

RANDOM FIELDS AND GEOMETRY

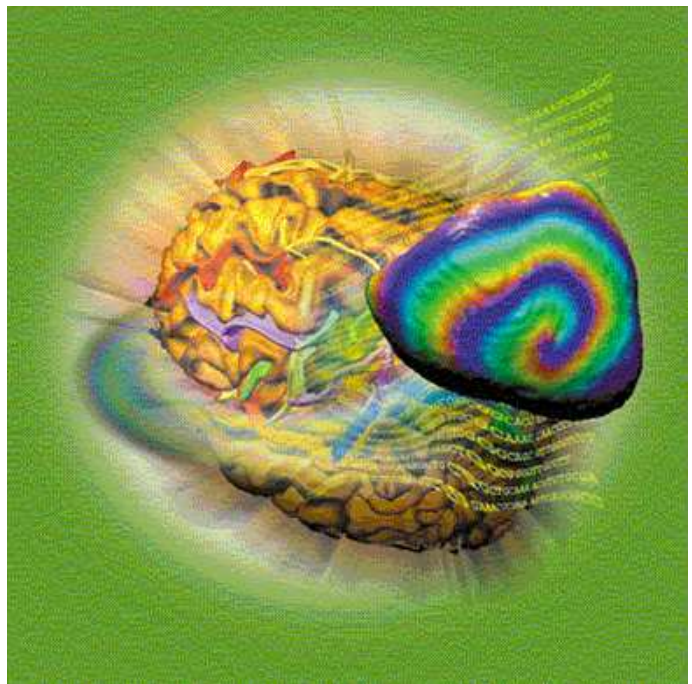
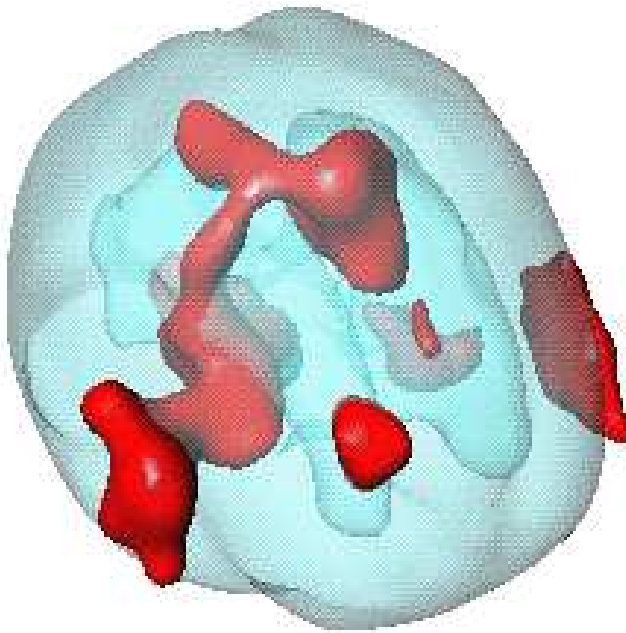
from the book of the same name by

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ie.technion.ac.il/Adler.phtml
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Mapping the Brain



- A result 70 years in the making: Under the model

$f(t) : M \rightarrow \mathbb{R}$ is smooth, Gaussian

$$\mathbb{E}\{f(t)\} \equiv 0.$$

$$\mathbb{E}\{f^2(t)\} \equiv \sigma^2 = 1.$$

$$C(s, t) \triangleq \mathbb{E}\{f(s)f(t)\} \text{ is known.}$$

$$\mathbb{P} \left\{ \sup_{t \in M} f_t > u \right\} \approx e^{-u^2/2} \sum_{j=0}^n C_j u^{\alpha-j} + o \left(e^{-u^2(1+\eta)/2} \right)$$

- THEOREM: For piecewise C^2 Whitney stratified manifolds M embedded in C^3 ambient manifolds \widetilde{M} , and with convex support cones

$$\liminf_{u \rightarrow \infty} -u^{-2} \log \left| \mathbb{P} - \sum_{j=0}^{\dim M} \mathcal{L}_j(M) \rho_j(u) \right| \geq \frac{1}{2} + \frac{1}{2\sigma_c^2(f)}.$$

The main Gaussian result

- Excursion sets

$$A_u(f, M) \triangleq \{t \in M : f(t) \geq u\}$$

- The result:

$$\sum_{j=0}^{\dim M} \mathcal{L}_j(M) \rho_j(u) = \mathbb{E} \{ \mathcal{L}_0(A_u(f, M)) \}$$

- where

$$\rho_j(u) = (2\pi)^{-(j+1)/2} H_{j-1}(u) e^{-\frac{u^2}{2}}$$

- H_j is the j -th Hermite polynomial

$$H_n(x) = n! \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j x^{n-2j}}{j! (n-2j)! 2^j}$$

$$H_{-1}(x) = e^{x^2/2} \int_x^\infty e^{-x^2/2} dx$$

- The $\mathcal{L}_j(M)$ are the Lipschitz-Killing curvatures of M

Non-Gaussian processes

- $f^1(t), \dots, f^k(t)$ i.i.d. Gaussian satisfying all the assumptions in force until now

- $F : \mathbb{R}^k \rightarrow \mathbb{R}$ twice differentiable defines

$$f(t) \triangleq F(f^1(t), \dots, f^k(t))$$

- Examples of F :

$$\sum_1^k x_i^2, \quad \frac{x_1 \sqrt{k-1}}{(\sum_2^k x_i^2)^{1/2}}, \quad \frac{m \sum_1^n x_i^2}{n \sum_{n+1}^{n+m} x_i^2}$$

- The result:

$$\begin{aligned} & \mathbb{E} \left\{ \mathcal{L}_j(A_u(f, M)) \right\} \\ &= \sum_{l=0}^{\dim M - j} \begin{bmatrix} j + l \\ l \end{bmatrix} (2\pi)^{-j/2} \mathcal{L}_{j+l}(\textcolor{red}{M}) \mathcal{M}_l^{(k)}(D_{F,u}) \end{aligned}$$

where

$$D_{F,u} = F^{-1}([u, \infty))$$

We cannot avoid geometry

- The result:

$$\sum_{l=0}^{\dim M - j} \begin{bmatrix} j + l \\ l \end{bmatrix} (2\pi)^{-j/2} \mathcal{L}_{j+l}(\textcolor{red}{M}) \mathcal{M}_l^{(k)}(D_{F,u})$$

$$D_{F,u} = F^{-1}([u, \infty))$$

Lipschitz-Killing curvatures \mathcal{L}_j ,

Whitney stratified manifolds M ,

Gaussian Minkowski functionals $\mathcal{M}_j^{(k)}$

- In dimension 1, with f stationary:

$$M = [0, T]$$

$$\mathcal{L}_0(A_u) = \mathbb{1}_{f(0) > u} + \# \left\{ t \in [0, T] : f(t) = u, f'(t) > 0 \right\}$$

$$\mathcal{L}_1(A_u) = \lambda_1 \{ t \in [0, T] : f(t) \geq u \}$$

$$\mathcal{L}_0(M) = 1$$

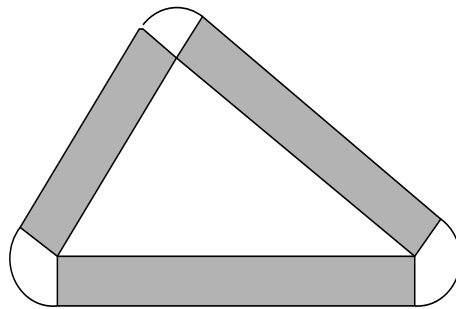
$$\mathcal{L}_1(M) = T$$

But the $\mathcal{M}_l^{(k)}$ remain!

Lipschitz-Killing curvatures: I

- The ‘tube’ in $\mathbb{R}^{N'}$ of radius ρ around an N dimensional M , ($N \leq N'$) is

$$\text{Tube}(M, \rho) \triangleq \{t \in M : d(t, M) \leq \rho\}$$



- For nice (e.g. convex) M , the volume of $\text{Tube}(M, \rho)$ is, for $\rho < r'_c(M)$, given by [Weyl's tube formula](#),

$$\lambda_{N'}(\text{Tube}(M, \rho)) = \sum_{j=0}^N \omega_{N'-j} \rho^{N'-j} \mathcal{L}_j(M)$$

- The \mathcal{L}_j can be defined via the tube formula
- The \mathcal{L}_j are [intrinsic](#)

Two examples:

- The solid ball $B^N(T)$:

$$\begin{aligned}
 \lambda_N \left(\text{Tube} \left(B^N(T), \rho \right) \right) &= (T + \rho)^N \omega_N \\
 &= \sum_{j=0}^N \binom{N}{j} T^j \rho^{N-j} \omega_N \\
 &= \sum_{j=0}^N \omega_{N-j} \rho^{N-j} \binom{N}{j} T^j \frac{\omega_N}{\omega_{N-j}}.
 \end{aligned}$$

so that

$$\mathcal{L}_j \left(B^N(T) \right) = \binom{N}{j} T^j \frac{\omega_N}{\omega_{N-j}}.$$

- The sphere $S^{N-1}(T) \equiv S_T(\mathbb{R}^N)$:

$$\text{Tube} \left(S^{N-1}(T), \rho \right) = B^N(T + \rho) - B^N(T - \rho)$$

yields

$$\mathcal{L}_j \left(S^{N-1}(T) \right) = 2 \binom{N}{j} \frac{\omega_N}{\omega_{N-j}} T^j$$

if $N - 1 - j$ is even, and 0 otherwise.

- Note the scaling in $T!$

Fundamental nature of \mathcal{L}_j

Let ψ be a real valued function on nice sets in \mathbb{R}^N which is

- **Invariant** under rigid motions.

- **Additive**, in that

$$\psi(M_1 \cup M_2) = \psi(M_1) + \psi(M_2) - \psi(M_1 \cap M_2)$$

- **Monotone**, in that

$$M_1 \subseteq M_2 \Rightarrow \psi(M_1) \leq \psi(M_2).$$

Then

$$\psi(M) = \sum_{j=0}^N c_j \mathcal{L}_j(M),$$

where c_0, \dots, c_N are non-negative (ψ -dependent) constants.

\mathcal{L}_0 : The Euler characteristic

- $M \subset \mathbb{R}^N$ is nice and “triangulisable”
- α_k is the number of k -dimensional simplices in the triangulation
 - α_0 = number of vertices
 - α_1 = number of lines
 - α_2 = number of triangles
 - α_3 = number of tetrahedra
 - α_4 = number of 4-simplices
 - α_5 = number of 5-simplices
 - α_6 = number of 6-simplices
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 - α_{100} = number of 100-simplices
- $\mathcal{L}_0(M) \equiv$ Euler characteristic of M is

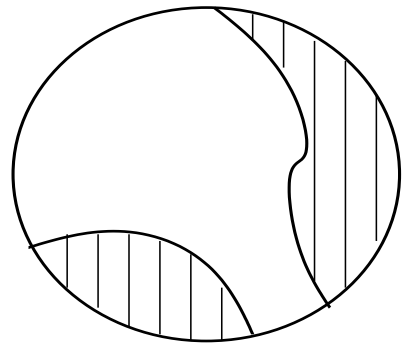
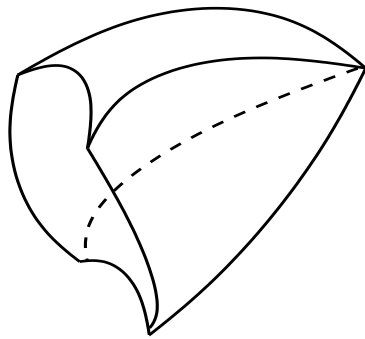
$$\varphi(A) = \alpha_0 - \alpha_1 + \cdots + (-1)^d \alpha_N$$

Whitney stratified manifolds

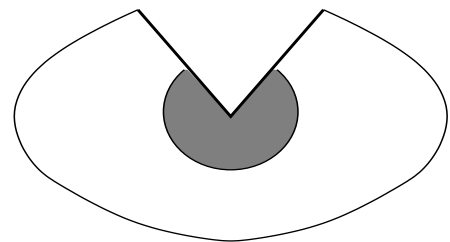
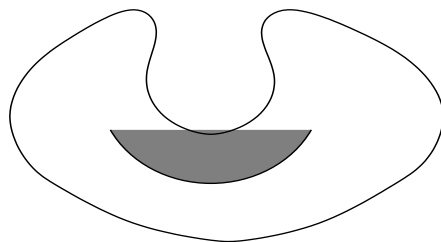
- WSM's can be written as

$$M = \bigcup_{k=0}^{\dim M} \partial_k M$$

with rules about glueing strata.



- Piecewise smooth manifolds are WSM's which have convex support cones



Riemannian manifolds

- Riemannian metrics. For each $t \in T$,

$$g_t : T_t M \times T_t M \rightarrow \mathbb{R}$$

is linear, positive definite, symmetric.

- $g_t(X_t, X_t) = 0 \iff X_t = 0$
- (M, g) is called a Riemannian manifold
- g is NOT a metric, but τ_g is:

$$\tau_g(s, t) = \inf_{c \in D^1([0, 1]; M)_{(s, t)}} L(c)$$

$$L(c) = \int_{[0, 1]} \sqrt{g_t(c'(t), c'(t))} dt$$

and $D^1([0, 1]; M)_{(s, t)}$ contains all piecewise C^1 maps $c : [0, 1] \rightarrow M$ with $c(0) = s$, $c(1) = t$.

- The canonical Gaussian metric:

$$g_t(X_t, Y_t) \triangleq \mathbb{E} \{X_t f, Y_t f(t)\}$$

Riemannian curvature

- Curvature operator:

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}.$$

- Curvature tensor:

$$R(X, Y, Z, W) = g(R(X, Y)Z, W)$$

- Second fundamental form S

$$S(X, Y) \triangleq \widehat{\nabla}_X Y - \nabla_X Y = P_{TM}^\perp (\widehat{\nabla}_X Y)$$

- Scalar second fundamental form S_ν If ν is a unit normal vector field on M , The *scalar second fundamental form* of M in \widehat{M} for ν is

$$S_\nu(X, Y) \triangleq \widehat{g}(S(X, Y), \nu)$$

- Shape operator \mathcal{S} : Defined by

$$\widehat{g}(\mathcal{S}_\nu(X), Y) = S_\nu(X, Y)$$

for $Y \in T(M)$, maps $T(M) \rightarrow T(M)$

Lipschitz-Killing curvatures II

1: The general case of LK measures:

$$\begin{aligned}
 & \mathcal{L}_i(M, A) \\
 &= \sum_{j=i}^N (2\pi)^{-(j-i)/2} \sum_{m=0}^{\lfloor \frac{j-i}{2} \rfloor} \frac{C(N-j, j-i-2m)}{(-1)^m m! (j-i-2m)!} \\
 & \quad \times \int_{\partial_j M \cap A} \int_{S(T_t \partial_j M^\perp)} \text{Tr}^{T_t \partial_j M} \left(\hat{R}^m \hat{S}_{\nu_{N-j}}^{j-i-2m} \right) \\
 & \quad \times \mathbb{1}_{N_t M}(-\nu_{N-j}) \mathcal{H}_{N-j-1}(d\nu_{N-j}) \mathcal{H}_j(dt)
 \end{aligned}$$

2: M embedded in \mathbb{R}^l with Euclidean metric:

$$\begin{aligned}
 & \mathcal{L}_i(M, A) \\
 &= \sum_{j=i}^N (2\pi)^{-(j-i)/2} C(l-j, j-i) \\
 & \quad \times \int_{\partial_j M \cap A} \int_{S(\mathbb{R}^{l-j})} \frac{1}{(j-i)!} \text{Tr}^{T_t \partial_j M} (S_\eta^{j-i}) \\
 & \quad \times \mathbb{1}_{\widehat{N_t M}}(-\eta) \mathcal{H}_{l-j-1}(d\eta) \mathcal{H}_j(dt)
 \end{aligned}$$

3: Lipschitz-Killing curvatures:

$$\mathcal{L}_j(M) = \mathcal{L}_j(M, M).$$

Tube formulae

- On \mathbb{R}^l :

$M \subset \mathbb{R}^l$ a piecewise smooth manifold.

For $\rho < \rho_c(M, \mathbb{R}^l)$

$$\mathcal{H}_l(\text{Tube}(M, \rho)) = \sum_{i=0}^N \rho^{l-i} \omega_{l-i} \mathcal{L}_i(M)$$

- On the sphere $S_\lambda(\mathbb{R}^l)$:

Similar: But the constants are different and we need a one-parameter family \mathcal{L}^λ of Lipschitz-Killing curvatures.

- On general manifolds:

Assuming piecewise smooth basic form remains, but constants change.

- General representation:

$$\psi(M) = \sum_{j=0}^N c_j \mathcal{L}_j(M),$$

for additive, monotone, ‘invariant’ ψ

A Gaussian tube formula

- Gauss measure γ_k is the (product) measure induced on \mathbb{R}^k by k i.i.d. standard Gaussian random variables.
- Tube formula for WSM's in \mathbb{R}^k :

$$\gamma_k(\text{Tube}(M, \rho)) = \gamma_k(M) + \sum_{j=1}^{\infty} \frac{\rho^j}{j!} \mathcal{M}_j^{(k)}(M)$$

- Example 1: $M = [u, \infty) \subset \mathbb{R}^1$

$$\mathcal{M}_j^{(1)}([u, \infty)) = H_{j-1}(u) \frac{e^{-u^2/2}}{\sqrt{2\pi}}.$$

where

$$H_n(x) = n! \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j x^{n-2j}}{j! (n-2j)! 2^j}, \quad n \geq 0$$

$$H_{-1}(x) = \sqrt{2\pi} \psi(x) e^{x^2/2},$$

- Example 2: $M = \mathbb{R}^k \setminus S_u(\mathbb{R}^k)$ hinges on calculations involving the χ_k^2 distribution, etc.

$$\sigma_c^2(f)$$

- Recall the Gaussian excursion problem:

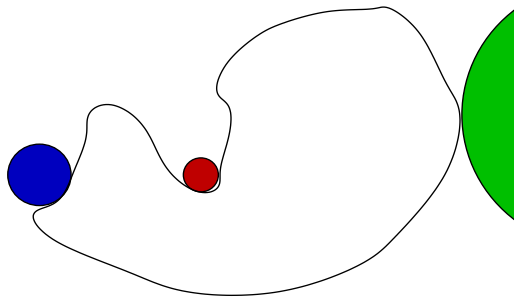
$$\liminf_{u \rightarrow \infty} -u^{-2} \log \left| \mathbb{P} \left\{ \sup_{t \in M} f_t > u \right\} - \sum_{j=0}^{\dim M} \mathcal{L}_j(M) \rho_j(u) \right|$$

$$= \liminf_{u \rightarrow \infty} -u^{-2} \log \left| \mathbb{P} \left\{ \sup_{t \in M} f_t > u \right\} - \mathbb{E} \{ \mathcal{L}_0(A_u(M, f)) \} \right|$$

$$\geq \frac{1}{2} + \frac{1}{2\sigma_c^2(f)}.$$

$\sigma_c^2(f)$: cont

- The critical radius, r_c , of a set M is the radius of the largest ball that can be rolled around ∂M so that, at each point, it touches ∂M only once.



- If $\varphi(M)$ has no boundary, then

$$\sigma_c^2(f) = (\cot(r_c))^2$$

- If $r_c = \pi/2$, then $\sigma_c^2 = 0$ and the error in the approximation is **zero**!
- If M is **convex**, f is **isotropic** (\Rightarrow **no** finite expansion) and **monotone** then

$$\sigma_c^2(f) = \text{Var}(f''(t)|f(t))$$

$\sigma_c^2(f)$: In general

- Reproducing kernel Hilbert space \mathcal{H}_f is the space of functions on M satisfying

$$\langle f(s), C(t, s) \rangle_{\mathcal{H}} = f(t),$$

where the inner product is determined by the covariance function C via

$$\langle C(s, \cdot), C(t, \cdot) \rangle_{\mathcal{H}} = C(s, t),$$

- An orthonormal basis $\{\varphi_n\}$ for \mathcal{H} will always give a orthonormal expansion for f . i.e.

$$f_t = \sum_{n=1}^{\infty} \xi_n \varphi_n(t)$$

- **THEOREM:** In general, $\sigma_c^2(f)$ can be defined in terms of the critical radius of $S(\mathcal{H})$, the unit ball of the RKHS.

Orthogonal expansions

- **Theorem:** If $\{\varphi_n\}_{n \geq 1}$ is an orthonormal basis for the reproducing kernel Hilbert space of C then f has the L^2 -representation

$$f_t = \sum_{n=1}^{\infty} \xi_n \varphi_n(t)$$

where the $\{\xi_n\}_{n \geq 1}$ are i.i.d. $N(0, 1)$.

- **Convergence:** Sum converges uniformly $\iff f$ is continuous (w.p. 1).
- **A crucial identity:**

$$1 = C(t, t) = \text{Var} f_t = \sum_1^{\infty} \varphi_j^2(t)$$

- **An observation:**

$$X_t f_t = \sum_{n=1}^{\infty} \xi_n X_t \varphi_n(t)$$

where $X_t \in T_t M$, assuming that M is a differentiable manifold and $f_t \in C^1(M)$.

The rôle of the sphere

- An astounding consequence: If C is smooth enough every Gaussian process with expansion of order K , $\dim(M) \leq K < \infty$ can be rewritten on a subset of S^{K-1} via

$$\begin{aligned} t &\equiv (\varphi_1(t), \dots, \varphi_K(t)) \\ f(t) &\equiv f'(\varphi_1(t), \dots, \varphi_K(t)) \end{aligned}$$

and

$$f'(u) \triangleq \langle u, \xi \rangle$$

- Change of parameter space: From M to $\varphi(M) \in S^{K-1}$
- Covariance structure of f' :

$$\begin{aligned} \mathbb{E} \{ f'(u) f'(v) \} &= \mathbb{E} \{ \langle u, \xi \rangle \langle v, \xi \rangle \} \\ &= \mathbb{E} \left\{ \sum_i u_i \xi_i \sum_j v_j \xi_j \right\} \\ &= \langle u, v \rangle \end{aligned}$$

- Consequence: If the expansion is finite, one case covers all.

Suprema and tubes

- Rewriting the canonical process:

$$\begin{aligned}
 f'_u &\sim \sum_{n=1}^K \xi_n u_n \\
 &= \left(\sum_{n=1}^K \xi_n^2 \right)^{1/2} \sum_{n=1}^K \frac{\xi_n}{\left(\sum_{n=1}^K \xi_n^2 \right)^{1/2}} u_n \\
 &= \sqrt{\chi_K^2} \cdot \sum_{n=1}^K U_n u_n
 \end{aligned}$$

for **uniform** U_n , independent of χ_K^2 .

- Consequently:

$$\begin{aligned}
 &\mathbb{P} \left\{ \sup_{u \in M} f'_u \geq \lambda \right\} \\
 &= \int_0^\infty \mathbb{P} \left\{ \sup_{u \in M} f'(u) > \lambda \mid \chi_K^2 = x \right\} \phi_{\chi_K^2}(x) dx \\
 &= \int_0^\infty \mathbb{P} \left\{ \sup_{u \in M} \langle U, u \rangle > \lambda / \sqrt{x} \right\} \phi_{\chi_K^2}(x) dx
 \end{aligned}$$

- BUT**, $\mathbb{P} \{ \sup_{u \in M} \langle U, u \rangle > y \}$ is a tube volume!!

Back to non-canonical f

- For the canonical process on the sphere we now know that excursion probabilities are related to tube volumes
- By Weyl's tube formula these are related to Lipschitz-Killing curvatures
- The supremum of a non-canonical process over M is the same as the canonical process over $\varphi(M)$.
- To get an answer in terms of M , we need to carry back the Riemannian (Euclidean) structure of $\varphi(M)$ (i.e. of S^{K-1}) to M and so need smooth φ .
- Computation: The induced Riemannian metric that f' induces on M under the map φ^{-1} is given by

$$\begin{aligned} g(X_t, Y_t) &= \sum_n X_t \varphi_n(t) \cdot Y_t \varphi_n(t) \\ &= \mathbb{E} \{X_t f_t \cdot Y_t f_t\} \end{aligned}$$

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www.math.mcgill.ca/~keith