Rare-event Analysis and Simulations for Gaussian and Its Related Processes

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Workshop on Extremes in Space and Time
University of Copenhagen

May 31, 2013

Overview

- Gaussian process and its related functions: supreme, general convex functions, more complicated structured functions.
- Asymptotic analysis
- Rare-event simulations

Gaussian Random Field

- Probability space $(\Omega, \mathcal{F}, P)$
- $f : T \times \Omega \to \mathbb{R}$, $f(t, \omega)$, short form: $f(t)$.
- $(t_1, ..., t_n) \subset T$, $(f(t_1), ..., f(t_n))$ is a multivariate Gaussian random vector.
- $T \subset \mathbb{R}^d$, e.g., $T = [0, 1]^d$.

Interesting quantities

- The tail probabilities of functions of $\Gamma(f(\cdot))$
- The supremum norm

$$\Gamma(f) = \sup_{t\in T} f(t)$$

- General convex functions, for instance,

$$\Gamma(f) = \int_{t\in T} e^{f(t)}\,dt$$

- Differential equations with random coefficients

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The analysis

- Bounds and asymptotic bounds
- Asymptotic approximations
  - Tail probability

$$\lim_{b\to\infty} \frac{P(\Gamma(f) > b)}{\alpha(b)} = 1, \qquad \lim_{b\to\infty} \frac{\log P(\Gamma(f) > b)}{\log \alpha(b)} = 1$$

  - Local results: approximations of the density functions, $g_\Gamma(x)$
- Simulation of the tail probability
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A brief summary of the results

	Approx.	Sim.	Cond. Dist.
$\sup_T f(t)$	A lot	A lot	limited
$\int_T e^{f(t)} dt$	limited	limited	very limited
SPDE	very limited	very limited	???

Asymptotic approximations of $P(\sup_T f(t) > b)$

Asymptotic Analysis of $\Gamma(f) = \sup_T f(t)$

- Logarithmic approximation

$$\lim_{u\to\infty} -\frac{\log P(\sup_T f(t) > u)}{u^2} = \frac{1}{2\sup_T \sigma^2(t)}.$$

- Sharp asymptotics under regularity conditions

$$P(\sup_T f(t) > u) = (1 + o(1)) \times C(T) \times u^{\beta} \times P(Z > u)$$

- Cramer and Leadbetter (1967), Pickands (1969), Adler (1981), Sun (1993), Piterbarg (1995), Azais and Wschebor (2005).

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The intuition

- $B_1, ..., B_n$ are independent events, where $P(B_i) = \alpha \approx 0$
- Then,

$$P(\cup_{i=1}^{n} B_i > 0) = 1 - (1-\alpha)^n \approx n\alpha.$$

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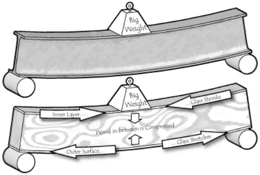
Random differential equations

Material Failure – one dimensional example

Physical meaning

- $u(x)$: the shape of the material
- $\nabla u(x)$: strain
- $p(x)$: pressure
- $a(x)$: material-specific coefficients



[1]

[1] The picture is published at http://www.guillemot-kayaks.com

Material Failure

- The partial differential equation: $x \in T$

$$\nabla \cdot \{a(x) \nabla u(x)\} = -p(x)$$

- The ordinary differential equation: $x \in [0, 1]$

$$\{a(x)u'(x)\}' = -p(x)$$

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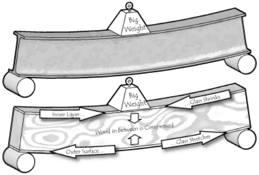

- Composite material characterized by the tensor $a(x)$
- Spatial variation: $a(x) = e^{f(x)}$, where $f(x)$ is a Gaussian process.

Material Failure

- Question: whether and where the material breaks.

The failure probability

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$$P\left(\sup_{x\in T}|\nabla u(x)| > b\right)$$

- The displacement $u(x)$ depends on the process $a(x)$.

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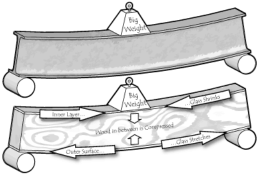
Material Failure – Dirichlet condition



- Dirichlet condition: $u(0) = u(1) = 0$
- The solution:
  $u(x) = \int_0^x F(y)a^{-1}(y)dy - \frac{\int_0^1 F(y)a^{-1}(dy)dy}{\int_0^1 a^{-1}(dy)} \int_0^x a^{-1}(y)dy,$
  where $F(x) = \int_0^x p(y)dy$.

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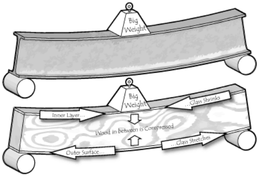

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  where $F(x) = \int_0^x p(y)dy$.

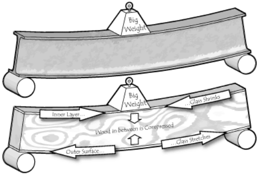
Material Failure – Dirichlet condition

- The strain

$$
\begin{aligned}
u'(x) &= a^{-1}(x)\left(F(x) - \frac{\int_0^1 F(y)a^{-1}(y)dy}{\int_0^1 a^{-1}(y)dy}\right) \\
&= a^{-1}(x)[F(x) - E_f(F(Y))]
\end{aligned}
$$

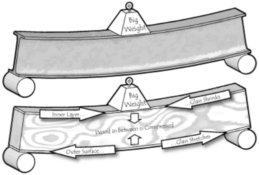
where $a^{-1}(x) = e^{f(x)}$.

The external force



- Delta external force: $p(x) = \delta_{x_*}(x), \quad F(x) = I(x \geq x_*)$.
- Continuous external force $p(x)$: $x_* = \arg\sup_{x \in T} |p(x)|$.

The external force



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Theorem: approximation of the Delta function (L. and Zhou 2011)

- Homogeneous, mean zero, and $C^3(T)$
- The covariance $C(t) = 1 - \frac{1}{2}t^2 + O(|t|^4)$.
- The external $F(x) = I(x \geq x_*)$, $\quad p(x) = \delta_{x_*}(x)$.

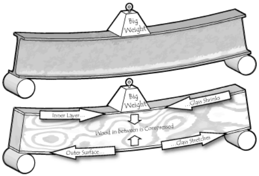
Theorem: approximation of the Delta function (L. and Zhou 2013)

- Use Z to denote a standard normal random variable. Define $H(x) = -\frac{x^2}{2} + \log P(Z \leq x)$, and $\kappa = \sup H(x)$.
- Let $r = \log b - \kappa$.

Then, we have the approximation

$$P\left(\sup_{x\in[0,1]} |u'(x)| > b\right) \sim D \times P(Z > r).$$

Key components of the conditional distribution



- Questions about the conditional distribution
  - Where does the break occur or $\arg\sup u'(x) = ?$
  - Where does $f(x)$ attain it maximum?
  - At what level?

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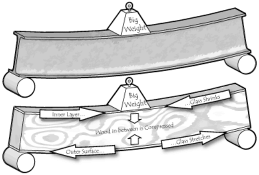

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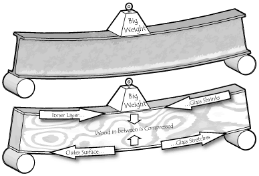

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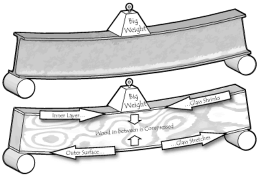

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Theorem: approximation for continuous force (L. and Zhou 2013)

- The external force $p(x)$ is a continuously differentiable function.
- $x_* = \arg\sup_x p(x)$.

Then, we have the approximation

$$P(\sup_{x\in[0,1]} |u'(x)| > b) \sim P(|u'(0)| > b) + P(|u'(1)| > b) + P(\sup_{|x-x_*|<\varepsilon} |u'(x)| > 0).$$

Exact asymptotic approximation for continuous body force

- Let $p(x_*)r^{-1}e^{r-\frac{1}{2}} = b$. Then,

$$P(\sup_{|x-x_*|<\varepsilon} |u'(x)| > 0) \sim \kappa_* \times r^{-1/2}\exp\{-r^2/2\}.$$

- Let $H_0 r_0^{-1/2} e^{r_0} = b$. Then,

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Conclusion

- Extremes of Gaussian processes
- Differential equations
- Understanding the tail events