Solvency Capital Requirement calculation for different hedge funds strategies.

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Outline

1. Motivation: realistic SCR calculation
2. Peaks Over Threshold approach
3. Precise large deviations in the iid case
4. Precise large deviations in the dependent case

Outline

Solvency II and the SCR

Definition (Solvency Capital Requirement)

"The SCR is the capital required to ensure that the (re)insurance company will be able to meet its obligations over the next 12 months with a probability of at least 99.5%." (Wikipedia)

Solvency II: a challenge for the mathematician

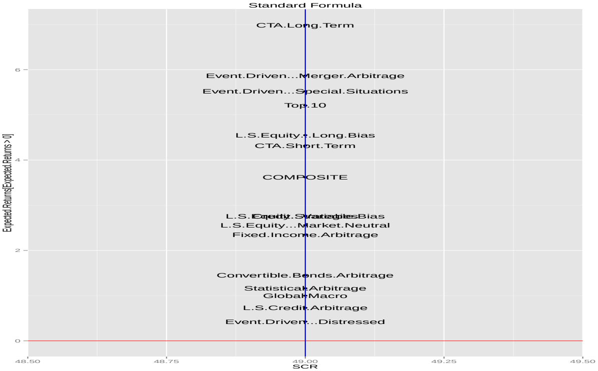
Use a standard formula or find a more realistic calculation of the SCR (quantiles, VaR) using an internal model.

Extrapolation of the magnitude of an event that occurs once per 200 years, i.e. that is not observed!

Motivation: insurance companies as Hedge Fund investors

Standard formula for "other equities"

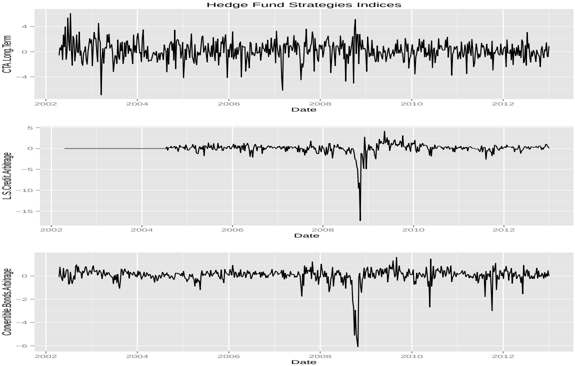
The capital requirement is 48% of the investment whatever is the HF strategy.



Annualized returns Lyxor indices

Limitation of the standard formula

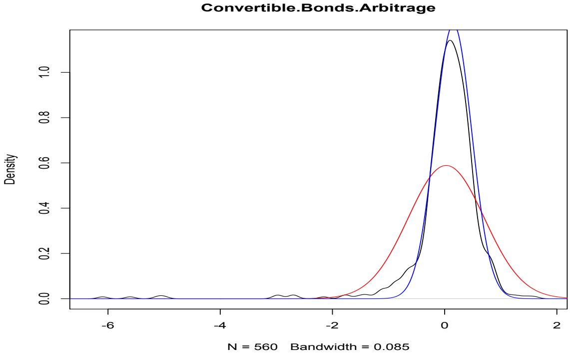
Log ratios $r_t = \log(P_{t+1}/P_t)$ where (P_t) are weakly prices of HF indices.



3 different strategies, same SCRs under standard formula

Outline

Gaussian modeling works on average

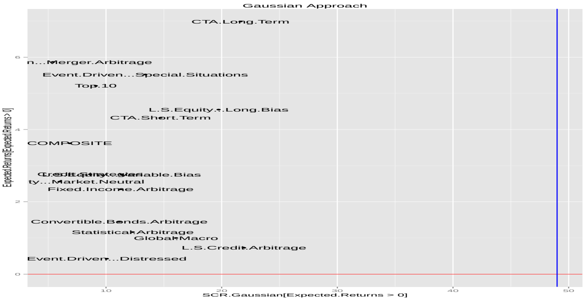


Dark: marginal density of the log-ratio of an index
Red: Gaussian model
Blue: Gaussian model excluding the 40 worst days

Gaussian modeling underestimate tails

$$\mathbb{P}\left(\frac{P_{T+1\text{ an}} - P_T}{P_T} \leqslant SCR\right) = 0.005 \quad <\approx> \quad \mathbb{P}\left(\log\left(\frac{P_{T+1\text{ an}}}{P_T}\right) \leqslant SCR\right) = 0.005$$

$$\approx \quad \mathbb{P}\left(\sum_{t=T}^{T+52} r_t \leqslant SCR\right) = 0.005.$$



Modern portfolio theory, Markowicz (1952)

Peaks Over Threshold approach

To extrapolate, "Let the tails speak by themselves"! (EKM, 1997)

Theorem (Pickands-Balkema-de Haan, 1975-1974)

Let (X_t) iid with distribution F. Denote F_u the distribution of the exceedances

$$F_u(x) = \mathbb{P}(X - u \leqslant x \mid X > u), \qquad x \geqslant 0, \quad X \sim F.$$

Then for a large class of underlying distribution functions F with endpoint x_F

$$\lim_{u \to x_F} \sup_{0 < x < x_F - u} |F_u(x) - G_{\xi,\sigma}(x)| = 0$$

where $G_{\xi,\sigma}$ is the Generalized Pareto Distribution

$$G_{\xi,\sigma}(x) = \begin{cases} 1 - (1 - \xi x/\sigma)^{-1/\xi}, & \text{if} \quad \xi \neq 0, \\ 1 - e^{-x/\sigma}, & \text{if} \quad \xi = 0. \end{cases}$$

Regularly varying distribution

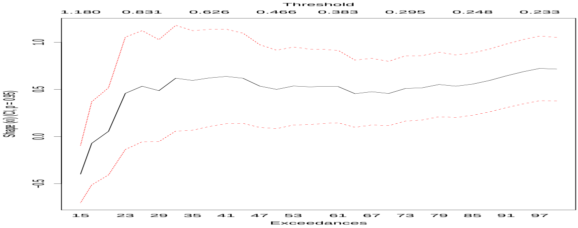
Extrapolation to high quantiles possible iff $\xi > 0$ iff $X \sim F$ is an $\alpha = \xi^{-1}$ regularly varying r.v.:

Definition (Feller, 1971)

$\exists\ p, q \geqslant 0$ with $p + q = 1$ and a slowly varying function L such that

$$\mathbb{P}(X > x) \sim p\,\frac{L(x)}{x^{\alpha}} \quad \text{and} \quad \mathbb{P}(X \leqslant -x) \sim q\,\frac{L(x)}{x^{\alpha}}, \quad x \to \infty.$$

POT approach: fit by ML a GPD on the exceedances under low thresholds. $\hat{\xi}_u$:



Outline

The iid case

Theorem (A.V. Nagaev, 1969)

(X_i) iid random variables with $\alpha > 0$ regularly varying (centered if $\alpha > 1$) distribution then $S_n = \sum_{i=1}^n X_i$ satisfies the precise large deviations relation

$$\lim_{n\to\infty} \sup_{x \geqslant b_n} \left| \frac{\mathbb{P}(S_n > x)}{n\,\mathbb{P}(|X| > x)} - p \right| = 0 \text{ and } \lim_{n\to\infty} \sup_{x \geqslant b_n} \left| \frac{\mathbb{P}(S_n \leqslant -x)}{n\,\mathbb{P}(|X| > x)} - q \right| = 0$$

with $b_n = n^{\delta + 1/(\alpha\wedge 2)}$ for any $\delta > 0$.

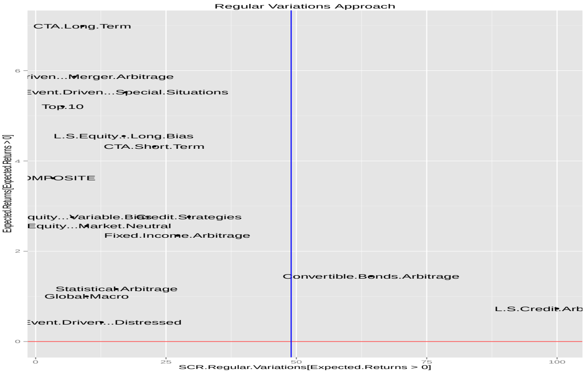
Assume (r_t) iid $\Longrightarrow \mathbb{P}\Big(\sum_{j=T}^{T+52} r_t \leqslant SCR\Big) = 0.005 <\approx> \mathbb{P}\Big(r_t \leqslant SCR\Big) = 0.0001.$

SCR extrapolation, iid case

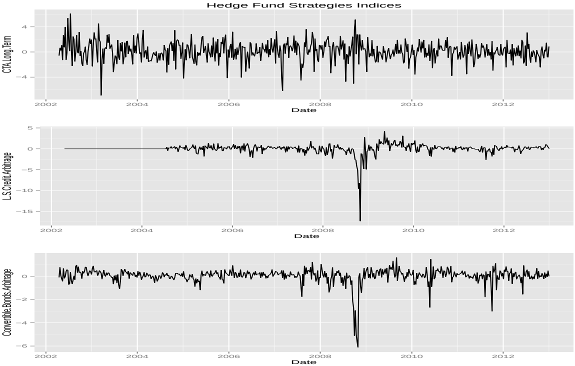
SCR with POT approach based on iid $(r_1, \ldots, r_n)$

$$\widehat{SCR} = \max_{1 \leqslant m \leqslant 40} u_m + \frac{\hat{\sigma}}{\hat{\xi}} \left[\left(\frac{n * 0.0001}{m} \right)^{-\hat{\xi}} - 1 \right]$$

where m is the number of exceedances below $u_m < 0$.



SCR extrapolation when extremes cluster



What is happening for dependent sequences for whom extremes cluster?

Outline

Regularly varying processes

Definition (Basrak & Segers, 2009)

A stationary sequence (X_t) is regularly varying of order $\alpha > 0$ if $\exists$ spectral tail process (Θ_t) defined for any $k \geqslant 0$, any $u > 0$ by the relation

$$\mathbb{P}(|X_0|^{-1}(X_0, \ldots, X_k) \in \cdot, |X_0| > ux \mid |X_0| > x) \xrightarrow[x\to\infty]{w} u^{-\alpha}\mathbb{P}((\Theta_0, \ldots, \Theta_k) \in \cdot).$$

Large deviations in the m-dependent case

Assume $(X_t,\ t \leqslant 0)$ is independent of $\sigma(X_t,\ t \geqslant m)$ then $\Theta_t = 0$ for $|t| \geqslant m$.

Theorem (Mikosch & W., 2012)

Assume (X_t) is $\alpha > 0$ regularly varying (centered if $\alpha > 1$) distribution then

$$\lim_{n\to\infty} \sup_{x \geqslant b_n} \left| \frac{\mathbb{P}(S_n > x)}{n\,\mathbb{P}(|X| > x)} - b_+ \right| = 0 \text{ and } \lim_{n\to\infty} \sup_{x \geqslant b_n} \left| \frac{\mathbb{P}(S_n \leqslant -x)}{n\,\mathbb{P}(|X| > x)} - b_- \right| = 0\,,$$

with $b_n = n^{\delta + 1/(\alpha\wedge 2)}$ for any $\delta > 0$ and cluster indices

$$b_\pm = \mathbb{E}\Big[\Big(\sum_{t=0}^{m-1} \Theta_t\Big)_\pm^\alpha - \Big(\sum_{t=1}^{m-1} \Theta_t\Big)_\pm^\alpha\Big] = \mathbb{E}\Big[\Big(\sum_{t=0}^{m-1} \Theta_t 1_{\Theta_{-j}=0,\,0<j<m}\Big)_\pm^\alpha\Big].$$

Examples

Definition (Conditional spectral tail process)

Define for m-dependent RV(α) processes $(\Theta_0', \ldots, \Theta_m') = (\Theta_0, \ldots, \Theta_k)$ conditionally to $\Theta_{-j} = 0, 0 < j < m$.

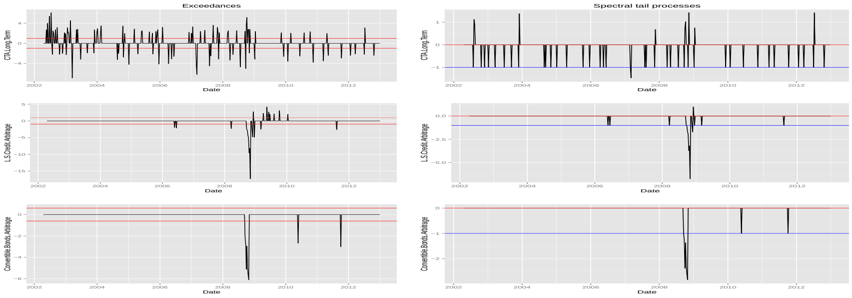
1. (X_t) iid then $\Theta_0' = 1$,
2. $X_t = \max(Z_{t-1}, Z_t)$ then
$$(\Theta_0, \Theta_1)1_{\Theta_{-1}=0} = \begin{cases} (\Theta_0', \Theta_1') = (1,1), & \text{if first exc. at 0, i.e. w.p. } 1/2, \\ (0,0). & \text{else,} \end{cases}$$
3. $X_t = Z_t + \frac{1}{2} Z_{t-1}$ then
$$(\Theta_0, \Theta_1)1_{\Theta_{-1}=0} = \begin{cases} (\Theta_0', \Theta_1') = (1, \frac{1}{2}), & \text{w.p.} = \text{extr. index } \theta_+, \\ (0,0). & \text{else.} \end{cases}$$

Application to risk management

Definition (Empirical conditional spectral tail process)

Define $(\hat{\Theta}'_j, \ldots, \hat{\Theta}'_{j+k}) = (r_j/|r_j|, \ldots, r_{j+k}/|r_j|)$ if $|r_t| > u$, $t \in I = \{j, \ldots, j+k\}$, $|r_{j-1}| < \varepsilon u$ and $|r_{j+k+1}| < u$.

Approximation of $(\Theta_0, \ldots, \Theta_k)$ conditionally on $\Theta_{-j} = 0$, $j \geqslant 1$.



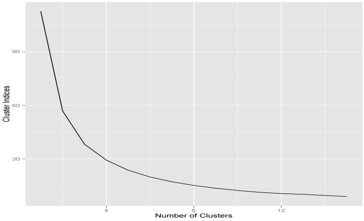
Approximation of the cluster index

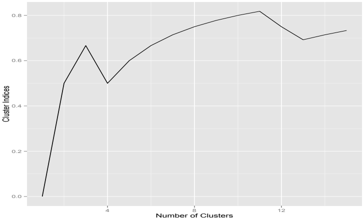
$$\mathbb{P}\Big(\sum_{j=T}^{T+52} r_t \leqslant SCR\Big) = 0.005 <\approx> \mathbb{P}\Big(|r_t| > SCR\Big) = 0.0001/b_-,$$

with $b_- = \mathbb{E}[(\sum_{t=0}^{m-1} \Theta_t 1_{\Theta_{-j}=0,\, 0<j<m})_-^{\alpha}] = \theta_+ \mathbb{E}[(\sum_{t=0}^{m-1} \Theta'_t)_-^{\alpha}]$.

Definition (Empirical cluster index)

$$\hat{b}_-^{\ell} = \frac{1}{\ell}\sum_{i=1}^{\ell}\frac{1}{k_i}\Big(\sum_{t=0}^{k_i}\hat{\Theta}'_{j_i+t}\Big)_-^{\hat{\alpha}} \text{ with } \max(X_t,\ t\in I_i) \geqslant \max(X_t,\ t\in I_{i+1}).$$

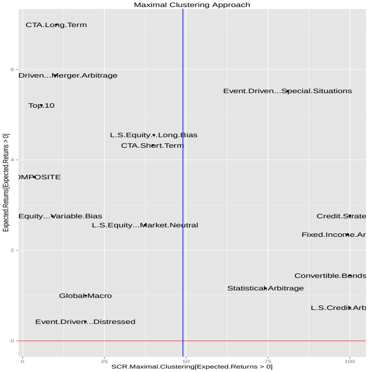


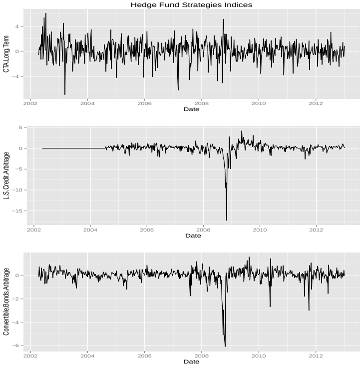


Calculation of the SCR when extremes cluster

Definition

$$\widehat{SCR} = \max_{15 \leqslant m \leqslant 40} \max_{\ell} u_m + \frac{\hat{\sigma}}{\hat{\xi}}\left[\left(\frac{n * 0.0001}{m * \hat{b}^{\ell}_{-}}\right)^{-\hat{\xi}} - 1\right].$$



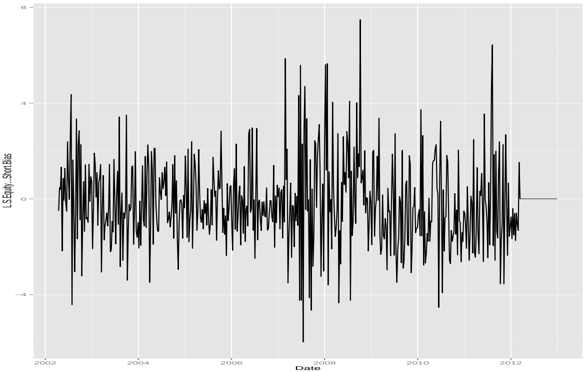


New calculation of the SCR

Strategies	SCR.Gaussian	SCR.POT	SCR.Clustering
Convertible.Bonds	11.1	64.3	**100**
CTA.Long.Term	**21.6**	9	10.3
CTA.Short.Term	14.7	22.7	**39.7**
Event.Distressed	10.1	12.7	**19.2**
Event.Arbitrage	5.4	7.5	**10.1**
Event.Situations	13.4	17.2	**81**
Fixed.Income.Arbitrage	11.3	27.2	**99.2**
Global.Macro	16.1	9.8	**19.3**
L.S.Credit.Arbitrage	21.9	**100**	**100**
L.S.Equity.Bias	19.7	16.9	**40.1**
L.S.Equity.Neutral	6	9.8	**37.4**
L.S.Equity.Bias	**11.3**	7	9.1
Statistical.Arbitrage	12.3	15.5	**74.3**
COMPOSITE	**6.9**	3.3	3.6
Top.10	**9.1**	5.2	5.8
Credit.Strategies	8.6	29.4	**100**
Average SCR	12.5	21	**46.8**

Conclusion

- Cluster indices $b_{\pm}$ determine the large deviations of the sums of m-dependent regularly varying sequences,
- A new approach of risk management based on clusters of extremes to take into account the dependence.



Thank you for your attention!