Asymptotic Independence of Stochastic Volatility Models

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Workshop on Extremes in Space and Time
May 2013

Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Overview

General definition of SV models

- Many common models for financial time series are of the form

$$X_t = \sigma_t \epsilon_t, \quad t \in \mathbb{Z},$$

where $\epsilon_t, t \in \mathbb{Z}$, are i.i.d. standardized innovations and $(\sigma_t)_{t\in\mathbb{Z}}$, is referred to as a "volatility" sequence.

- Sometimes

$$\sigma_t \in \sigma(X_t, X_{t-1}, \ldots, \sigma_{t-1}, \sigma_{t-2}, \ldots), \quad t \in \mathbb{Z},$$

e.g. for GARCH models.

- Alternative: Volatility sequence $(\sigma_t)_{t\in\mathbb{Z}}$ depends on an additional source of randomness $\Rightarrow$ SV models!

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Taylor's SV model

A very common specification is

Taylor's lognormal SV model (1982)

$$X_t = \sigma_t \epsilon_t, \quad t \in \mathbb{Z},$$

$$\log(\sigma_t) - \mu = \phi(\log(\sigma_{t-1}) - \mu) + \xi_t, \quad t \in \mathbb{Z},$$

where $\xi_t, t \in \mathbb{Z}$, are i.i.d. standard normal, independent of $(\epsilon_t)_{t\in\mathbb{Z}}$ and $|\phi| < 1$.

$\Rightarrow$ Volatility sequence has a log-normal distribution.

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SV models with heavy-tailed innovation sequence

Breiman's lemma - "the heaviest tail wins"

- If $|\epsilon_t|$ is regularly varying with index $-\alpha$, i.e.

$$c(u)P(|\epsilon_t| > u) \to 1, \quad u \to \infty,$$

  for a regularly varying function $c(\cdot)$ with index α
- and $\sigma_t \geq 0$ independent of ϵ_t with $E(\sigma_t^{\alpha+\delta}) < \infty$ for some $\delta > 0$, it holds that

$$c(u)P(\sigma_t|\epsilon_t| > u) \to E(\sigma_t^{\alpha}), \quad u \to \infty,$$

i.e. $|X_t| = \sigma_t|\epsilon_t|$ is tail-equivalent to $|\epsilon_t|$.

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Extremal dependence structure

What do we know about joint extremal behavior of

$$\begin{pmatrix} X_0 \\ X_h \end{pmatrix} = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_h \end{pmatrix} \begin{pmatrix} \epsilon_0 \\ \epsilon_h \end{pmatrix}, \quad h > 0?$$

Multivariate Breiman (Basrak, Davis, Mikosch (2002))

- Random vector $\mathbf{X} \in \mathbb{R}^d$ multivariate regularly varying with index $-\alpha$, i.e.

$$c(u)P(u^{-1}\mathbf{X} \in \cdot) \xrightarrow{v} \mu(\cdot)$$

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- random $q \times d$ matrix $\mathbf{A}$, independent of $\mathbf{X}$, with $0 < E(\|\mathbf{A}\|^{\alpha+\delta}) < \infty$ for some $\delta > 0$. Then

$$c(u)P(u^{-1}\mathbf{A}\mathbf{X} \in \cdot) \xrightarrow{v} \tilde{\mu}(\cdot) := E\left[\mu \circ \mathbf{A}^{-1}(\cdot)\right].$$

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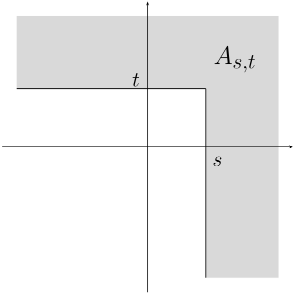
$$c(u)P(u^{-1}\mathbf{X} \in \cdot) \xrightarrow{v} \mu(\cdot) \Rightarrow c(u)P(u^{-1}\mathbf{AX} \in \cdot) \xrightarrow{v} \tilde{\mu}(\cdot) := E\left[\mu \circ \mathbf{A}^{-1}(\cdot)\right]$$

Application to Taylor's log-normal SV model:

- $\begin{pmatrix} X_0 \\ X_h \end{pmatrix} = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_h \end{pmatrix} \begin{pmatrix} \epsilon_0 \\ \epsilon_h \end{pmatrix}$
- (ϵ_0, ϵ_t) bivariate regularly varying with μ on $[-\infty, \infty] \times [-\infty, \infty] \setminus \{(0,0)\}$ concentrated on the axes $\Rightarrow \mu(A_{s,t}) = c(s^{-\alpha} + t^{-\alpha})$.



$\Rightarrow (X_0, X_h)$ is regularly varying with

$$\tilde{\mu}(A_{s,t}) = E\left[\mu \circ \begin{pmatrix} \sigma_0^{-1} & 0 \\ 0 & \sigma_h^{-1} \end{pmatrix} (A_{s,t})\right] = cE(\sigma_h^{\alpha})(s^{-\alpha} + t^{-\alpha})$$

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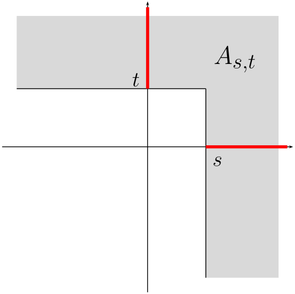

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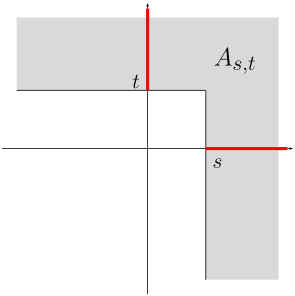

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Hidden regular variation and coefficient of tail dependence

Hidden regular variation (Resnick (2002))

A multivariate regularly varying vector $\mathbf{X} \in \mathbb{R}_+^d$ with limit measure μ concentrated on the axes shows hidden regular variation (HRV) on $(0, \infty]^d$ if a non-zero measure μ^0 on $(0, \infty]^d$ exists, such that

$$c^0(u)P(u^{-1}\mathbf{X} \in \cdot) \xrightarrow{v} \mu^0(\cdot), \quad u \to \infty,$$

for a regularly varying function $c^0(\cdot)$ with index α^0.

Coefficient of tail dependence (Ledford & Tawn (1998))

If $\mathbf{X}$ is standardized to index -1 of regular variation, we call $\eta = 1/\alpha^0 \in (0, 1]$ the coefficient of tail dependence.

$\Rightarrow$ Stochastic independence of X_1, X_2 implies $\eta = 1/2$ for (X_1, X_2) since $c^0(u) = (P(X_1 > u)P(X_2 > u))^{-1}$ is regularly varying with index 2.

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Breiman's lemma for hidden regular variation

- Remember the multivariate version of Breiman's lemma for a multivariate regularly varying vector and a random matrix. Does there exist an analogue for HRV?
- In the MRV setting, sets must be bounded away from $\mathbf{0}$, for HRV they must be bounded away from the axes. Set $\mathbb{F}^d = \{\mathbf{x} \in \mathbb{R}^d_{0,+} : \min(x_1, \ldots, x_d) = 0\}$.
- Define $d(\mathbf{x}, B) := \min_{\mathbf{y} \in B} \|\mathbf{x} - \mathbf{y}\|$ for $\mathbf{x} \in \mathbb{R}^d, B \subset \mathbb{R}^d$, and $\mathcal{N}^d := \{\mathbf{x} \in \mathbb{R}^d_{0,+} : d(\mathbf{x}, \mathbb{F}^d) = 1\}$.

  $\Rightarrow$ For a $d \times d$ matrix $\mathbf{A}$ define

$$\tau(\mathbf{A}) := \sup_{\mathbf{x} \in \mathcal{N}^d} d(\mathbf{A}\mathbf{x}, \mathbb{F}^d) \in [0, \infty].$$

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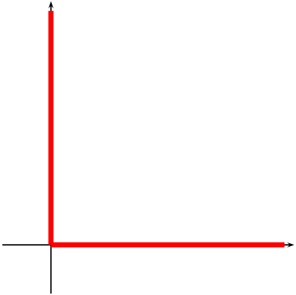

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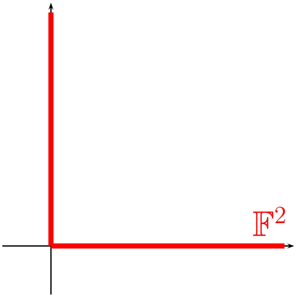

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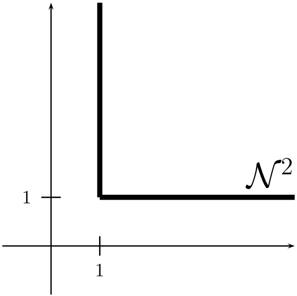

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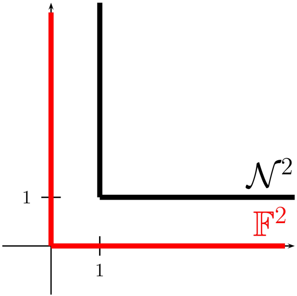

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Implications for "classical" SV models

- Let $(\sigma_t)_{t\in\mathbb{Z}}$ a light-tailed volatility sequence and $(\epsilon_t)_{t\in\mathbb{Z}}$ i.i.d. standardized regularly varying innovations independent of the volatilities.
- For $h > 0$, vector (ϵ_0, ϵ_h) shows HRV with coefficient of tail dependence $\eta = 1/2$ $(\alpha^0 = 2)$.
- For invertible 2×2-matrix $\mathbf{\Sigma}_h = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_h \end{pmatrix}$ one can show that $\tau(\mathbf{\Sigma}_h) = \max(\sigma_0, \sigma_h)$, thus $E(\tau(\mathbf{\Sigma}_h)^{2+\delta})$ exists for light-tailed volatilities.

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Two choices for η

- Thus, classic stochastic volatility models lead to $\eta = 1/2$ ("complete" asymptotic independence) while $\eta = 1$ for GARCH(p, q) models (asymptotic dependence).
- However, estimators of η for real data see η somewhere between those two values (project of Holger Drees).
- Are there models which allow for more flexibility in the modelling of η?

$\Rightarrow$ A heavy-tailed volatility sequence and light-tailed innovations would offer us more flexibility with respect to the finer modeling of the extremal dependence structure.
cf. also Mikosch and Rezapur (2013)

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- Innovations $\epsilon_t, t \in \mathbb{Z}$, i.i.d. such that $E(|\epsilon_t|^{1+\delta}) < \infty$.
- $\xi_t, t \in \mathbb{Z}$, i.i.d. and independent of (ϵ_t) with distribution such that

$$P(\xi_t > z) \sim K z^{\alpha} e^{-z}, \quad z \to \infty,$$

  for a real constant $\alpha \neq -1$ and a positive constant K and $P(\xi_t < z) = o\left(e^z\right), z \to -\infty$ (i.e. Exponential distribution).
- with $\alpha_i \in [0,1], \max_{i \in \mathbb{N}}\{\alpha_i\} = 1$, $\alpha_i = o(i^{-\theta}), i \to \infty$ for some $\theta > 1$.

Special case: Weibull-type AR(1) log-volatilities

- Assume that

$$X_t = \sigma_t \epsilon_t, \quad t \in \mathbb{Z},$$
$$\log(\sigma_t) - \mu = \phi(\log(\sigma_{t-1}) - \mu) + \xi_t, \quad t \in \mathbb{Z}.$$

  with the same assumptions on the distributions of $\epsilon_t, \xi_t, t \in \mathbb{Z}$ as before and $\phi \in (0, 1)$.
- This may be regarded as an extension of Taylor's "standard" SV model.

Stationary distribution of this model

- It follows from Rootzén (1986) that the corresponding MA(∞) process is well defined and that

$$P(\ln(\sigma_t) - \mu > z) \sim \hat{K} z^{\hat{\alpha}} e^{-z}, \quad z \to \infty,$$

for certain constants $\hat{K} > 0, \hat{\alpha} \in \mathbb{R}$.
- Thus, σ_t is regularly varying with index -1 (model can be generalized to index $-\alpha$ by writing $\frac{1}{\alpha}\xi_t$ instead ξ_t).
- Extremal behavior? Follows again from Rootzén that $(\ln(\sigma_0), \ln(\sigma_h))$ is asymptotically independent for all $h > 0$, same holds true for (σ_0, σ_h) and then by multivariate Breiman also for (X_0, X_h)

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Second order behavior of this model

- We are interested in the asymptotic behavior of

$$\begin{aligned} & P(\sigma_t > x, \sigma_{t+h} > x) \\ \stackrel{\mu=0}{=} & P\left(e^{\sum_{j=0}^{\infty} \xi_{t-j}\alpha_j} > x, e^{\sum_{j=0}^{\infty} \xi_{t+h-j}\alpha_j} > x\right) \\ = & P\left(\prod_{j=0}^{\infty}\left(e^{\xi_{t-j}}\right)^{\alpha_j} > x, \prod_{j=0}^{\infty}\left(e^{\xi_{t+h-j}}\right)^{\alpha_j} > x\right), \end{aligned}$$

where we know that $e^{\xi_t}, t \in \mathbb{Z}$, are i.i.d. regularly varying with index -1.

A general result for weighted power products

Let $Y_1, Y_2, \ldots$ be i.i.d. regularly varying random variables with index -1. Let $\alpha_i, \beta_i, i \in \mathbb{N}$, be two non-negative sequences. Then

$$P(\prod_{i=1}^{\infty} Y_i^{\alpha_i} > x, \prod_{j=1}^{\infty} Y_j^{\beta_j} > x) \sim cP(Y_s > x^{\kappa_s})P(Y_t > x^{\kappa_t})$$

where $s, t \in \mathbb{N}, \kappa_s, \kappa_t \geq 0$ are such that

$$\alpha_s\kappa_s + \alpha_t\kappa_t \geq 1, \;\; \beta_s\kappa_s + \beta_t\kappa_t \geq 1$$

and
$$\kappa_s + \kappa_t \to \min!$$

if a unique solution to this optimization problem exists.

"The *most efficient* tail combination wins".

$\Rightarrow$ In our AR(1) model, this gives us that the coefficient of tail dependence for vectors of lag h is equal to $\frac{1}{2-\phi^h}$.

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Résumé

- "Classic" SV models with heavy-tailed innovations are (just like GARCH(p, q) models) limited to a very specific range of extremal behavior.
- SV models with heavy-tailed volatility sequence share nice probabilistic properties of well-known models while allowing for a finer modelling of the extremal dependence structure.

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