RANDOM FIELDS AND GEOMETRY

from the book of the same name by

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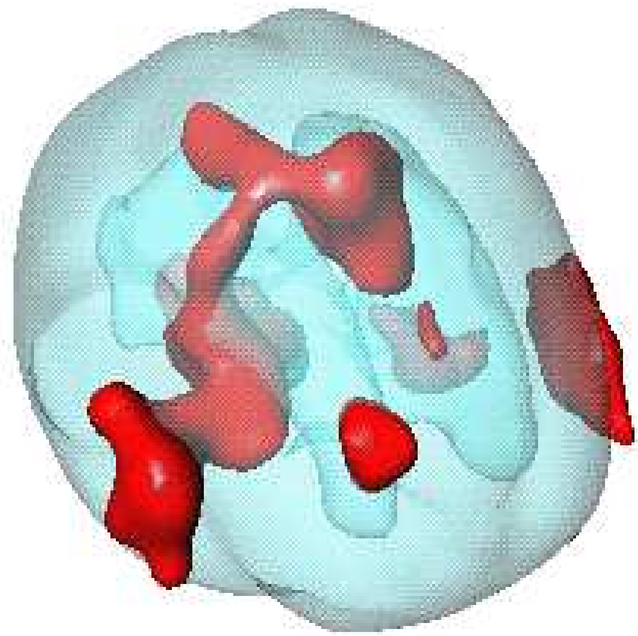
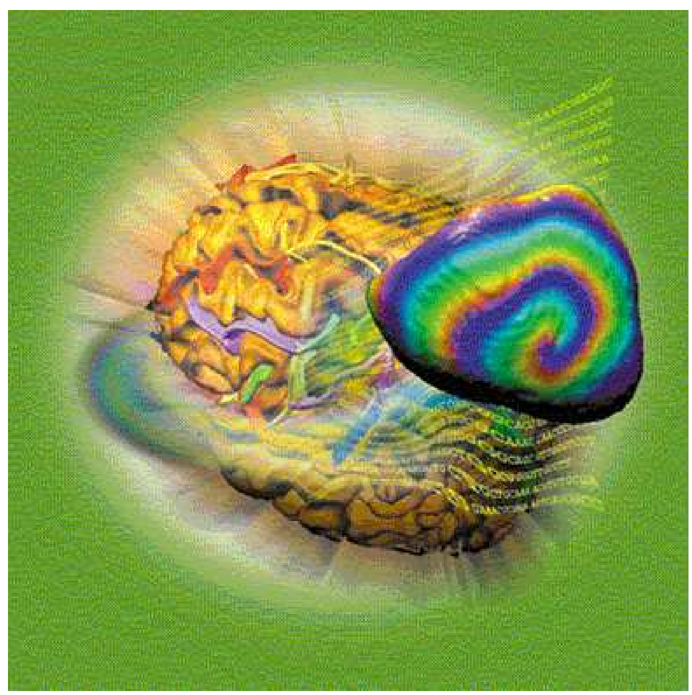
Mapping the Brain





- A result 70 years in the making: Under the model

$$
\begin{aligned}
f(t) : M &\rightarrow \mathbb{R} \text{ is smooth, Gaussian} \\
\mathbb{E}\{f(t)\} &\equiv 0. \\
\mathbb{E}\{f^2(t)\} &\equiv \sigma^2 \;=\; 1. \\
C(s,t) &\triangleq \mathbb{E}\{f(s)f(t)\} \text{ is known.}
\end{aligned}
$$

$$
\mathbb{P}\left\{\sup_{t\in M} f_t > u\right\}
$$

$$
\approx e^{-u^2/2} \sum_{j=0}^{n} C_j u^{\alpha-j} + o\left(e^{-u^2(1+\eta)/2}\right)
$$

- THEOREM: For piecewise C^2 Whitney stratified manifolds M embedded in C^3 ambient manifolds $\widetilde{M}$, and with convex support cones

$$
\liminf_{u\to\infty} -u^{-2} \log \left| \mathbb{P} - \sum_{j=0}^{\dim M} \mathcal{L}_j(M)\rho_j(u) \right|
$$

$$
\geq \frac{1}{2} + \frac{1}{2\sigma_c^2(f)}.
$$

The main Gaussian result

- Excursion sets

$$A_u(f, M) \;\stackrel{\Delta}{=}\; \{t \in M : f(t) \geq u\}$$

- The result:

$$\sum_{j=0}^{\dim M} \mathcal{L}_j(M)\rho_j(u) \;=\; \mathbb{E}\{\mathcal{L}_0(A_u(f, M))\}$$

- where

$$\rho_j(u) \;=\; (2\pi)^{-(j+1)/2} H_{j-1}(u) e^{-\frac{u^2}{2}}$$

- H_j is the j-th Hermite polynomial

$$H_n(x) \;=\; n! \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j x^{n-2j}}{j!\,(n-2j)!\,2^j}$$

$$H_{-1}(x) \;=\; e^{x^2/2} \int_x^{\infty} e^{-x^2/2}\, dx$$

- The $\mathcal{L}_j(M)$ are the Lipschitz-Killing curvatures of M

Non-Gaussian processes

- $f^1(t), \ldots, f^k(t)$ i.i.d. Gaussian satisfying all the assumptions in force until now

- $F : \mathbb{R}^k \to \mathbb{R}$ twice differentiable defines

$$f(t) \triangleq F\left(f^1(t), \ldots, f^k(t)\right)$$

- Examples of F:

$$\sum_1^k x_i^2, \qquad \frac{x_1\sqrt{k-1}}{(\sum_2^k x_i^2)^{1/2}}, \qquad \frac{m\sum_1^n x_i^2}{n\sum_{n+1}^{n+m} x_i^2}$$

- The result:

$$\mathbb{E}\left\{\mathcal{L}_j(A_u(f,M))\right\}$$

$$= \sum_{l=0}^{\dim M - j} \begin{bmatrix} j+l \\ l \end{bmatrix} (2\pi)^{-j/2} \mathcal{L}_{j+l}(M)\, \mathcal{M}_l^{(k)}\left(D_{F,u}\right)$$

where

$$D_{F,u} = F^{-1}([u,\infty))$$

We cannot avoid geometry

- The result:

$$\sum_{l=0}^{\dim M - j} \begin{bmatrix} j+l \\ l \end{bmatrix} (2\pi)^{-j/2} \mathcal{L}_{j+l}(M) \, \mathcal{M}_l^{(k)} \left(D_{F,u} \right)$$

$$D_{F,u} = F^{-1}([u,\infty))$$

Lipschitz-Killing curvatures $\mathcal{L}_j$,
Whitney stratified manifolds M,
Gaussian Minkowski functionals $\mathcal{M}_j^{(k)}$

- In dimension 1, with f stationary:

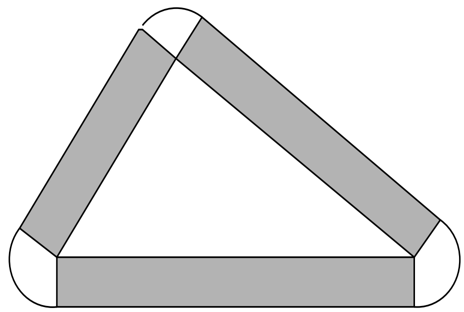
$$\begin{aligned} M &= [0,T] \\ \mathcal{L}_0(A_u) &= \mathbb{1}_{f(0)>u} + \\ &\quad \#\left\{ t \in [0,T] : \, f(t) = u, \, f'(t) > 0 \right\} \\ \mathcal{L}_1(A_u) &= \lambda_1 \{ t \in [0,T] : \, f(t) \geq u \} \\ \mathcal{L}_0(M) &= 1 \\ \mathcal{L}_1(M) &= T \end{aligned}$$

But the $\mathcal{M}_l^{(k)}$ remain!

Lipschitz-Killing curvatures: I

- The 'tube' in $\mathbb{R}^{N'}$ of radius ρ around an N dimensional M, $(N \leq N')$ is

$$\text{Tube}(M,\rho) \triangleq \{t \in M : d(t,M) \leq \rho\}$$



- For nice (e.g. convex) M, the volume of Tube(M,ρ) is, for $\rho < r_c'(M)$, given by Weyl's tube formula,

$$\lambda_{N'}(\text{Tube}(M,\rho)) = \sum_{j=0}^{N} \omega_{N'-j}\rho^{N'-j}\mathcal{L}_j(M)$$

- The $\mathcal{L}_j$ can be defined via the tube formula

- The $\mathcal{L}_j$ are intrinsic

Two examples:

- The solid ball $B^N(T)$:

$$\begin{aligned}
&\lambda_N\left(\text{Tube}\left(B^N(T),\rho\right)\right)\\
&\quad = (T+\rho)^N\omega_N\\
&\quad = \sum_{j=0}^{N}\binom{N}{j}T^j\rho^{N-j}\omega_N\\
&\quad = \sum_{j=0}^{N}\omega_{N-j}\rho^{N-j}\binom{N}{j}T^j\frac{\omega_N}{\omega_{N-j}}.
\end{aligned}$$

so that

$$\mathcal{L}_j\left(B^N(T)\right) = \binom{N}{j}T^j\frac{\omega_N}{\omega_{N-j}}.$$

- The sphere $S^{N-1}(T) \equiv S_T(\mathbb{R}^N)$:

$$\text{Tube}\left(S^{N-1}(T),\rho\right) = B^N(T+\rho) - B^N(T-\rho)$$

yields

$$\mathcal{L}_j\left(S^{N-1}(T)\right) = 2\binom{N}{j}\frac{\omega_N}{\omega_{N-j}}T^j$$

if $N-1-j$ is even, and 0 otherwise.

- Note the scaling in T!

Fundamental nature of $\mathcal{L}_j$

Let ψ be a real valued function on nice sets in $\mathbb{R}^N$ which is

- Invariant under rigid motions.

- Additive, in that

  $$\psi(M_1 \cup M_2) = \psi(M_1) + \psi(M_2) - \psi(M_1 \cap M_2)$$

- Monotone, in that

  $$M_1 \subseteq M_2 \;\Rightarrow\; \psi(M_1) \leq \psi(M_2).$$

Then

$$\psi(M) \;=\; \sum_{j=0}^{N} c_j \mathcal{L}_j(M),$$

where $c_0, \ldots, c_N$ are non-negative (ψ-dependent) constants.

$\mathcal{L}_0$: The Euler characteristic

- $M \subset \mathbb{R}^N$ is nice and "triangulisable"

- α_k is the number of k-dimensional simplices in the triangulation

  ◦ α_0 = number of vertices

  ◦ α_1 = number of lines

  ....

  ◦ α_k = number of "full" simplices

- $\mathcal{L}_0(M) \equiv$ Euler characteristic of M is

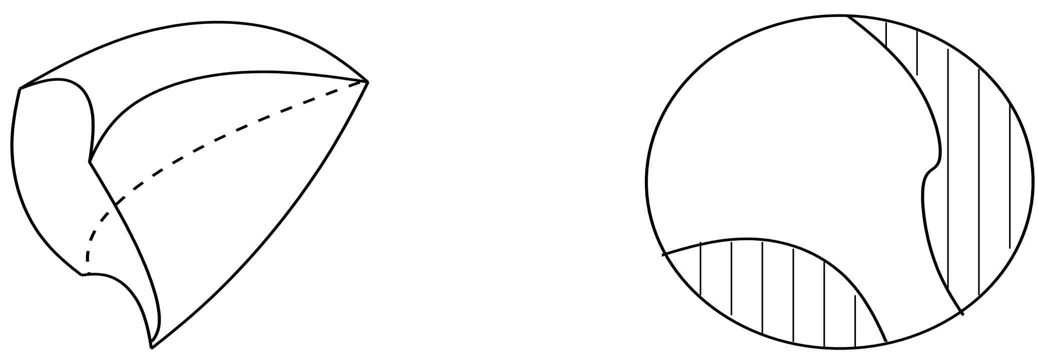
$$\varphi(A) \ = \ \alpha_0 - \alpha_1 + \cdots + (-1)^d \alpha_N$$

Whitney stratified manifolds

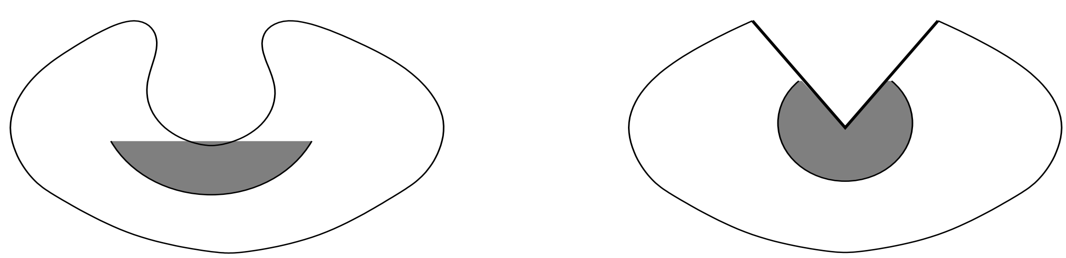
- WSM's can be written as

$$M = \bigcup_{k=0}^{\dim M} \partial_k M$$

with rules about glueing strata.



- Piecewise smooth manifolds are WSM's which have convex support cones



Riemannian manifolds

- Riemannian metrics. For each $t \in T$,

$$g_t : T_tM \times T_tM \;\to\; \mathbb{R}$$

is linear, positive definite, symmetric.

- $g_t(X_t, X_t) = 0 \iff X_t = 0$

- (M, g) is called a Riemannian manifold

- g is NOT a metric, but τ_g is:

$$\tau_g(s,t) \;=\; \inf_{c \in D^1([0,1];M)_{(s,t)}} L(c)$$

$$L(c) \;=\; \int_{[0,1]} \sqrt{g_t(c', c')(t)}\, dt$$

and $D^1([0,1];M)_{(s,t)}$ contains all piecewise C^1 maps $c : [0,1] \to M$ with $c(0) = s$, $c(1) = t$.

- The canonical Gaussian metric:

$$g_t(X_t, Y_t) \;\triangleq\; \mathbb{E}\left\{X_t f, Y_t f(t)\right\}$$

Riemannian curvature

- Curvature operator:

$$R(X,Y) \;=\; \nabla_X\nabla_Y \;-\; \nabla_Y\nabla_X \;-\; \nabla_{[X,Y]}.$$

- Curvature tensor:,

$$R(X,Y,Z,W) \;=\; g(R(X,Y)Z,W)$$

- Second fundamental form S

$$S(X,Y) \;\triangleq\; \widehat{\nabla}_X Y - \nabla_X Y \;=\; P_{TM}^{\perp}\left(\widehat{\nabla}_X Y\right)$$

- Scalar second fundamental form S_ν If ν is a unit normal vector field on M, The *scalar second fundamental form* of M in $\widehat{M}$ for ν is

$$S_\nu(X,Y) \;\triangleq\; \widehat{g}\left(S(X,Y),\nu\right)$$

- Shape operator $\mathcal{S}$: Defined by

$$\widehat{g}(\mathcal{S}_\nu(X),Y) = S_\nu(X,Y)$$

for $Y \in T(M)$, maps $T(M) \to T(M)$

.

Lipschitz-Killing curvatures II

1: The general case of LK measures:

$$\mathcal{L}_i(M,A) = \sum_{j=i}^{N} (2\pi)^{-(j-i)/2} \sum_{m=0}^{\lfloor \frac{j-i}{2} \rfloor} \frac{C(N-j, j-i-2m)}{(-1)^m m!\,(j-i-2m)!} \times \int_{\partial_j M \cap A} \int_{S(T_t\partial_j M^\perp)} \mathrm{Tr}^{T_t\partial_j M}\left(\widehat{R}^m \widehat{S}_{\nu_{N-j}}^{j-i-2m}\right) \times \mathbb{1}_{N_t M}(-\nu_{N-j})\, \mathcal{H}_{N-j-1}(d\nu_{N-j})\mathcal{H}_j(dt)$$

2: M embedded in $\mathbb{R}^l$ with Euclidean metric:

$$\mathcal{L}_i(M,A) = \sum_{j=i}^{N} (2\pi)^{-(j-i)/2} C(l-j, j-i) \times \int_{\partial_j M \cap A} \int_{S(\mathbb{R}^{l-j})} \frac{1}{(j-i)!} \mathrm{Tr}^{T_t\partial_j M}(S_\eta^{j-i}) \times \mathbb{1}_{\widehat{N_t M}}(-\eta)\mathcal{H}_{l-j-1}(d\eta)\mathcal{H}_j(dt)$$

3: Lipschitz-Killing curvatures:

$$\mathcal{L}_j(M) = \mathcal{L}_j(M,M).$$

Tube formulae

- On $\mathbb{R}^l$:
  $M \subset \mathbb{R}^l$ a piecewise smooth manifold. For $\rho < \rho_c(M, \mathbb{R}^l)$

$$\mathcal{H}_l\left(\text{Tube}(M,\rho)\right) = \sum_{i=0}^{N} \rho^{l-i} \omega_{l-i} \mathcal{L}_i(M)$$

- On the sphere $S_\lambda(\mathbb{R}^l)$:
  Similar: But the constants are different and we need a one-parameter family $\mathcal{L}^\lambda$ of Lipschitz-Killing curvatures.

- On general manifolds:
  Assuming piecewise smooth basic form remains, but constants change.

- General representation:

$$\psi\left(M\right) = \sum_{j=0}^{N} c_j \mathcal{L}_j(M),$$

for additive, monotone, 'invariant' ψ

A Gaussian tube formula

- Gauss measure γ_k is the (product) measure induced on $\mathbb{R}^k$ by k i.i.d. standard Gaussian random variables.

- Tube formula for WSM's in $\mathbb{R}^k$:

$$\gamma_k(\text{Tube}(M,\rho)) = \gamma_k(M) + \sum_{j=1}^{\infty} \frac{\rho^j}{j!} \mathcal{M}_j^{(k)}(M)$$

- Example 1: $M = [u,\infty) \subset \mathbb{R}^1$

$$\mathcal{M}_j^{(1)}([u,\infty)) = H_{j-1}(u) \frac{e^{-u^2/2}}{\sqrt{2\pi}}.$$

where

$$H_n(x) = n! \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j x^{n-2j}}{j!\,(n-2j)!\,2^j}, \qquad n \geq 0$$

$$H_{-1}(x) = \sqrt{2\pi}\,\Psi(x)\,e^{x^2/2},$$

- Example 2: $M = \mathbb{R}^k \setminus S_u(\mathbb{R}^k)$ hinges on calculations involving the χ_k^2 distribution, etc.

$\sigma_c^2(f)$

- Recall the Gaussian excursion problem:

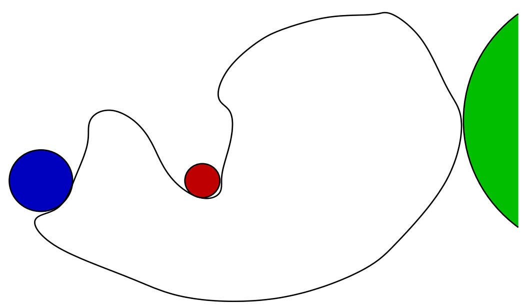
$$\liminf_{u\to\infty} -u^{-2}\log\left|\mathbb{P}\left\{\sup_{t\in M} f_t > u\right\} - \sum_{j=0}^{\dim M} \mathcal{L}_j(M)\rho_j(u)\right|$$

$$= \liminf_{u\to\infty} -u^{-2}\log\left|\mathbb{P}\left\{\sup_{t\in M} f_t > u\right\} - \mathbb{E}\left\{\mathcal{L}_0\left(A_u(M,f)\right)\right\}\right|$$

$$\geq \frac{1}{2} + \frac{1}{2\sigma_c^2(f)}.$$

$\sigma_c^2(f)$: cont

- The critical radius, r_c, of a set M is the radius of the largest ball that can be rolled around ∂M so that, at each point, it touches ∂M only once.



- If $\varphi(M)$ has no boundary, then

$$\sigma_c^2(f) \;=\; (\cot(r_c))^2$$

- If $r_c = \pi/2$, then $\sigma_c^2 = 0$ and the error in the approximation is zero!

- If M is convex, f is isotropic ($\Rightarrow$ no finite expansion) and monotone then

$$\sigma_c^2(f) \;=\; \mathrm{Var}\left(f''(t)\middle| f(t)\right)$$

$\sigma_c^2(f)$: In general

- Reproducing kernel Hilbert space $\mathcal{H}_f$ is the space of functions on M satisfying

$$\langle f(s), C(t,s) \rangle_{\mathcal{H}} = f(t),$$

where the inner product is determined by the covariance function C via

$$\langle C(s,\cdot), C(t,\cdot) \rangle_{\mathcal{H}} = C(s,t),$$

- An orthonormal basis $\{\varphi_n\}$ for $\mathcal{H}$ will always give a orthonormal expansion for f. i.e.

$$f_t = \sum_{n=1}^{\infty} \xi_n \varphi_n(t)$$

- THEOREM: In general, $\sigma_c^2(f)$ can be defined in terms of the critical radius of $S(\mathcal{H})$, the unit ball of the RKHS.

Orthogonal expansions

- Theorem: If $\{\varphi_n\}_{n\geq 1}$ is an orthonormal basis for the reproducing kernel Hilbert space of C then f has the L^2-representation

$$f_t = \sum_{n=1}^{\infty} \xi_n \varphi_n(t)$$

where the $\{\xi_n\}_{n\geq 1}$ are i.i.d. $N(0,1)$.

- Convergence: Sum converges uniformly $\Longleftrightarrow$ f is continuous (w.p. 1).

- A crucial identity:

$$1 = C(t,t) = \mathrm{Var} f_t = \sum_{1}^{\infty} \varphi_j^2(t)$$

- An observation:

$$X_t f_t = \sum_{n=1}^{\infty} \xi_n \, X_t \varphi_n(t)$$

where $X_t \in T_t M$, assuming that M is a differentiable manifold and $f_t \in C^1(M)$.

The rôle of the sphere

- An astounding consequence: If C is smooth enough every Gaussian process with expansion of order K, $\dim(M) \le K < \infty$ can be rewritten on a subset of S^{K-1} via

$$\begin{aligned} t &\equiv (\varphi_1(t), \ldots, \varphi_K(t)) \\ f(t) &\equiv f'(\varphi_1(t), \ldots, \varphi_K(t)) \end{aligned}$$

  and

$$f'(u) \triangleq \langle u, \xi \rangle$$

- Change of parameter space: From M to $\varphi(M) \in S^{K-1}$

- Covariance structure of f':

$$\begin{aligned} \mathbb{E}\left\{ f'(u) f'(v) \right\} &= \mathbb{E}\left\{ \langle u, \xi \rangle \langle v, \xi \rangle \right\} \\ &= \mathbb{E}\Big\{ \sum_i u_i \xi_i \ \sum_j v_j \xi_j \Big\} \\ &= \langle u, v \rangle \end{aligned}$$

- Consequence: If the expansion is finite, one case covers all.

Suprema and tubes

- Rewriting the canonical process:

$$
\begin{aligned}
f'_u &\sim \sum_{n=1}^{K} \xi_n u_n \\
&= \left(\sum_{n=1}^{K} \xi_n^2\right)^{1/2} \sum_{n=1}^{K} \frac{\xi_n}{\left(\sum_{n=1}^{K} \xi_n^2\right)^{1/2}} u_n \\
&= \sqrt{\chi_K^2} \cdot \sum_{n=1}^{K} U_n \, u_n
\end{aligned}
$$

for uniform U_n, independent of χ_K^2.

- Consequently:

$$
\begin{aligned}
&\mathbb{P}\left\{\sup_{u \in M} f'_u \geq \lambda\right\} \\
&= \int_0^\infty \mathbb{P}\left\{\sup_{u \in M} f'(u) > \lambda \Big| \chi_K^2 = x\right\} \phi_{\xi_K^2}(x)\, dx \\
&= \int_0^\infty \mathbb{P}\left\{\sup_{u \in M} \langle U, u \rangle > \lambda / \sqrt{x}\right\} \phi_{\xi_K^2}(x)\, dx
\end{aligned}
$$

- BUT, $\mathbb{P}\{\sup_{u \in M} \langle U, u \rangle > y\}$ is a tube volume!!

Back to non-canonical f

- For the canonical process on the sphere we now know that excursion probabilities are related to tube volumes
- By Weyl's tube formula these are related to Lipschitz-Killing curvatures
- The supremum of a non-canonical process over M is the same as the canonical process over $\varphi(M)$.
- To get an answer in terms of M, we need to carry back the Riemannian (Euclidean) structure of $\varphi(M)$ (i.e. of S^{K-1}) to M and so need smooth φ.
- Computation: The induced Riemannian metric that f' induces on M under the map φ^{-1} is given by

$$\begin{aligned} g(X_t, Y_t) &= \sum_n X_t\varphi_n(t) \cdot Y_t\varphi_n(t) \\ &= \mathbb{E}\{X_t f_t \cdot Y_t f_t\} \end{aligned}$$

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