Efficient Importance Sampling in a Credit Risk Model with Contagion

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University of Copenhagen, May 31, 2013
joint work with Pierre Nyquist

Outline

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A Simple Credit Risk Model with Contagion

- A credit portfolio consisting of n obligors is divided into d groups.
- Let $w_1, \dots, w_d$ be the fraction of obligors in each group, $w_j > 0$, $\sum w_j = 1$.
- Let $Q^n(t) = (Q_1^n(t), \dots, Q_d^n(t))$ be number of defaults in each group by time t.
- Q^n is modeled as a continuous time pure birth Markov chain with intensity $n\lambda(Q^n(t)/n)$ where

$$\lambda(x) = (\lambda_1(x), \dots, \lambda_d(x)), \quad \lambda_j(x) = (w_j - x_j)ae^{b\sum_{k=1}^d x_k}.$$

A Simple Credit Risk Model with Contagion

- Let $X^n(t) = Q^n(t)/n$.
- **Objective:** Compute the probability that at least a fraction z has defaulted by time 1:

$$p_n = P\left\{\sum_{j=1}^{d} Q_j^n(1) \geq nz\right\} = P\left\{\sum_{j=1}^{d} X_j^n(1) \geq z\right\}.$$

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Large Deviations

The processes $\{X^n\}$ satisfy a Laplace principle: for any bounded continuous function h: $\mathscr{D}[0,1] \to \mathscr{R}$

$$\lim_{n\to\infty} -\frac{1}{n} \log E_x[\exp\{-nh(X^n)\}] = \inf_{\psi}\{I_x(\psi) + h(\psi)\},$$

with rate function I_x given by

$$I_x(\psi) = \int_0^1 L(\psi(t), \dot{\psi}(t))dt,$$

for all nonnegative and absolutely continuous $\psi \in \mathscr{C}[0,1]$ with $\psi(0) = x$, and

$$L(x, \beta) = \langle \beta, \log \frac{\beta}{\lambda(x)} \rangle - \langle \beta - \lambda(x), \mathbf{1} \rangle.$$

Large Deviations

In particular, with

$$h(\psi) = \begin{cases} 0, & \text{if } \sum_{j=1}^d \psi_j(1) \geq z, \\ \infty, & \text{if } \sum_{j=1}^d \psi_j(1) < z, \end{cases}$$

and $B_z = \{\psi : \sum_{j=1}^d \psi_j(1) \geq z\}$ it follows that

$$-\frac{1}{n} \log p_n = -\frac{1}{n} \log P\left\{ \sum_{j=1}^d X_j(1) \geq z \right\} = \inf_{\psi \in B_z} I_0(\psi) =: \gamma.$$

Outline

Importance Sampling

- Sample $\widetilde{X}_1^n, \dots, \widetilde{X}_N^n$ from $\widetilde{F}_n$ with $F_n \ll \widetilde{F}_n$ on B.
- Form the weighted empirical measure

$$\widetilde{\mathbf{F}}_n^w(\cdot) = \frac{1}{N} \sum_{k=1}^{N} \frac{dF_n}{d\widetilde{F}_n}(\widetilde{X}_k^n)\delta_{\widetilde{X}_k^n}(\cdot).$$

- Use the plug-in estimator

$$\widehat{p}_n = \widetilde{\mathbf{F}}_n^w(B).$$

Efficient Importance Sampling

- The choice of sampling distribution $\widetilde{F}_n$ is good if $\mathrm{Std}(\widehat{p}_n)$ is of roughly the same size as p_n.
- We say that $\widetilde{F}_n$ is efficient if

$$\lim_{n\to\infty} -\frac{1}{n}\log \widetilde{E}[\widehat{p}_n^2] = 2\gamma.$$

- Note that since $\widetilde{E}[\widehat{p}_n^2] \geq p_n^2$ it is sufficient to show

$$\liminf_{n\to\infty} -\frac{1}{n}\log \widetilde{E}[\widehat{p}_n^2] \geq 2\gamma.$$

Outline

The Credit Risk Model

- In the credit risk example $\{X^n\}$ is a continuous time Markov chain.
- The only way to select the sampling distribution $\widetilde{F}_n$ is to use transition intensities $\widetilde{\lambda}(x,t)$ in place of $\lambda(x)$, at state $X^n(t) = x$.
- Challenge: how to find appropriate $\widetilde{\lambda}$?

Some Remarks

- This model has been studied by R. Carmona and S. Crépey (Int. J. Theor. Appl. Fin., 2010).
- They apply a state-independent change of measure and show by numerical experiments that importance sampling performs poorly in the presence of contagion ($b > 0$).
- We will solve the problem by means of the subsolution approach developed by P. Dupuis and H. Wang (Brown U.).

Outline

The Subsolution Approach

Overview

- View the change of measure as a control problem.
- The value function of the control problem satisfies a dynamic programming principle.
- Send $n \to \infty$ and derive a limiting control problem.
- The value function of the limiting control problem W satisfies a partial differential equation (Isaacs equation).
- By constructing a subsolution $\bar{W}$ to the PDE a change of measure can be based on $D\bar{W}(x,t)$ (gradient) and the performance of the corresponding algorithm is given by $\bar{W}(0,0)$.
- The algorithm based on a subsolution $\bar{W}$ is asymptotically optimal if $\bar{W}(0,0) = 2\gamma$.

The Isaac's Equation for the Credit Risk Problem

- In the credit risk problem the Isaac's equation reduces to a Hamilton-Jacobi equation:

$$W_t(x,t) - 2H(x, -DW(x,t)/2) = 0,$$

$$W(x,1) = 0, \text{ for } \sum_{j=1}^{d} x_j \geq z,$$

where the Hamiltonian is

$$H(x,p) = \sum_{j=1}^{d} \lambda_j(x)(e^{p_j} - 1).$$

Subsolutions

- We aim to find a subsolution $\bar{W}(x,t)$, i.e., a continuously differentiable function such that

$$\bar{W}_t(x,t) - 2H(x, -D\bar{W}(x,t)/2) \geq 0,$$

$$\bar{W}(x,1) \leq 0, \text{ for } \sum_{j=1}^{d} x_j \geq z, \text{ and}$$

$$\bar{W}(0,0) \geq 2\gamma.$$

- The corresponding importance sampling algorithm uses the intensity $\widetilde{\lambda}(x,t) = \lambda(x)\exp\{-D\bar{W}(x,t)/2\}$.

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Finding a Subsolition

One-dimensional case

- Candidate subsolution is of the form:

$$\bar{W}(x,t;\mu) = 2\int_x^z \log\left(1+\frac{\mu}{\lambda(y)}\right)dy - 2\mu(1-t).$$

It has

$$W_t(x,t) = 2\mu, \quad DW(x,t) = -2\log\left(1+\frac{\mu}{\lambda(x)}\right),$$

and

$$\bar{W}_t(x,t;\mu) - 2H(x,-D\bar{W}(x,t;\mu)/2) = 0,$$
$$\bar{W}(x,1;\mu) = 0, \text{ for } x \geq z.$$

Finding a Subsolution

One-dimensional case

- The optimal candidate is found by maximizing $\bar{W}(0,0;\mu)$ over μ:

$$\mu^* = \text{argmax}\ \bar{W}(0,0;\mu) = \text{argmax}\ 2\int_0^z \log\left(1+\frac{\mu}{\lambda(y)}\right)dy - 2\mu.$$

- We claim that $\bar{W}(0,0;\mu^*) = 2\gamma$. That is, $\bar{W}(x,t;\mu^*)$ determines an asymptotically optimal importance sampling algorithm.

Outline

Illustration - Monte Carlo

$a = 0.01$, $b = 5$, $d = 2$, $z = 0.25$, $N = 10000$

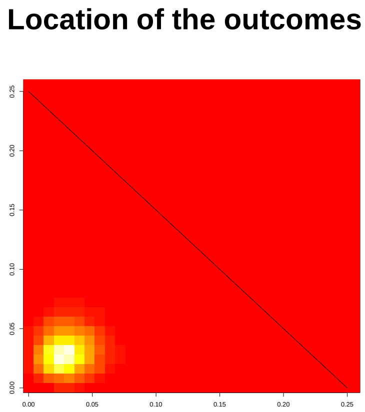
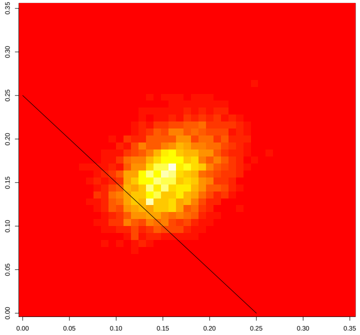


Illustration - Importance Sampling

$a = 0.01$, $b = 5$, $d = 2$, $z = 0.25$, $N = 10000$

Location of the outcomes



Weighted empirical measure