Large deviation estimates for exceedances of perpetuity sequences

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Part I: Exceedances of stochastic fixed point equations

Suppose:

$$\boxed{V \stackrel{d}{=} f(V).}$$

Basic problem: Estimate large deviation tail asymptotics for

$$\mathbf{P}\{V > u\} \quad \text{as} \quad u \to \infty.$$

Examples and applications

Quasi-linear SFPEs ($V \overset{d}{\approx} AV + B$) arise in many applications:

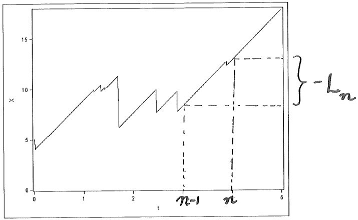
- Stationary tail for reflected random walk (GI/G/1 queue).
- Ruin problems in non-life insurance.
- Perpetuities (cash flows) in life insurance.
- GARCH(1,1) and ARCH(1) processes in finance.
- AR(1) processes with random coefficients.
- Branching processes with random environment.

Related non-homogeneous SFPEs ($V \overset{d}{=} \sum_{i=1}^{N} A_i V_i + B$) arise in:

- Quicksort algorithm in computer science.
- Branching random walk.
- Mandelbrot cascades.

Example: Ruin in insurance.

Lundberg's (1903) insurance model:



$$X_t = u + ct - \sum_{i=1}^{N_t} \zeta_i.$$

Consider *discrete* losses at time n:

$$L_n := -(X_n - X_{n-1}) \qquad (= \text{claims losses - premiums income}).$$

Investment returns:

$$R_n = (1 + r_n), \qquad \text{i.i.d.}$$

Total capital at time n:

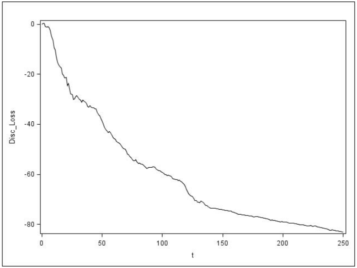
$$Y_n = R_n Y_{n-1} - L_n, \qquad n = 1, 2, \ldots, \quad Y_0 = u.$$

Ruin problem (cont.)

Cumulative discounted loss process:

$$\mathfrak{L}_n = L_1 + A_1 L_2 + \cdots + (A_1 \cdots A_n) L_n,$$

where $A_n = 1/R_n$ are discounted returns. ("Perpetuity seq.")



$\leftarrow \mathfrak{L}_n$

Probability of ruin (following Cramér, 1930):

$$\Psi(u) := \mathbf{P}\{Y_n < 0, \text{ some } n\} = \mathbf{P}\Big\{\sup_n \mathfrak{L}_n > u\Big\}.$$

Ruin problem (cont.)

Want to determine tail of $\mathfrak{L} := \sup_n \mathfrak{L}_n$ as $u \to \infty$.
Can show $\mathfrak{L}$ satisfies a *stochastic fixed point equation*:

$$\mathfrak{L} \stackrel{d}{=} A \max\{0, \mathfrak{L}\} + L,$$

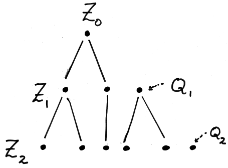
i.e., a special case of the equation

$$V \stackrel{d}{=} f(V).$$

Example: Branching process in random environment

Assume

$$Z_n = \left(\sum_{j=1}^{Z_{n-1}} \xi_{n,j} \right) + Q_n$$



where

$$\xi_{n,j} \sim \mathbf{p}(\zeta_n) \quad \text{–children in } n^{\text{th}} \text{ generation;}$$

$$Q_n \sim \mathbf{q}(\zeta_n) \quad \text{–immigrants in } n^{\text{th}} \text{ generation.}$$

Here, the distribution functions $\{\mathbf{p}(\zeta_n)\}$ are *random*, dependent on i.i.d. environment $\{\zeta_n\}$ (Solomon, Kesten).

Let $\mathfrak{F}_n = \sigma(\zeta_0, \ldots, \zeta_n)$, and consider

$$Y_n := \mathbf{E}\left[Z_n | Z_{n-1}, \mathfrak{F}_n\right] = \mathbf{E}\left[\xi_{n,1} | \zeta_n\right] Z_{n-1} + \mathbf{E}\left[Q_n | \zeta_n\right].$$

Branching in random environment (cont.)

Then $V_n := \mathbf{E}[Z_n | \mathfrak{F}_n]$ satisfies the equation

$$V_n = m(\zeta_n) V_{n-1} + \lambda(\zeta_n), \quad n = 1, 2, \ldots,$$

where $(m(\zeta_n), \lambda(\zeta_n))$ are *random*. Thus

$$\boxed{V \overset{d}{=} m(\zeta) V + \lambda(\zeta)} \qquad \text{"linear recursion,"}$$

i.e. $V \overset{d}{=} AV + B$. (Kesten '73, for multi-type BP.)

Closely related: tree-indexed random walk.

Stochastic fixed point equations

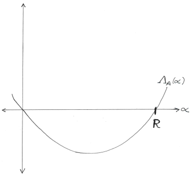
In general, would like to solve the SFPE

$$\boxed{V \overset{d}{=} f(V),} \qquad f(V) \approx AV + B.$$

Using implicit renewal theory (Kesten '73, Goldie '91):

$$\boxed{\mathbf{P}\left\{V > u\right\} \sim Cu^{-R} \quad \text{as} \quad u \to \infty,}$$

where $R > 0$ satisfies $\Lambda_A(R) = 0$.



$$\Lambda_A(\alpha) = \log \mathbf{E}\left[e^{\alpha \log A}\right]$$

Implicit renewal theory

Basic idea: Note

$$e^{Rv}\mathbf{P}\{V > e^v\} = e^{Rv}\Big(\mathbf{P}\{V > e^v\} - \mathbf{P}\{AV > e^v\}\Big) + e^{Rx}\int_{\mathbb{R}} \mathbf{P}\left\{V > e^{v-x}\right\} d\mu(x),$$

where $\mu \sim \mathcal{L}(\log A)$. That is,

$$\boxed{Z(v) = z(v) + Z * \mu_R(v),} \qquad \text{where} \quad d\mu_R(x) = e^{Rx} d\mu(x).$$

Many unanswered questions:

- Characterize const. C, where $\mathbf{P}\{V > u\} \sim Cu^{-R}$.
- Extend to more general processes.
- Large deviation path behavior.
- Rare event simulation. Etc.

A new approach

Start with a general SFPE,

$$V \stackrel{d}{=} F_Y(V).$$

Begin with *quasi-linear recursion* (Letac's "Model E"):

$$\boxed{V \stackrel{d}{=} A \max\{V, D\} + B,} \qquad \text{where } Y = (A, B, D).$$

- Includes standard applications (ruin, branching, GARCH(1,1), perpetuities).
- Useful *approximation* for more general quasi-linear processes: Iterated random maps $V_n = G_n(V_{n-1})$ (Mirek '10) under "cancellation condition"

$$F_{\tilde{Y}_n}(v) \le G_n(v) \le F_{Y_n}(v).$$

Letac-Furstenberg principle

The *forward recursive sequence* generated by $V \stackrel{d}{=} F_Y(V)$ is given by

$$V_n(v) = F_{Y_n} \circ F_{Y_{n-1}} \circ \cdots \circ F_{Y_1}(v), \quad V_0 = v.$$

The *backward recursive seq.* generated by this SFPE is

$$Z_n(v) = F_{Y_1} \circ F_{Y_2} \circ \cdots \circ F_{Y_n}(v), \quad V_0 = v.$$

Here, $\{Y_n\}$ is the *driving sequence* and is i.i.d.

<u>*Principle*</u>: The limiting distribution of $\{Z_n\}$ is unique and is equal to the limiting distribution of $\{V_n\}$.

Forward and backward sequences

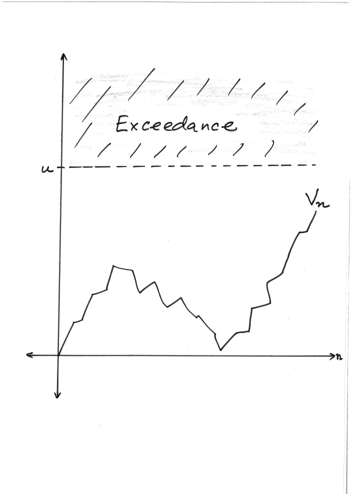


Figure : Forward sequence.

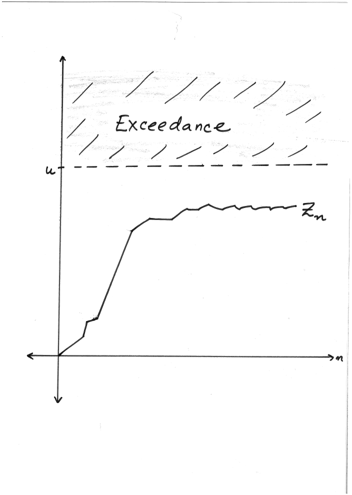


Figure : Backward sequence.

General approach

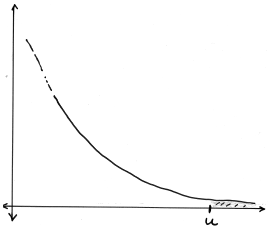
Observe: $\{V_n\}$ is a Harris rec. *Markov chain* (while $\{Z_n\}$ is *not*).

Thus, to study the SFPE $\boxed{V \stackrel{d}{=} F_Y(V),}$ generate the forward recursive sequence

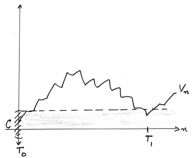
$$V_n := F_{Y_n}(V_{n-1}), \quad n = 1, 2, \ldots .$$

Set $V = \lim_{n\to\infty} V_n$, and determine

$$\lim_{u\to\infty} \mathbf{P}\{V > u\} \quad \text{as} \quad u \to \infty.$$



Regeneration



Suppose $\{V_n\}$ is a Markov chain satisfying the *minorization condition*

$$\delta \mathbf{1}_{\mathcal{C}}(x)\nu(dy) \leq P(x, dy).$$

Then:

Lemma (Athreya-Ney, Nummelin '78)

There exists a sequence of random times $0 \leq T_0 < T_1 < \cdots$ such that:
(i) $\{T_i - T_{i-1}\}$ is i.i.d.
(ii) The random blocks $\{V_{T_{i-1}}, \ldots, V_{T_i - 1}\}$ are independent.
(iii) $V_{T_i} \sim \nu$.

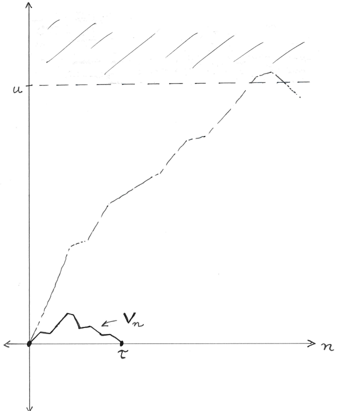
Large deviation approach

- Since $\{V_n\}$ is a Markov chain, "regenerates" at $\mathcal{C}$, so

$$\mathbf{P}\{V > u\} = \frac{\mathbf{E}[N_u]}{\mathbf{E}[\tau]}.$$

Regeneration cycle of $\{V_n\}$ (e.g., returns to 0).

Estimate exceedances above level u:

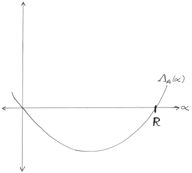


Large deviation approach (cont.)

- The event $\{V_n > u\}$ is a *rare* event.
- Introduce a "stopped" *large deviation* change of measure to determine this probability:
  Let μ denote the probab. law of $(\log A, B, D)$, and set

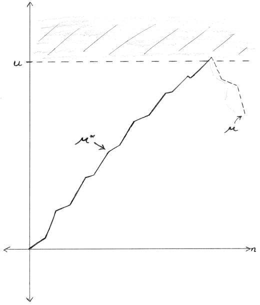
$$d\mu_R(x,y,z) = e^{Rx} d\mu(x,y,z)$$

when $n \le \inf\{\tilde{n} : V_{\tilde{n}} > u\}$.
Here, $R > 0$ solves the eqn. $\Lambda_A(\alpha) = 0$.
(Cramér transform.)



Large deviation approach

The process $\{V_n\}$
under the LD
change of measure
$\mu^* \equiv \mu_R$ (followed by μ):



Computing, as $u \to \infty$,

$$\mathbf{E}\left[N_u\right] \sim \mathbf{E}^*\left[W^R \mathbf{1}_{\{\tau=\infty\}}\right] \mathbf{E}^*\left[N_u e^{-R(S_{T_u} - \log u)}\right],$$

where $S_{T_u} = \log V_{T_u}$, and we have (approximately) that W is a perpetuity sequence:

$$Z^{(p)} := V_0 + \frac{B_1}{A_1} + \frac{B_2}{A_1 A_2} + \frac{B_3}{A_1 A_2 A_3} + \cdots .$$

(Relates to moments of return time of $\{V_n\}$ to its regeneration set.)

Connections with nonlinear renewal theory

$\{S_n\} \equiv \{\log V_n\}$ can be viewed as a perturbed random walk:

$$S_n = \sum_{i=1}^{n} \log A_i + \epsilon_n, \quad \text{where } \epsilon_n \text{ "small."}$$

($\{\epsilon_n\}$ slowly changing, $\epsilon_n / n \to 0$ a.s.)
Nonlinear renewal theory (Siegmund, Lai, Woodroofe) describes

$$S_{T_u} - \log u \quad \text{as} \quad u \to \infty,$$

and hence

$$\mathbf{E}^* \left[N_u e^{-R(S_{T_u} - \log u)} \right] \quad \text{as} \quad u \to \infty.$$

Main result

Assume $\mathbf{E}\left[\log A\right] < 0$ and $\mathbf{E}\left[(|B| + A|D|)^R\right] < \infty$, etc.,
and $A > 0$ has abs. cont. component.

Theorem (J.C.-A.Vidyashankar '13)

We have

$$\mathbf{P}\{V > u\} \sim Cu^{-R} \quad \textit{as} \quad u \to \infty,$$

where

$$C = \frac{1}{R\lambda'(R)\mathbf{E}[\tau]}\mathbf{E}^*\left[W_n^R\right] + o(e^{-\epsilon n})$$

and $W_n := \left(Z_n^{(p)} - Z_n^{(c)}\right)^+ \mathbf{1}_{\{\tau > n\}}$.

The constant C is explicit and computable.

"Usually" $Z^{(c)} \equiv 0$, leaving the "perpetuity seq."

$$\boxed{Z^{(p)} := V_0 + \frac{B_1}{A_1} + \frac{B_2}{A_1A_2} + \frac{B_3}{A_1A_2A_3} + \cdots, \qquad V_0 \sim \nu.}$$

Extensions

- Lundberg-type strict upper bound for $\mathbf{P}\{V > u\}$.
- General random maps: $V_n = G_n(V_{n-1})$.
- Markov-dependent recursions.
- Importance sampling: exact computational est. for $\mathbf{P}\{V > u\}$.
- With some modifications, non-homogeneous recursions:

$$V \stackrel{d}{=} \sum_{i=1}^{N} A_i V_i + B_i.$$

See J.C.-A.Vidyashankar '13 (several papers),
J.C. '09 (Markov case).

Extensions (cont.)

Extremal index: For forward process $V_n = F_{Y_n}(V_{n-1})$, obtain *closed-form* expression:

$$\Theta = \frac{1 - \mathbf{E}\left[e^{RS_{\tau^*}}\right]}{\mathbf{E}[\tau^*]},$$

where $\tau^* = \inf\{n \geq 1 : S_n \leq 0\}$ and $S_n^* = \sum_{i=1}^n \log A_i$ (*cf.* Iglehart '72).

In contrast, for $V_n = A_n V_{n-1} + B_n$, de Haan et al. '89 showed:

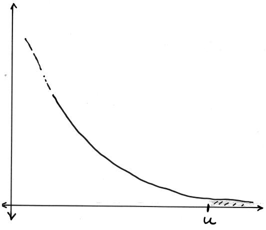
$$\Theta = \int_1^\infty \mathbf{P}\left\{\bigvee_{j=1}^{\infty} \prod_{i=1}^{j} A_i \leq y^{-1}\right\} R y^{-R-1} dy.$$

Extensions: importance sampling

Goal: to *simulate* the "rare event" tail probability

$$\mathbf{P}\{V > u\}, \quad \text{for large } u,$$

where $V \stackrel{d}{=} A \max\{V, D\} + B$.



- Rare event probability: suggests importance sampling, i.e., simulate under a *different* distribution than true probability distribution.
- We simulate *forward process* generated by given SFPE.
- The "dual" change of measure (for theoretical estimate) yields an efficient importance sampling algorithm.

References (Part I)

COLLAMORE, J.F. and VIDYASHANKAR, A.N. (2013). Tail estimates for stochastic fixed point equations via nonlinear renewal theory. *Stoch. Process. Appl.* **123** 3378-3429.

COLLAMORE, J.F. (2009). Random recurrence equations and ruin in a Markov-dependent stochastic economic environment. *Ann. Appl. Probab.* **19** 1404-1458. (Markov version of Goldie's Theorem.)

COLLAMORE, J.F. and VIDYASHANKAR, A.N. (2013). Large deviation tail estimates and related limit laws for stochastic fixed point equations. In *Random Matrices and Iterated Random Functions* (Alsmeyer, Löwe, eds.), Springer. (Markov and explosive cases.)

COLLAMORE, J.F., DIAO, G., VIDYASHANKAR, A.N. (2013). Rare event simulation for processes generated via stochastic fixed point equations. Submitted, 37 pp.

Part II: Path properties of perpetuity sequences

Now specialize to perpetuity sequence,

$$Z_n = B_1 + A_1 B_2 + \cdots + (A_1 \cdots A_{n-1}) B_n.$$

Thus, in particular,

$$Z_\infty \stackrel{d}{=} A Z_\infty + B.$$

What is the large deviation path behavior of $\{Z_n\}$?
(*Cf.* J.C.'98 and several classical large deviation papers.)

See our forthcoming paper:

COLLAMORE, J.F., DAMEK, E., BURACZEWSKI, D., ZIENKIEWICZ, J. (2013). Ruin times and related large deviation path behavior of perpetuity sequences. In preparation.