# FATIGUE LIFE PREDICTION FOR A VESSEL SAILING THE NORTH ATLANTIC ROUTE

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# Introduction

Reliability of a vessel depends on the fatigue strentgh of the material, whose properties are determined by experiments. Based on a damage accumulation rule, we derive the asymptotic distribution of the damage accumulated by the material and use it to derive the probability distribution of the fatigue life prediction

# Definitions-Assumptions

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- Damage accumulation rule - Palmgren-Miner

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- Amplitude def - Rainflow cycle count (RFC)

# Expected nominal damage

- For $N(u,v;t)$ : nb of RFC-cycles with $max > u$ and $min < v$, bdd fun. of $u$ and $N(u,v;0) = 0$

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- For $\mu(u, v; t) = E[N(u, v; t)]$

$$E[D_X(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{u} \beta(\beta - 1)(u - v)^{\beta - 2} \mu(u, v; t)\, dv\, du$$

# Upper bound for the damage intensity

$\mu(u, v; t)$ is decreas. for $u$, increas. for $v$, hence for $u \geq v$

$$\mu(u, v; t) \leq min\{\mu(u, u; t), \mu(v, v; t)\}$$

After some lengthy derivations we can show that

$$\frac{\partial \mu(u, v; t)}{\partial t} \leq min\{\mu_t^+(u), \mu_t^+(v)\}$$

where $\mu_t^+(u) = \frac{\partial \mu(u,u;t)}{\partial t}$ - upcrossing intensity of $u$ at $t$

## Upper bound for expected nominal damage

- Damage intens. $d_X(t) = \frac{d(E[D_X(t)])}{dt}$ bdd from above

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- For $X(t)$ zero-mean, loc. stat., Gaussian load, $C[X(t), \dot{X}(t)] \ll min\{\sigma_X(t), \sigma_{\dot{X}}(t)\} \to C[X(t), \dot{X}(t)] = 0$

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- Hence

$$\tilde{E}[D_X(t)] = 2^{\frac{3\beta}{2}} \Gamma\left(\frac{\beta}{2}+1\right) \int_0^t \frac{\sigma_{\dot{X}}(\tau)\sigma_X^{\beta-1}(\tau)}{2\pi}\, d\tau$$

## Upper bound for expected damage

Suppose $\sigma_X(t)$ and $\sigma_{\dot{X}}(t)$ known for $[0, t]$. Then, for $t$ large enough the difference

$$D_X(t) - \tilde{E}[D_X(t)] \approx 0$$

hence

$$D(t) \approx K \int_0^t \frac{\sigma_{\dot{X}}(\tau)\sigma_X^{\beta-1}(\tau)}{2\pi} \, d\tau$$

with $\log(K) \in N(m_K, \sigma_K^2)$

## Fatigue life distribution

Failure time:

$$P(T^f \leq t) = P(D(t) \geq d^{crt})$$

If $D_X(t) \in N(m(t), s^2(t))$

$$P[T^f \leq t] = \int_{-\infty}^{\infty} \Phi\left(\frac{m_K + \log m(t) + \log(1 + \frac{s(t)}{m(t)}z) - \log d^{crt}}{\sigma_K}\right) \varphi(z)d$$

## Fatigue damage accumulated by a vessel

Let $X(t)$ be a variable load applied at a vessel. Then, after certain considerations / simplifications

$$D(t) = K \int_0^t H_s(\tau)^\alpha \, d\tau$$

where $H_s(t)$ - significant wave height process encountered by the vessel

# Notation

- $X(t)$ - random load, e.g. stress at some point on the vessel
- $t_0$ - departure time
- $T$ - duration of trip
- $\mathbf{V}(t) = (V_x, V_y)$ - velocity of vessel, known in advance
- $\mathbf{s}(t) = (x(t), y(t))$ - deterministic position of vessel at time $t$

## The mean and variance of $\int_{t_0}^{t_0+T} H_s(t)^\alpha \, dt$ during one voyage

Let $\log(H_s(\mathbf{s}, t))$ be a locally stat. Gaussian r.f. with

- $\mu(\mathbf{s}, t) = E[\log(H_s(\mathbf{s}, t))]$
- $r((\mathbf{s}_1, t_1), (\mathbf{s}_2, t_2)) = C[\log(H_s(\mathbf{s}_1, t_1)), \log(H_s(\mathbf{s}_2, t_2))]$

Then encountered pr. $Y(t) = \log(H_s(t)) = \log(H_s(\mathbf{s}(t), t))$ is locally stat. Gaussian with

- $\mu(t) = E[Y(t)] = E[\log(H_s(t))] = \mu(\mathbf{s}(t), t)$
- $r(t_1, t_2) = C[Y(t_1), Y(t_2)] = C[\log(H_s(t_1)), \log(H_s(t_2))] = r((\mathbf{s}(t_1), t_1), (\mathbf{s}(t_2), t_2))$

## The mean and variance of $\int_{t_0}^{t_0+T} H_s(t)^\alpha \, dt$ during one voyage

$$m(T) = E[\int_{t_0}^{t_0+T} H_s(t)^\alpha dt] := \int_{t_0}^{t_0+T} h(t)\, dt$$

with $h(t) = \exp\left(\alpha\mu(t) + \alpha^2 \frac{\sigma^2(t)}{2}\right)$ and $\sigma^2 = r(t, t)$

$$s^2(T) = V[\int_{t_0}^{t_0+T} H_s(t)^\alpha \, dt] = \int_{t_0}^{t_0+T} \int_{t_0}^{t_0+T} h(t_1)h(t_2) \left(e^{\alpha^2 r(t_1,t_2)} - 1\right) d$$

# Distribution for $D(n)$ - damage accumulated during $n$ voyages

$t_{ijr}, T_{ijr}, j = 1, \ldots, 12, r = 1, 2$ and $i = 1, \ldots, n_{rj}$: departure, duration of $i^{th}$ voyage in $j^{th}$ month in $r^{th}$ dir. $n_{1j} = n_{2j}$ or $n_{1j} = n_{2j} + -1$ and $T_{ijr}$ indep. of $t. H_s(t) = 0, t \notin [t_{ijr}, t_{ijr} + T_{ijr}]$ and vessel stays at port enough time for $Y(t)$ to become independent. Then

- $Z^i_{jr} = \int_{t_{irj}}^{t_{irj}+T_{irj}} H_s(t)^\alpha \, dt$: indep., mean $m_{jr}$ and var. $s^2_{jr}$

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- $Z^i_{jr} = \int_{t_{irj}}^{t_{irj}+T_{irj}} H_s(t)^\alpha \, dt$: indep., mean $m_{jr}$ and var. $s^2_{jr}$
- CLT $\rightarrow D_{jr} = \sum_{i=1}^{n_{rj}} Z^i_{jr} \approx N(n_{rj} m_{jr}, n_{rj} s^2_{jr})$

# Distribution for $D(n)$ - damage accumulated during $n$ voyages

Hence

$$D_X(n) = \sum_{j=1}^{12} \sum_{r=1}^{2} D_{jr} = \int_{t_0}^{t_0+T} H_s(t)^{\alpha}\, dt \approx N(\sum_{j=1}^{12} \sum_{r=1}^{2} n_{rj} m_{jr}, \sum_{j=1}^{12} \sum_{r=1}^{2} n_{rj}$$

with $n = \sum_{j=1}^{12} \sum_{r=1}^{2} n_{rj}$ and

$$P[T^f \le n] = \int_{-\infty}^{\infty} \Phi\left(\frac{m_K + \log(m(n)) + \log(1 + \frac{s(n)}{m(n)} z)}{\sigma_K}\right) \varphi(z)\, dz$$

## Remarks

- If $r(t_1, t_2)$ is unknown but $> 0$, setting $r(t_1, t_2) = 0$ results to an underestimation of $s^2(T)$ which consequently leads to a fatigue failure time more concentrated about the median
- Most of damage occurs during such operations as loading cargo or supplying with fuel, but these loads are not considered here. Obviously though, this damage could also be included in the analysis by allowing the function $h(t)$ to take on suitable values during the times between travelling.

# Fatigue life prediction for a vessel sailing the NAr

Assume a vessel departs from $\mathbf{s}_0 = (x_0, y_0)$ at $t_0$ and travels with $\mathbf{V} = (V_1, V_2) = (0.5046, 0)$ (deg/hr)

$(H_s)_{t_0}(t) = H_s(x(t), y(t), t), \quad t_0 \le t \le t_0 + T$

where

$x(t) = x_0 + V_1(t - t_0), y(t) = y_0 + V_2(t - t_0), \mathbf{s}(t) = (x(t), y(t))$

Assuming duration of voyage is too short so time variability of mean $\mu(\mathbf{s}, t)$ can be neglected

$$\mu(t) = \mu(x_0 + V_1(t - t_0), y_0 + V_2(t - t_0), t_0), \quad t_0 \le t \le t_0 + T$$

and

$$r(t_1, t_2) = r(x(t_2) - x(t_1), y(t_2) - y(t_1), t_2 - t_1) = r(t),$$

for $\mathbf{s}(t_1)$ and $\mathbf{s}(t_2)$ inside the same stationarity region and $t = t_2 - t_1$, otherwise $r(t_1, t_2) = 0$

# Fatigue life prediction for a vessel sailing the NAr

Each crossings lasts around 4.5 days, short compared to time variability of covariance parameters and covariance depends only $t_0$. Hence,

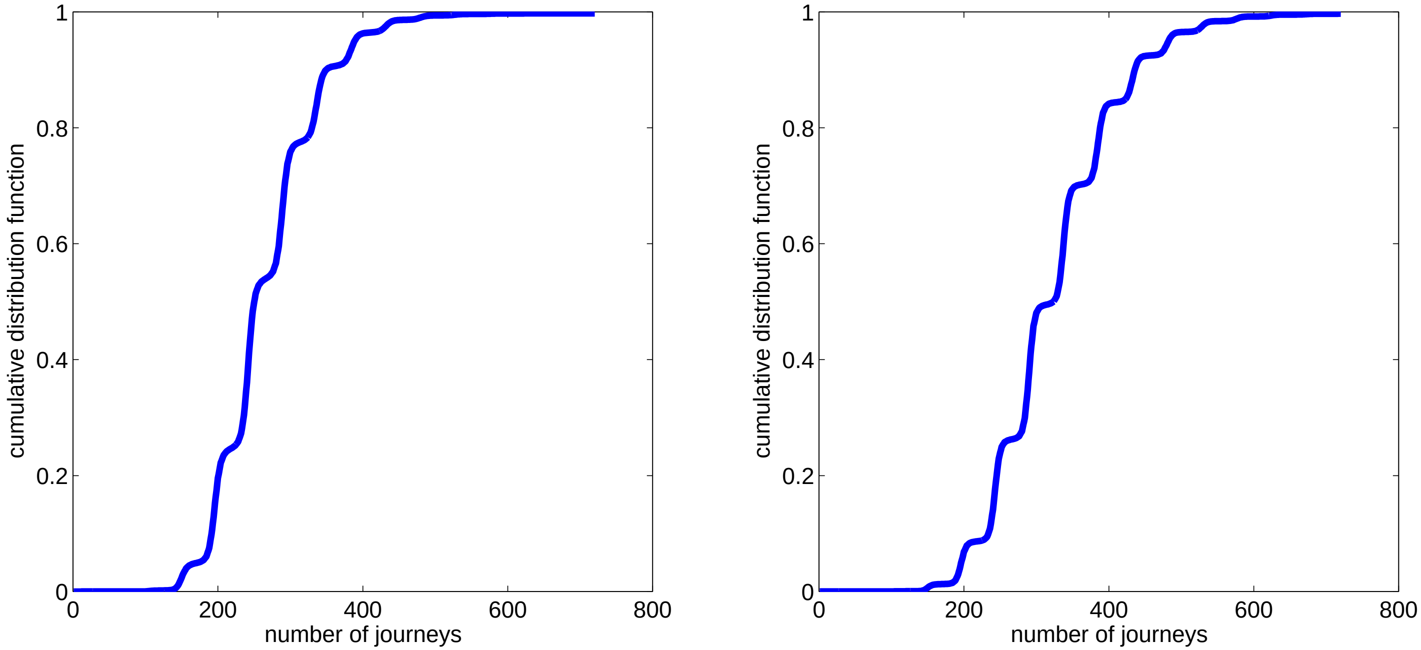
$$r(t) = \sigma^2 \left( p e^{-\frac{\lambda_{200}^1}{2\sigma^2}[(V_1 - v_x)^2 + V_2^2]t^2 - c|t|} + (1-p) e^{-\frac{\lambda_{200}^2}{2\sigma^2}[(V_1 - \tilde{v}_x)^2 + V_2^2]t^2 - c|t|} \right)$$

Consequently,

$$E[D_X(T)] = \int_{t_0}^{t_0+T} e^{\alpha\mu(t) + \alpha^2 \frac{\sigma^2(t)}{2}}\, dt$$

$$V[D_X(T)] = \int_{t_0}^{t_0+T} \int_{t_0}^{t_0+T} e^{\alpha\mu(t) + \alpha^2 \frac{\sigma^2(t)}{2}} e^{\alpha\mu(s) + \alpha^2 \frac{\sigma^2(s)}{2}} \cdot \left( e^{\alpha^2 r(s,t)} - 1 \right) ds\, dt$$

# Fatigue life distribution




*Left:* Fatigue life distribution for a vessel with $\beta = 4$ and $\log(K) \in N(6.4523 \cdot 10^{-8}, 0.06)$
*Right:* Fatigue life distribution for a vessel with $\log(K) \in N(4.5625 \cdot 10^{-7}, 0.06)$