LARGE DEVIATION ESTIMATES FOR CERTAIN HEAVY-TAILED DEPENDENT SEQUENCES ARISING IN RISK MANAGEMENT

by

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The classical Ruin Problem

Let $S_t =$ capital gain of an insurance co. by time t

$$\text{capital outflow (claims)} = \sum_{i=1}^{N(t)} Z_i$$

$$\text{capital inflow (premiums)} = ct$$

$$\longrightarrow S_t = -\sum_{i=1}^{N(t)} Z_i + ct$$

Find $\mathbf{P}\{S_t \textit{ ever} < -u\} = \mathbf{P}\{\text{ruin}\}$ (Lundberg, '03).

Theorem (Cramér, '30). *If $\{S_t\}$ has positive drift, "light–tailed" claim sizes, then*

$$\mathbf{P}\Big\{S_t < -u, \text{some } t\Big\} \sim Ce^{-Ru} \text{ as } u \to \infty.$$

Some extensions:

(i) "Heavy–tailed" claims:

$$\mathbf{P}\Big\{S_t < -u, \text{ some } t\Big\} \sim \tilde{C} \int_u^\infty \bar{F}_Z(s)ds.$$

(ii) Finite-time estimates for light tails (Arfwedsen'55):

$$\mathbf{P}\{\text{ruin } \underline{\text{before}} \text{ time } \tau u\} \sim \frac{D}{\sqrt{u}} e^{-uJ(\tau)}.$$

A modified ruin problem

Now consider discrete-time process,

$$S_n = \xi_1 + \cdots + \xi_n,$$

where $\mathbf{E}\xi_i > 0$, and assume:

I. Subexponential claims ("heavy-tails").

$$\Longrightarrow \mathbf{E}\left[e^{\epsilon \xi_i}\right] = \infty, \text{ all } \epsilon > 0.$$

II. Positive barrier for ruin.

Ruin occurs if $S_n > u$, some $n \leq \delta u$.

Thus, finite-time ruin est. (cf. Arfwedsen).

III. Markov dependence in *general* state space:

$$\xi_i = f(X_i),$$

where:

- $f(\cdot)$ is a random function,
- $\{X_i\} \subset \mathbb{S}$ is a general (e.g. infinite) state M.C.

Motivating examples

I. Operational risk losses.

(E.g., back office errors at a bank.)

- "Claims" arrive at a Poisson rate.
- Claim sizes are heavy-tailed.
- Frequency of claims depends on traded *volume* in the stock market.

For example, if

$X_i =$ traded volume at time i,

then could model $\{X_i\}$ as pos.-drift AR(1) pr. (say).

Losses at time i:

$$\xi_i = f(X_i) = \sum_{j=1}^{N(X_i)} Z_{i,j},$$

where, for each i, $\{Z_{i,j}\}_{j\geq 1}$ is i.i.d., heavy-tailed.

Study total loss by time n:

$$S_n = f(X_1) + \cdots + f(X_n).$$

Related work (Rogers-Zane '05):
$S_n =$ price increase in high-freq. financial market;
$N(X_i) =$ number of quotes (price changes) during interval i (where $N(\cdot)$ is Poisson, Markov-dep.).

II. Financial losses (GARCH(1,1) model).

Log. returns on a stock:

$$R_i = \sigma_i Z_i, \qquad \text{for } Z_i \sim N(0,1),$$

where

$$\sigma_i^2 = a_0 + b_1 \sigma_{i-1}^2 + a_1 R_{i-1}^2.$$

Motivation. Volatility shows:

Correlation with *absolute* log. returns
(and previous volatility);
Little correlation with *actual* log. returns (R_{i-1}).

Set: $\sigma_i^2 = X_i, \quad A_i = (b_1 + a_1 Z_{i-1}^2), \quad B_i = a_0.$

Then above model becomes:

$$(*) \qquad \boxed{X_i = A_i X_{i-1} + B_i,}$$

where $\{(A_i, B_i)\}$ is i.i.d., $\mathbf{E}[\log A_i] < 0$.

$(*)$ is called a "stochastic recurrence equation."

Note:

$\{X_i\}$ is a Markov chain on $\mathbb{R}$.

Consider:

$$\boxed{S_n = X_1 + \cdots + X_n.}$$

(cf. Mikosch–Konstantinides '05).

General Problem

Now suppose

$$S_n = f(X_1) + \cdots + f(X_n),$$

where:

- $f(\cdot)$ is a random function.
- $\{X_i\}$ is an underlying Markov chain (on $\mathbb{R}$, or $\mathbb{S}$).

Nummelin-Athreya-Ney regeneration method:

Assume $\{X_i\}$ satisfies:

Minorization.

$$\textbf{(M)}\ h(x)\nu(A) \le P^k(x, A) \equiv \mathbf{P}\left\{X_{n+k} \in A | X_n = x\right\}.$$

Then:

- $\tau_i \equiv T_i - T_{i-1}$ "inter-regen. times" exist, i.i.d.
- $U_i \equiv S_{T_{i+1}} - S_{T_i}$ i.i.d.
- Probab. law of S_{T_i} is $\nu(\cdot)$.

Results

Objective: Determine

$$\mathbf{P}\left\{S_n > u, \text{ some } n \le \delta u\right\},$$

where $S_n = f(X_1) + \cdots + f(X_n)$,

and $\mu \equiv \mathbf{E}_\pi\left[f(X)\right] > 0$ (positive drift).

Let $\boxed{U \stackrel{d}{=} S_{T_{i+1}} - S_{T_i}.}$

Assumptions:

(A1) U is *subexponential*.

(A2) $\mathbf{P}\left\{U^- < -u\right\} = o\left(\mathbf{P}\left\{U > u\right\}\right),\ u \to \infty.$

(A3) Markov chain is *geometrically recurrent*, i.e.,

$$\mathbf{E}\left[e^{\epsilon(T_{i+1}-T_i)}\right] < \infty, \text{ some } \epsilon > 0.$$

Thm. (C-H., '05). *Assume M.C. satisfies (M), and (A1)-(A3) hold. Then*

$$\mathbf{P}\left\{S_n > u, \text{ some } n \le \delta u\right\} \sim \frac{\delta u}{\mathbf{E}\tau} \cdot \mathbf{P}\left\{U > (1-\delta\mu)u\right\}.$$

Characterizing exceedence over regeneration cycle

Case 1: *Operational risk losses.*

For this case,

$$S_n = f(X_1) + \cdots + f(X_n),$$

where $\boxed{f(X_i) = \sum_{i=1}^{N(X_i)} Z_{i,j}.}$

Here, $N(x) \sim$ Poisson($\lambda(x)$).

Assumption:

(A4) $\Lambda(\alpha) < \infty$, some $\alpha > 0$,

where

$$\Lambda(\alpha) = \lim_{n\to\infty} \frac{1}{n} \log \mathbf{E}\left[e^{\alpha(\lambda(X_1)+\cdots+\lambda(X_n))}\right].$$

(Spectral radius, "Gärtner-Ellis limit.")

Means: the intensity process $\{\lambda(X_i)\}$ has light tails.

Proposition 1 (C-H.,'05). *Assume cond's. of prev. thm., and that (A4) holds. Then*

$$\mathbf{P}\{U > u\} \sim \mathbf{E}\tau\mathbf{E}_\pi[\lambda(X)]\,\bar{F}_Z(u) \text{ as } u \to \infty.$$

Equiv.: $\mathbf{P}\{U > u\} \sim \mathbf{E}\tau\mathbf{P}_\pi\{f(X) > u\}$.

Case 2: *Stochastic recurrence eqn's.*

Here,

$$X_i = A_i X_{i-1} + B_i,$$

$$\text{and } S_n = X_1 + \cdots + X_n.$$

Suppose: $\mathbf{E}[\log A_i] < 1$ ($A_i < 1$ "on average").
Define:

$$\Lambda_A(\alpha) = \log \mathbf{E}\left[e^{\alpha \log A_i}\right],$$

(c.g.f. of $\log A$); and let $\Lambda_B(\cdot) =$ c.g.f. of $\log B$.

Assumptions:

(A5) $\Lambda_A(\kappa) = 0$ some $\kappa > 0$.
(A6) $\Lambda_A(\alpha)$, $\Lambda_B(\alpha)$ finite for $\alpha \in \mathfrak{N}(\kappa)$.

Proposition 2 (C-H.,'05). *Assume (A5) and (A6). Then*

$$\mathbf{P}\{U > u\} \sim Cu^{-\kappa} \text{ as } u \to \infty.$$

(Build-up of $\log A_i$'s over long interval of length $= \rho \cdot \log u$.)

Summarizing:

$$\boxed{\mathbf{P}\Big\{S_n > u,\ \text{some } n \le \delta u\Big\} \sim Cu\mathbf{P}_\pi\Big\{f(X) > (1-\delta\mu)u\Big\};}$$

but C (and its derivation) is different in the two separate cases.

Related extension (cf. Mikosch-Konstantinides '05): In GARCH(1,1) case, but with neg. drift, consider

$$\mathbf{P}\{S_n > u,\ \text{some } n\} = \mathbf{P}\{\text{ruin}\}.$$

Then a simple application of Prop. 2 yields

$$\mathbf{P}\{\text{ruin}\} \sim Du^{-(\kappa-1)}, \quad u \to \infty.$$

Reference:

COLLAMORE, J. F. and HÖING, A. (2005). Small-time ruin for a financial process modulated by a Harris recurrent Markov chain. Submitted. (*Available from* http://www.math.ku.dk/~collamore/)