A CLASS OF DISTRIBUTION FUNCTIONS WITH UNBIASED ESTIMATORS FOR THE EXTREME VALUE INDEX

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Introduction

Let $X_1, X_2, \ldots$ be i.i.d. random variables with d.f. F.

Let F be in the domain of attraction of an extreme value distribution, i.e. for some $\gamma \in \mathbb{R}$ (the **extreme value index**) and sequences a_n and b_n

$$\lim_{n\to\infty} F^n(a_n x + b_n) = \exp\{-(1+\gamma)^{-1/\gamma}\} \qquad (1)$$

for all x with $1 + \gamma x > 0$.

In terms of the function

$$U := (1/(1-F))^{\leftarrow}$$

the convergence (1) becomes

$$\lim_{t\to\infty} \frac{U(tx)-U(t)}{a(t)} = \frac{x^{\gamma}-1}{\gamma} \qquad (2)$$

for some positive function a and all $x > 0$.

The most used estimators for γ are constructed as **functionals** of

$$(X_{n-k,n}, X_{n-k+1,n}, \ldots, X_{n,n})$$

and it is well known that they are **consistent** under (2) provided

$$k = k(n) \to \infty, \quad k(n)/n \to 0, \quad \text{as} \quad n \to \infty.$$

- Pickands estimator (Pickands 1975)
- maximum likelihood estimator (R. Smith 1987, Drees, de Haan and Li 2002)
- moment estimator (Dekkers, Einmahl and de Haan 1989)

In order to get **asymptotic normality** it is very useful to work under a somehow stronger condition than (2), the **second order condition** (de Haan and Stadtmüller 1996, Drees 1998):

Suppose that there exists a positive or negative function A with $\lim_{t\to\infty} A(t) = 0$ such that for all $x > 0$

$$\lim_{t\to\infty} \frac{\frac{U(tx)-U(t)}{a(t)} - \frac{x^\gamma-1}{\gamma}}{A(t)} = \frac{1}{\rho}\left(\frac{x^{\gamma+\rho}-1}{\gamma+\rho} - \frac{x^\gamma - 1}{\gamma}\right) \qquad (3)$$

where $\rho \leq 0$ is the **second order parameter**.

If (3) holds, the function $|A|$ is regularly varying of order ρ that is, the convergence in (2) is basically at a **polynomial rate**.

Under condition (3) one can prove that any of the mentioned estimators is asymptotically normal provided $\sqrt{k}A(n/k) = O(1)$, $n \to \infty$, i.e., roughly speaking $k(n) = O\left(n^{1-\frac{1}{1-2\rho}}\right)$.

A bias appears when $\sqrt{k}A(n/k) \to \lambda \neq 0$

There is no asymptotic normality when $\sqrt{k}A(n/k) \to \infty$.

The asymptotic normality result follows relatively easily from the following weighted approximation to the tail quantile function valid under the second order condition (Drees 1998):

There exists a sequence of Brownian motions $\{W_n\}$ *such that for a suitable choice of functions* a_0 *and* A_0 *(as in (3)) and for each* $\epsilon > 0$

$$\sup_{0<s\leq 1} s^{\gamma+1/2+\epsilon}\left|\sqrt{k}\left(\frac{X_{n-[ks],n} - B_0\left(\frac{n}{k}\right)}{a_0\left(\frac{n}{k}\right)} - \frac{s^{-\gamma}-1}{\gamma}\right)\right.$$

$$\left. - s^{-\gamma-1}W_n(s) - \sqrt{k}A_0\left(\frac{n}{k}\right)\Psi_{\gamma,\rho}(s^{-1})\right| \xrightarrow{P} 0, \qquad (4)$$

where

$$B_0\left(\frac{n}{k}\right) := \begin{cases} U\left(\frac{n}{k}\right) & if \quad \gamma \geq -\frac{1}{2} \\ X_{n,n} + \frac{a_0\left(\frac{n}{k}\right)}{\gamma} & if \quad \gamma < -\frac{1}{2} \end{cases}$$

The term involving W_n accounts for the random limit and the term involving Ψ accounts for the bias.

New results

Let us now look at what happens if the speed of convergence in (2) is **faster than polynomial**, i.e., if

$$\lim_{t\to\infty} t^{\alpha}\left(\frac{U(tx)-U(t)}{a(t)} - \frac{x^{\gamma}-1}{\gamma}\right) = 0 \tag{5}$$

for all x and **for all** $\alpha > 0$.

Rewriting (5) in a most convenient way we obtain, for $\gamma \neq 0$ and for all x

$$U(tx) - U(t) = a(t)\,\frac{x^{\gamma}-1}{\gamma} + o(t^{-\alpha}) \tag{6}$$

for all $\alpha > 0$.

Fix $x, y > 0$. We have

$$U(txy) - U(tx) = a(tx)\,\frac{y^\gamma - 1}{\gamma} + o(t^{-\alpha})$$

$$U(txy) - U(t) = a(t)\,\frac{(xy)^\gamma - 1}{\gamma} + o(t^{-\alpha})$$

$$U(tx) - U(t) = a(t)\,\frac{x^\gamma - 1}{\gamma} + o(t^{-\alpha})$$

It follows that

$$a(tx)\,\frac{y^\gamma - 1}{\gamma} - a(t)\,\frac{(xy)^\gamma - 1}{\gamma} + a(t)\,\frac{x^\gamma - 1}{\gamma} = o(t^{-\alpha})$$

and hence $\qquad a(tx) - a(t)\,x^\gamma = o(t^{-\alpha}) \qquad$ for any $\alpha > 0$.

This can be written respectively as

$$\frac{a(tx) - a(t)}{\gamma} - a(t)\,\frac{x^{\gamma} - 1}{\gamma} = o(t^{-\alpha}) \tag{7}$$

and

$$(tx)^{-\gamma}a(tx) - t^{-\gamma}a(t) = o(t^{-(\alpha+\gamma)}) \tag{8}$$

Now look again at (6): $U(tx) - U(t) = a(t)\,\frac{x^{\gamma}-1}{\gamma} + o(t^{-\alpha})$

Combination with (7) yields

$$\left(U(tx) - \frac{a(tx)}{\gamma}\right) - \left(U(t) - \frac{a(t)}{\gamma}\right) = o(t^{-\alpha}) \tag{9}$$

for any $\alpha, x > 0$.

It is convenient to use the following result (Ash, Erdös and Rubel (1974)):

If $f(tx) - f(t) = o(t^{-\alpha})$, $t \to \infty$, for some $\alpha > 0$ and all $x > 0$, then

$$C := \lim_{t \to \infty} f(t)$$

exists (finite) and

$$C - f(t) = o(t^{-\alpha})$$

as $t \to \infty$.

If applied to (8): $(tx)^{-\gamma}a(tx) - t^{-\gamma}a(t) = o(t^{-(\alpha+\gamma)})$ we find

$$t^{-\gamma}a(t) \to c_0 \quad \text{and} \quad c_0 - t^{-\gamma}a(t) = o(t^{-(\alpha+\gamma)})$$

as $t \to \infty$, for any $\alpha > 0$.

In view of (2) the constant c_0 can not be zero and, in fact, has to be positive (since a is regularly varying with index γ). We find

$$a(t) = c_0 t^{\gamma} + o(t^{-\alpha}), \tag{10}$$

as $t \to \infty$, for any $\alpha > 0$.

Similarly when applying the result of Ash, Erdös and Rubel (1974) to (9): $\left(U(tx) - \frac{a(tx)}{\gamma}\right) - \left(U(t) - \frac{a(t)}{\gamma}\right) = o(t^{-\alpha})$ we obtain

$$U(t) - \frac{a(t)}{\gamma} \to d \quad \text{and} \quad U(t) - \frac{a(t)}{\gamma} - d = o(t^{-\alpha})$$

as $t \to \infty$, for any $\alpha > 0$. When combined with (10) it follows that

$$U(t) = c_1 + c_0 \frac{t^{\gamma} - 1}{\gamma} + o(t^{-\alpha}), \tag{11}$$

as $t \to \infty$.

We now get the same expansion of U for $\gamma = 0$.

We have, for $x > 0$,

$$U(tx) - U(t) - a(t) \log x = o(t^{-\alpha})$$

which implies for $x, y > 0$

$$U(txy) - U(tx) - U(ty) + U(t) = o(t^{-\alpha}).$$

Hence for each $y > 0$ the function $U(ty) - U(t)$ satisfies the conditions of the result of Ash, Erdös and Rubel (1974).

It follows that for some $c(y)$

$$U(ty) - U(t) - c(y) = o(t^{-\alpha}) \tag{12}$$

and so

$$\frac{U(ty) - U(t)}{a(t)} = \frac{c(y)}{a(t)} + o(t^{-\alpha}).$$

Since by (2) $\frac{U(ty)-U(t)}{a(t)}$ converges to $\log y$, it follows that $c(y)/a(t) \to \log y$. Hence $a(t) \to c_o$, $t \to \infty$, with $c_0 > 0$, and $c(y) = c_0 \log y$.

Then for $y > 0$, by (12)

$$U(ty) - U(t) - c_0 \log t = \{U(ty) - c_0 \log(ty)\} - \{U(t) - c_0 \log(t)\} = o(t^{-\alpha})$$

and so by a second application of the result of Ash, Erdös and Rubel (1974)

$$U(t) = c_1 + c_2 \log t + o(t^{-\alpha})$$

as $t \to \infty$, for any $\alpha > 0$.

Now the following Theorem can be proved:

Theorem 1. *Assume condition (5). There is a positive constant c_0 and a real constant c_1 such that for each $\epsilon > 0$ as $n \to \infty$*

$$\sup_{0<s\leq 1} s^{\gamma+1/2+\epsilon}\left|\sqrt{k}\left(\frac{X_{n-[ks],n} - B\left(\frac{n}{k}\right)}{c_0\left(\frac{n}{k}\right)^{\gamma}} - \frac{s^{-\gamma}-1}{\gamma}\right) - s^{-\gamma-1}W_n(s)\right| \xrightarrow{P} 0$$

where $\{W_n\}$ is a sequence of Brownian motions and

$$B\Big(\frac{n}{k}\Big) := \begin{cases} c_1 + c_0\frac{\left(\frac{n}{k}\right)^{\gamma}-1}{\gamma} & if \quad \gamma \geq -\frac{1}{2} \\ X_{n,n} + \frac{c_0\left(\frac{n}{k}\right)^{\gamma}}{\gamma} & if \quad \gamma < -\frac{1}{2} \end{cases}$$

provided there is a $\delta > 0$ such that $k = k(n) = o\Big(n^{1-\delta}\Big)$, $n \to \infty$.

Remark 1. It follows that under (5) any of the mentioned estimators is asymptotically normal without bias provided $k(n) = o\left(n^{1-\delta}\right)$, $n \to \infty$.

For the proof observe that

$$\left\{X_{n-[ks],n}\right\}_s \overset{d}{=} \left\{U\left(Y_{n-[ks],n}\right)\right\}_s.$$

where Y is standard Pareto, i.e. with d.f. $1 - 1/x$, $x \geq 1$.

By the expansion of U that we have obtained it follows that

$$U\left(Y_{n-[ks],n}\right) = c_1 + c_0 \, \frac{Y^{\gamma}_{n-[ks],n} - 1}{\gamma} + o_p\left(Y^{-\alpha}_{n-[ks],n}\right).$$

Hence

$$\frac{U\left(Y_{n-[ks],n}\right) - c_1}{c_0 \left(\frac{n}{k}\right)^{\gamma}} = \frac{Y^{\gamma}_{n-[ks],n} - 1}{\gamma \left(\frac{n}{k}\right)^{\gamma}} + \left(\frac{n}{k}\right)^{-\gamma} Y^{-\alpha}_{n-[ks],n} \, o_p(1) \quad (13)$$

The main idea now is to use the version of (4) that corresponds

to the random variable Y^{γ}.

For the error term we do the following: using a uniform bound for the order statistics from Shorack and Wellner (1986)

$$\max_{0 \le i \le n-1} \frac{n}{i\, Y_{n-i,n}} \xrightarrow{d} R\,,$$

with R a finite positive random variable, we show that

$$\left(\frac{n}{k}\right)^{-\gamma} Y_{n-[ks],n}^{-\alpha} = O_p\left(\left(\frac{n}{k}\right)^{-(\alpha+\gamma)} s^{\alpha}\right)$$

for all $\alpha > 0$.

Hence

$$\frac{X_{n-[ks],n} - c_1}{c_0 \left(\frac{n}{k}\right)^{\gamma}} \stackrel{d}{=} \frac{Y_{n-[ks],n}^{\gamma} - 1}{\gamma \left(\frac{n}{k}\right)^{\gamma}} + \left(\frac{n}{k}\right)^{-(\alpha+\gamma)} s^{\alpha}\, o_p(1) \tag{14}$$

for all $\alpha > 0$.

Example

Consider the mixture distribution $F = p\,F_1 + (1-p)\,F_2$, with $0 < p < 1$, where $F_1(t) = 1 - 1/t$, $t \geq 1$ and $F_2(t) = 1 - e^{-t}$, $t \geq 0$.

Then for $\alpha > 0$

$$\frac{1}{1 - F(t)} = t + o(t^{-\alpha}),\ t \to \infty,$$

hence for $\alpha > 0$

$$U(t) = t + o(t^{-\alpha}),\ t \to \infty,$$

and condition (11) holds.

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