n]}}(N_c(0,u])^2\,du\right)\\ &= J_1(n)+J_2(n). \end{aligned}$$

We have for any $T>0$,

$$\begin{aligned} \int_0^T (N_c(0,u])^2\,du &= \sum_{n=1}^{[(T/b)^{1/\beta}]\wedge K} n^2(T_n^{(0)}-T_{n-1}^{(0)}) + K^2(T-bK^\beta)_+\\ &= b\sum_{n=1}^{[(T/b)^{1/\beta}]\wedge K} n^2(n^\beta-(n-1)^\beta) + K^2(T-bK^\beta)_+. \end{aligned}$$

Hence,

$$\begin{aligned} J_1(n) &= \frac{n}{b_n^2}E\left(I\{K\le b_n\} b\sum_{n=1}^{[(\lambda_n t/b)^{1/\beta}]\wedge K} n^2(n^\beta-(n-1)^\beta)\right) + \frac{n}{b_n^2}E(I\{K\le b_n\}K^2(\lambda_n t - bK^\beta)_+)\\ &= J_{11}(n)+J_{12}(n). \end{aligned}$$

Clearly, $J_{12}(n)\le (n/b_n^2)E(K^2 I\{K\le (t/b)^{1/\beta}\lambda_n^{1/\beta}\}) \le (n\lambda_n^{1/\beta}/b_n^2)(t/b)^{1/\beta}EK\to 0$. Moreover, by Karamata's theorem,

$$\begin{aligned} J_{11}(n) &\sim b\beta\frac{n}{b_n^2}E\left(I\{K\le b_n\}\sum_{n=1}^{[(\lambda_n t/b)^{1/\beta}]\wedge K} n^{1+\beta}\right)\\ &\sim b\frac{\beta}{\beta+2}\frac{n}{b_n^2}E(I\{K\le b_n\}((\lambda_n t/b)^{1/\beta}\wedge K)^{2+\beta})\\ &\sim b\frac{\beta}{2+\beta}\frac{n}{b_n^2}\int_0^{(\lambda_n t/b)^{(2+\beta)/\beta}} P(K>y^{1/(2+\beta)})\,dy\\ &\sim b\frac{\beta}{2+\beta}\frac{n}{b_n^2}\frac{2+\beta}{2+\beta-\alpha}(\lambda_n t/b)^{(2+\beta)/\beta}P(K>(\lambda_n t/b)^{1/\beta})\\ &\sim t^{2H}\sigma_1^2. \end{aligned}$$

Collecting the above estimates, we conclude that $J_1(n) \sim t^{2H}\sigma_1^2$. Now we turn to $J_2(n)$. Because $\lambda_n/h(b_n) \to 0$, we have

$$\begin{aligned} J_2(n) &\le \frac{n}{b_n^2} E\left(I\{K > b_n\} b \sum_{n=1}^{[(\lambda_n t/b)^{1/\beta}]} n^2 (n^\beta - (n-1)^\beta) \right) \\ &\le C \frac{n}{b_n^2} E(I\{K > b_n\}) \sum_{n=1}^{[(\lambda_n t/b)^{1/\beta}]} n^{1+\beta} \\ &\le C \frac{n}{b_n^2} \lambda_n^{(2+\beta)/\beta} P(K > b_n) = C \frac{P(K > b_n)}{P(K > \lambda_n^{1/\beta})} \to 0. \end{aligned}$$

Taking into account the above bounds, we conclude that (107) is satisfied. Therefore, we have checked the convergence assumption (69) in Theorem 5.2.

For the remaining assumptions, one needs to use the slow growth conditions (90)–(92), as is seen below. The assumption (70) in Theorem 5.2 follows because by the slow growth condition (90) we have $\lim_{n\to\infty}(n\lambda_n)^{1/\gamma}/b_n = 0$. It remains to check the assumption (65) of Theorem 5.2. Notice that, as in the proof of Proposition 5.6,

$$\mu(z, \infty) \le zP\left(\sum_{j=0}^{K} Z_j > z \right) + \int_z^\infty P(A^{(c)}(x) > z)\, dx. \tag{108}$$

Because (89) implied that $\gamma > \alpha$, we can apply Proposition 4.3 in Faÿ et al. [11] to conclude that

$$zP\left(\sum_{j=0}^{K} Z_j > z \right) \sim (EZ)^\alpha z P(K > z), \quad z \to \infty.$$

The slow growth condition (91) guarantees that the contribution of this term to the tail of $I_i^*(0)$ vanishes in the limit in (65). For the second term on the right-hand side in (108) we observe that for $x > z$ large enough, for some $C > 0$

$$\begin{aligned} P(A^{(c)}(x) > z) &\le P(Z > z/2) + P\left(\max_{j \le b^{1/\beta} x^{1/\beta}} Z_j > z/2 + C^{-1} x^{1-1/\beta} \right) \\ &= a_1(x, z) + a_2(x, z). \end{aligned}$$

Notice that for $0 < \rho < \gamma - 1$,

$$a_1(x, z) \le Cz^{-\gamma} \le Cx^{-(1+\rho)} z^{-(\gamma-1-\rho)}.$$

Therefore,

$$\int_z^\infty a_1(x, z)\, dx \le Cz^{-(\gamma-1)},$$

and because $\gamma > \alpha$, we see once again by the slow growth condition (91) that the contribution of this term to the tail of $I_i^*(0)$ vanishes in the limit in (65). Furthermore,

$$\int_z^\infty a_2(x, z)\, dx \le C \int_z^\infty x^{1/\beta} (z + x^{1-1/\beta})^{-\gamma}\, dx \le Cz^{-(\gamma-(\beta+1)/(\beta-1))}.$$

The slow growth condition (92) guarantees that the contribution of this term to the tail of $I_i^*(0)$ vanishes in the limit in (65) as well. This completes the proof of (65) and, hence, of the proposition. □

Acknowledgments. A large part of this research was done with support of Institut Mittag-Leffler of the Royal Swedish Academy of Sciences where the authors participated in the Fall 2004 program on Queuing Theory and Teletraffic Theory. The first author's research was also partially supported by the Danish Research Council (FNU) Grants 21-04-0400 and 272-06-0442. The second author's research is also partially supported by NSF grant DMS-0303493 and NSA grant MSPF-02G-183 at Cornell University. The authors would like to thank the referees and the associate editor for carefully reading the paper. Their suggestions led to a significant improvement of the presentation.

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