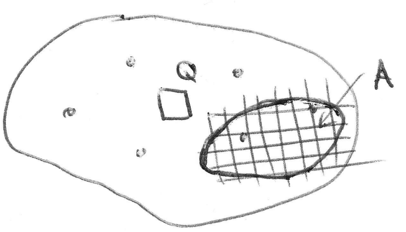
Max-stable random fields: Theory and examples.

Poisson processes.



Stars in the sky.

Cube Q. Volume $\varepsilon \approx 0$.

$P[\text{Star in } Q] \approx \lambda \varepsilon$, $\lambda > 0$

Consider set A. Vol of A: $|A|$. $N(A)$ = number of stars in A.

Distribution of $N(A)$? Cover A by $\approx \frac{|A|}{\varepsilon}$ cubes.

$N(A) \sim \mathrm{Bin}\left(\frac{|A|}{\varepsilon}, \lambda\varepsilon\right) \underset{\varepsilon \downarrow 0}{\approx} \mathrm{Poi}(\lambda |A|)$.

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Def. Homogeneous PPP with intensity $\lambda > 0$ is a random collection of points in $\mathbb{R}^d$ s.t.

1. If $A_1, \dots, A_n \subset \mathbb{R}^d$ (Borel) are disjoint $\Rightarrow N(A_1), \dots, N(A_n)$ are independent r.v.
2. $N(A) \sim \mathrm{Poi}(\lambda |A|)$.

Def. Inhomogeneous PPP. Let $(E, \mathcal{A}, \lambda)$ be a measure space. λ σ-finite measure.

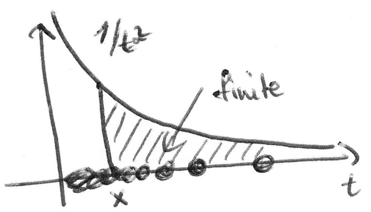
PPP(λ) is a random point configuration $\{X_i\}$ s.t.

1. If $A_1, \dots, A_n \in \mathcal{A}$ are disjoint $\Rightarrow N(A_1), \dots, N(A_n)$ are indep. r.v.
2. $N(A) \sim \mathrm{Poi}(\underbrace{\lambda(A)}_{\text{meas of } A})$ for $\forall A \in \mathcal{A}$. **Rem.** $\mathrm{Poi}(\infty)$ takes value $+\infty$ with probab 1.

Here, $N(A) = \sum_i \mathbb{1}_{X_i \in A}$ is the number of points in A.

Rem. $\mathbb{E} N(A) = \lambda(A)$

Example. $E = \mathbb{R}_+$, $\mathcal{A}$ = Borel σ-Algebra, $\lambda(dt) = \frac{dt}{t^2}$, that is $\lambda(A) = \int_A \frac{dt}{t^2}$.



(a) $\lambda((x, +\infty)) = \int_x^\infty \frac{dt}{t^2} = \frac{1}{x} < \infty$, $x > 0$

$N((x, +\infty)) \sim \mathrm{Poi}\left(\frac{1}{x}\right)$

$\mathbb{E} N((x, +\infty)) = \frac{1}{x}$.

(b) $\lambda((0, x)) = \int_0^x \frac{dt}{t^2} = \infty$

$N((0, x)) = +\infty$ a.s.

Extreme-value theory

$X, X_1, X_2, \dots$ iid r.v.

Sums. CLT: $\frac{X_1 + \dots + X_n - \mathbb{E}X_1 \cdot n}{\sqrt{n \operatorname{Var} X_1}} \xrightarrow[n\to\infty]{d} N(0,1)$ if $\operatorname{Var} X_1 < \infty$

If $\operatorname{Var} X_1 = \infty \Rightarrow$ stable: $\frac{X_1 + \dots + X_n - a_n}{b_n} \xrightarrow[n\to\infty]{d}$ stable laws.

Maxima. $M_n = \max\{X_1, \dots, X_n\} = X_1 \vee \dots \vee X_n$.

Question: $\exists a_n, b_n$ s.t. $\frac{M_n - a_n}{b_n} \xrightarrow[n\to\infty]{d} G(x)$ (non-degenerate)?

Fisher-Tippett – Gnedenko: Possible limits (called extreme-value distributions)

$\Lambda(x) = e^{-e^{-x}}$, $x \in \mathbb{R}$ Gumbel

$\Phi_\alpha(x) = e^{-1/x^\alpha}$, $x > 0$, $\alpha > 0$ Fréchet

$\Psi_\alpha(x) = e^{-(-x)^\alpha}$, $x < 0$, $\alpha > 0$ Weibull

or linear transformation of one of these laws.

Det. A r.v. X is max-stable if $\forall n\ \exists a_n, b_n$ s.t. $\frac{X_1 \vee \dots \vee X_n - a_n}{b_n} \stackrel{d}{=} X$

Max-stable $\Leftrightarrow$ extreme-value distribution $\Leftrightarrow \Lambda, \Phi_\alpha, \Psi_\alpha$ or linear transformation.

Let $\{X(t), t \in T\}$ be a random field. Let $X_1, X_2, \dots$ be iid copies of X.

Def. X is called max-stable (with 1-Fréchet margins) if

$$\frac{X_1 \vee \dots \vee X_n}{n} \stackrel{fdd}{=} X \quad \forall n \in \mathbb{N}.$$

Here, $(X_1 \vee \dots \vee X_n)(t) = X_1(t) \vee \dots \vee X_n(t)$, pointwise maximum.

Rem. $\forall t \in T$: $X(t)$ has distr. func. $e^{-c(t)/x}$, $x > 0$, for some $c(t) > 0$.

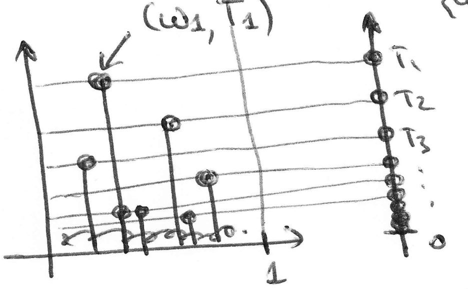
De Haan Representation of max-stable processes.

Fréchet noise $(E, \mathcal{A}, \lambda)$ σ-finite measure space. Main example: $E=[0,1]$, $\lambda = \text{Leb}$.

Def. Fréchet noise is a PPP on $E \times \mathbb{R}_+$ with intensity $\lambda \times \frac{dt}{t^2}$.

Points are denoted by $\{(\omega_i, T_i)\}_{i \in \mathbb{N}}$.

For $E=[0,1]$, $\lambda = \text{Leb}$: Let $\{T_i\}$ be PPP$(\frac{dt}{t^2})$ on $\mathbb{R}_+$ $\}$ indep
$\{\omega_i\}_{i\in\mathbb{N}}$ be iid uniform on $[0,1]$.



Def (Stochastic integral) Let $f \in L^1_+(E)$, that is $f \geq 0$, $\int_E f < \infty$.

$$\int^\vee f \overset{\text{def}}{=} \sup_{i \in \mathbb{N}} \left(f(\omega_i) \cdot T_i \right).$$

Properties:

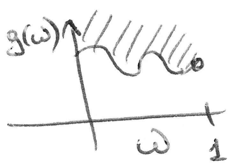
① Max-linearity: $\int^\vee cf = c \int^\vee f$, $\quad \int^\vee (f \vee g) = \left(\int^\vee f\right) \vee \left(\int^\vee g\right)$.

② Independence: if $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset \Rightarrow \int^\vee f$ is indep of $\int^\vee g$.

③ $\int^\vee f$ is 1-Fréchet: $\mathbb{P}\left[\int^\vee f \leq x\right] = \exp\left\{-\int_E f(t)\lambda(dt) \cdot \frac{1}{x}\right\}$, $x > 0$.

Proof.
$$\mathbb{P}\left[\int^\vee f \leq x\right] = \mathbb{P}\left[\forall i: T_i \leq \frac{x}{f(\omega_i)}\right]$$
$$= \mathbb{P}[\text{no point above } g]$$
$$= \exp\left\{-\int_0^1 \int_{g(\omega)}^\infty \frac{1}{t^2} \, dt \, d\omega\right\}$$
$$= \exp\left\{-\int_0^1 \frac{d\omega}{g(\omega)}\right\}$$
$$= \exp\left\{-\int_0^1 f(\omega)\, d\omega \cdot x^{-1}\right\}. \qquad \square$$

Let $g(\omega) = \frac{x}{f(\omega)}$



④ T any set, $\{f_t, t \in T\} \subset L^1_+$.
Then $\left\{\int^\vee f_t : t \in T\right\}$ is a max-stable process (with 1-Fréchet margins).

<u>Theorem</u> (De Haan, 1984). Let $\{X(t), t\in T\}$ be a max-stable process (with 1-Fréchet margins). Assume that

T is countable

or $T=\mathbb{R}^d$ and X is continuous in probability.

Then, $\exists \{f_t : t\in T\} \subset L_1^+[0,1]$ s.t.

$$\{X(t): t\in T\} \overset{fdd}{=} \left\{\int^{\vee} f_t : t\in T\right\}.$$

<u>Example</u>. Let $\{X(t), t\in \mathbb{N}\}$ be iid with d.f. $e^{-1/x}$, $x>0$.

Representation: Let $\sum c_i = 1$, $c_i>0$

$I_t := [c_1+\dots+c_{t-1},\ c_1+\dots+c_t]$

Let $f_t = \frac{1}{c_t}\cdot \mathbb{1}_{I_t}$

$\Rightarrow \{\int^{\vee} f_t : t\in T\}$ are iid Fréchet with d.f. $e^{-1/x}$.

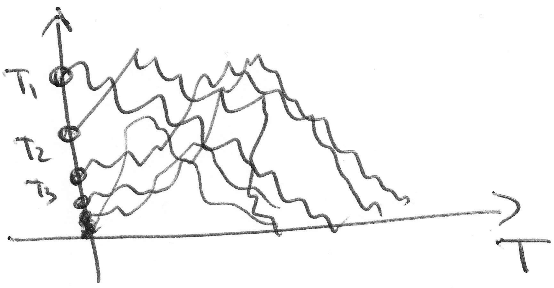
View $[0,1]$ as a probability space and $\{f_t, t\in T\}$ as a stochastic process.

If $\omega \sim \mathcal{U}[0,1] \Rightarrow F(t;\omega) = f_t(\omega)$ is a stoch. process.

$F_1, F_2, \dots$ $\quad F_i(t;\omega) = f_t(\omega_i)$ are iid copies of F ($\omega_i \sim \mathcal{U}[0,1]$ iid).

Then,

$$\{X(t): t\in T\} \overset{f.d.d}{=} \left\{\sup_{i\in\mathbb{N}} (\Pi_i \cdot F_i(t)) : t\in T\right\}.$$



Hüsler-Reiss distributions and Brown-Resnick processes.

First we recall de Haan's representation in the setting of Gumbel margins.

$\{X(t), t\in T\}$ random field. $X_1, X_2, \dots$ copies of X.

Def. X is called max-stable (with Gumbel margins) if

$$\{X_1 \vee \dots \vee X_n(t) - \log n, t\in T\} \overset{fdd}{=} \{X(t), t\in T\}.$$

Rem. $\forall t\in T$ $X(t)$ has distr. function $e^{-e^{-(x-\mu(t))}}$, $\mu(t)\in\mathbb{R}$.

Rem. $X(t)$ max-stable Gumbel $\Leftrightarrow$ $e^{X(t)}$ max-stable Fréchet.

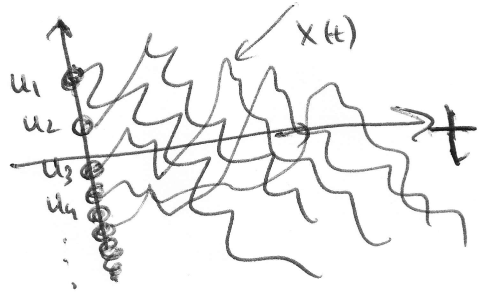
De Haan Representation:

$\exists$ process $\{F(t), t\in T\}$ with values $[-\infty,\infty)$ s.t. $\mathbb{E}e^{F(t)}<\infty$ $\forall t\in T$.

$F_1, F_2, \dots$ iid copies of F

$\{U_i\}\sim PPP(e^{-t}dt)$ on $\mathbb{R}$

$$\{X(t), t\in T\} \overset{fdd}{=} \{\sup_{i\in\mathbb{N}} (U_i + F_i(T)), t\in T\}$$

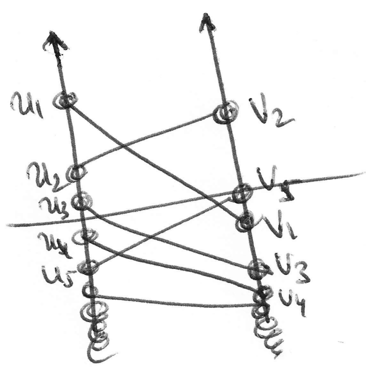


2 Dim Hüsler-Reiss distr.

Example $\{U_i\}\sim PPP(e^{-t}dt)$, $\xi_i\sim N(-\frac{\delta^2}{2},\delta^2)$ iid, indep.

$\underbrace{-\frac{\delta^2}{2}}_{\text{negative drift}}$

$V_i := U_i + \xi_i$



Claim: $\{V_i\}\sim PPP(e^{-t}dt)$

Proof Intensity of V_i at v $= \int_{\mathbb{R}} du\, e^{-u}\,\mathbb{P}[\xi = \underbrace{v-u}_{z}] = \int_{\mathbb{R}} dz\, e^{z}\,\mathbb{P}[\xi=z]\cdot e^{-v}$

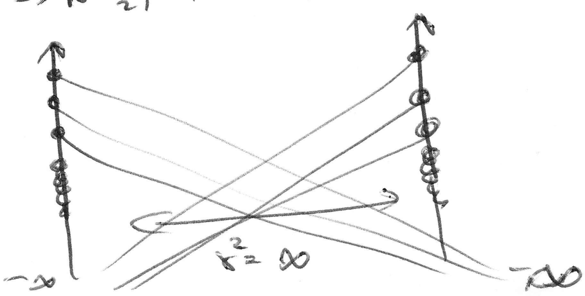
$= e^{-v}\cdot\mathbb{E}e^{\xi} = 1$ since $\mathbb{E}e^{\xi} = \mathbb{E}e^{\delta N - \delta^2/2} = e^{\delta^2/2-\delta^2/2} = 1.$ $\square$

$N\sim N(0,1)$

Special cases:

$\delta = 0 \Rightarrow$ complete dependence

$\delta^2 = +\infty \Rightarrow N(\frac{\delta^2}{2},\delta^2) = -\infty$ a.s.: complete independence.



Def. The vector $(\max U_i, \max V_i)$ has the two-dim Hüsler-Reiss distr. with parameter δ^2.

2Dim Hüsler-Reiss distr. as a limit

1 Station: Measurements $X_1, \dots, X_n \sim N(0,1)$ iid.

$M_n^X = \max\{X_1, \dots, X_n\}$

$\exists b_n \sim \sqrt{2\log n}$: $b_n(M_n^X - b_n) \xrightarrow[n\to\infty]{d} e^{-e^{-x}}$.

$\{b_n(X_i - b_n)\}_{i=1}^{n} \xrightarrow[n\to\infty]{d} PPP(e^{-t}\,dt)$ on $\mathbb{R}$.

2 Stations Measurements $X_1, \dots, X_n$
$Y_1, \dots, Y_n$

On day i: (X_i, Y_i).

Assumptions: $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$ are indep. biv. Gaussian

$X_1, \dots, X_n \sim N(0,1)$, $\quad Y_1, \dots, Y_n \sim N(0,1)$

$\operatorname{corr}(X_i, Y_i) = \rho < 1$.

$M_n^X = \max_{i=1,\dots,n} X_i \qquad M_n^Y = \max_{i=1,\dots,n} Y_i$.

Sibuya: $\underbrace{(b_n(M_n^X - b_n),\ b_n(M_n^Y - b_n))}_{\text{dependent}} \xrightarrow[n\to\infty]{d} \underbrace{(e^{-e^{-x}},\ e^{-e^{-y}})}_{\text{independent!}}$.

Asymptotic independence of M_n^X and M_n^Y for $\forall \rho < 1$.

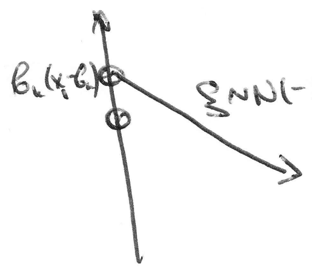
Hüsler-Reiss suggested: $\rho = \rho_n = 1 - \frac{\delta^2}{4\log n} + o\left(\frac{1}{\log n}\right) \xrightarrow[n\to\infty]{} 1$.

We compute the limiting distr. of $(b_n(M_n^X - b_n)\ b_n(M_n^Y - b_n))$.

Condition on $X_1 = b_n + \frac{x}{b_n}$, $x \in \mathbb{R}$ fixed

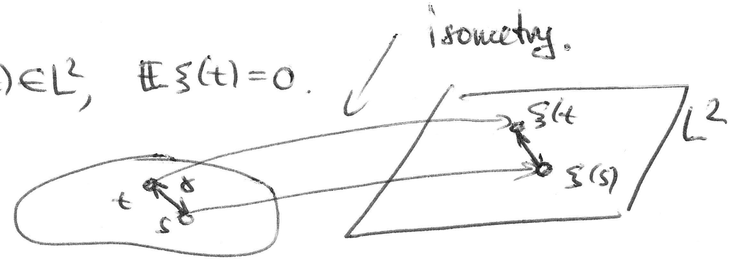
$$\mathbb{E}\left[b_n(Y_1 - b_n) \mid X_1 = b_n + \frac{x}{b_n}\right] = \rho_n \cdot b_n\left(b_n + \frac{x}{b_n}\right) - b_n^2 = \dots = -\frac{\delta^2}{2} + o(1)$$

$$\operatorname{Var}\left[b_n(Y_1 - b_n) \mid X_1 = b_n + \frac{x}{b_n}\right] = (1 - \rho_n^2)\, b_n^2 = \delta^2 + o(1).$$



$$\Longrightarrow \left(b_n(M_n^X - b_n),\ b_n(M_n^Y - b_n)\right) \xrightarrow[n\to\infty]{d} \overbrace{(\max U_i, \max V_i)}^{\text{Hüsler-Reiss distr.}}$$

Brown-Resnick processes.



Let $\{\xi(t), t \in T\}$ be a stoch process, $\xi(t) \in L^2$, $\mathbb{E}\xi(t) = 0$.

The incremental variance of ξ is

$\delta^2(t,s) := \mathrm{Var}\,[\xi(t) - \xi(s)]$

Note: $\delta = \sqrt{\delta^2}$ is a pseudometric on T which is isometrically embeddable in L^2.

δ^2 is a negative definite kernel on T meaning that $\forall t_1, \dots, t_n \in T$, $\forall a_1, \dots, a_n \in \mathbb{R}$

s.t. $\sum a_i = 0$ we have $\sum_{i,j} a_i a_j \delta^2(t_i, t_j) \le 0$.

Incremental variance $\Leftrightarrow$ pseudometric embeddable into Hilbert space $\Leftrightarrow$ negative definite kernel.

Construction of Brown-Resnick processes.

1) Let T be a set
2) Let $\delta^2(\cdot,\cdot)$ be negative definite kernel on T
3) Let $\{\xi(t), t \in T\}$ be a Gauss process s.t. $\mathbb{E}\xi(t) = 0$, $\mathrm{Var}\,[\xi(t) - \xi(s)] = \delta^2(t,s)$.

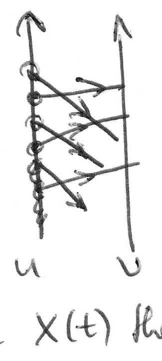
   Let $\sigma^2(t) = \mathrm{Var}\,\xi(t)$.

<u>Rem.</u> $\xi(t)$ is non-unique. $\forall t_0 \in T$ $\exists \xi$ such that $\xi(t_0) = 0$.

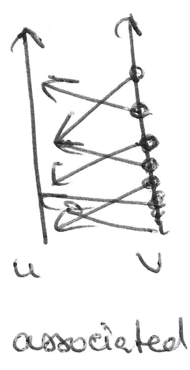
4) Let $\xi_1, \xi_2, \dots$ be iid copies of ξ.
5) Let $\{u_i\}$ be PPP$(e^{-t}dt)$ on $\mathbb{R}$.

<u>Thm.</u> The process $X(t) = \max\{u_i + \xi_i(t) - \frac{\sigma^2(t)}{2}\}$, $t \in T$ is max-stable (follows from de Haan's repr.) and its f.d.d depend on δ^2 only (and not on the choice of ξ).

<u>Example</u>



gives the same as

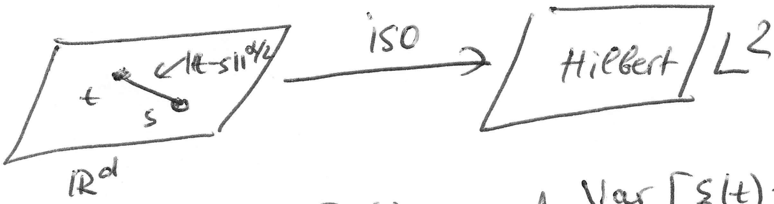


We call $X(t)$ the Brown-Resnick process associated to δ^2.

Stationary Brown–Resnick processes. Let $T=\mathbb{R}^d$

Consider $\gamma^2(t,s)=\gamma^2(t-s)$ depending only on $t-s$.

~~[illegible]~~ Example: $\gamma^2(t)=\|t\|_2^{\alpha}$, $\alpha\in(0,2]$.



Gaussian process with $\mathbb{E}\xi(t)=0$ and $\operatorname{Var}[\xi(t)-\xi(s)]=\|t-s\|_2^{\alpha}$ is called (Lévy) fractional Brownian Motion (fBM).

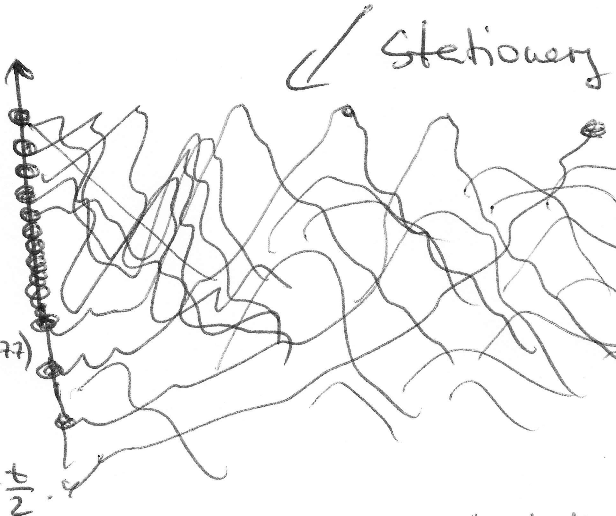
1) Let $\{\xi(t), t\in\mathbb{R}^d\}$ be a Gaussian process with $\mathbb{E}\xi(t)=0$, $\operatorname{Var}[\xi(t)-\xi(s)]$ $=\gamma^2(t-s)$. Note: ξ has stationary increments. Let $\operatorname{Var}\xi(t)=\sigma^2(t)$.

2) Let $\xi_1,\xi_2,\dots$ be iid copies of ξ

3) Let $\{u_i\}\sim PPP(e^{-t}dt)$ on $\mathbb{R}$.

<u>Thm</u>. The process $X(t)=\max\limits_{i\in\mathbb{N}}\left\{u_i+\xi_i(t)-\frac{\sigma^2(t)}{2}\right\}$ is max-stable and <u>stationary</u>.

<u>Proof</u>. $\gamma^2(t+h,s+h)=\gamma^2(t,s)$. $\square$



<u>Ex 1</u>. $\gamma^2(t)=|t|$, $T=\mathbb{R}$.

$\xi(t)$ is Brownian Motion on $\mathbb{R}$.

$$X(t)=\max_{i\in\mathbb{N}}\left\{u_i+\xi_i(t)-\frac{|t|}{2}\right\}$$

(original Brown–Resnick process, 1977)

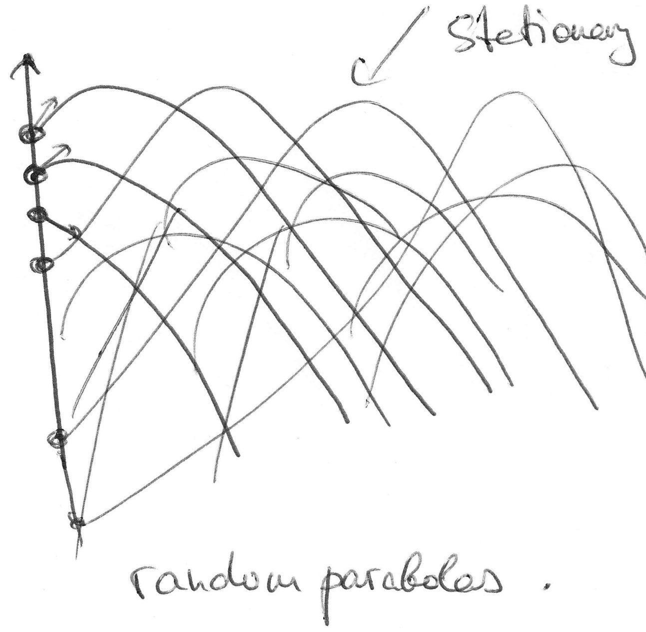
Explanation of stationarity:

$e^{-t}dt$ is invariant measure for $B(t)-\frac{t}{2}$.

<u>Ex 2</u>. $\xi(t)=Nt$, $N\sim N(0,1)$

$\gamma^2(t)=t^2$.

$$\xi(t)=\max_{i\in\mathbb{N}}\left\{u_i+Nt-\frac{t^2}{2}\right\}$$



random parabolas.

= Smith process.

<u>Ex 3</u>. $\gamma(t)=|t|^{\alpha}$. Systems of fBM's.

$\alpha=1\Rightarrow$ Ex 1.

$\alpha=2\Rightarrow$ Ex 2.