On a storage process for fluid networks with multiple Lévy input

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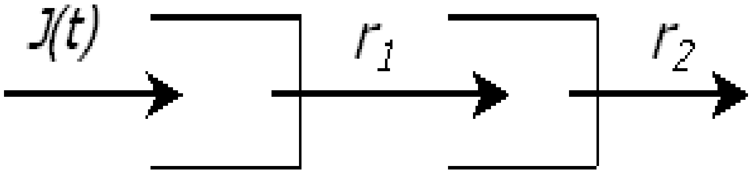
Outline of the talk

- *Two-node tandem network (Mandjes, van Uitert, K.D.)*
  - *New representation for* Q_2
  - *Lévy case: distribution of* Q_2
  - *Examples*
- *n-node tandem network (Dieker, Rolski, K.D.)*
  - *Skorokhod problem*
  - *Stationary representation*
  - *Laplace transform*

Two-node tandem network



- $r_1 > r_2 > 0$
- $J(t)$ - *with stationary increments,* $\mathbb{E}J(1) < r_2$

We are interested in

$$\mathbb{P}(Q_2 > u)$$

- Q_2 - stationary buffer content at the second node

- Kella, Whitt, Rubin, Shalmon, Mandjes, van Uitert,...

Representation for Q_2

Following Reich's representation we have

$$Q_1 =_{\mathrm{d}} \sup_{t \geq 0} \{J(t) - r_1 t\}$$

and

$$Q_{\mathrm{total}} =_{\mathrm{d}} \sup_{t \geq 0} \{J(t) - r_2 t\}.$$

Hence

$$Q_2 =_{\mathrm{d}} \sup_{t \geq 0} \{J(t) - r_2 t\} - \sup_{t \geq 0} \{J(t) - r_1 t\}.$$

New representation for Q_2

Let

$$t_u = \frac{u}{r_1 - r_2}.$$

Theorem 1. *For each* $u \geq 0$,

$$\mathbb{P}(Q_2 > u) = \mathbb{P}\left(\sup_{t \in [t_u, \infty)} \{J(t) - r_2 t\} - \sup_{t \in [0, t_u]} \{J(t) - r_1 t\} > u \right).$$

This representation enables us to analyze the distribution of Q_2 for following classes of input processes:

- processes with independent increments
- Gaussian processes

Input with stationary independent increments

Theorem 2. *Let* $\{J(t), t \in \mathbb{R}\}$ *be a stochastic process with stationary independent increments and let* $\mu = \mathbb{E}J(1) < r_2$. *Then for each* $u \geq 0$, *and* $J_1(\cdot)$ *and* $J_2(\cdot)$ *independent copies of the process* $J(\cdot)$,

$$\mathbb{P}(Q_2 > u) = \mathbb{P}\left(\sup_{t\in[0,\infty)}\{J_1(t) - r_2 t\} > \sup_{t\in[0,t_u]}\{-J_2(t) + r_1 t\}\right).$$

Input with spectrally positive Lévy process

Let $J(t)$ be a spectrally positive Lévy process. Introduce

$$\theta(s) := \log(\mathbb{E}e^{-s(J(1)-r_1)}).$$

Theorem 3. *Let $\{J(t), t \in \mathbb{R}\}$ be a spectrally positive Lévy process with $\mu := \mathbb{E}J(1) < r_2$. Then, for each $x > 0$,*

$$\mathbb{E}e^{-xQ_2} = \frac{r_2 - \mu}{r_1 - r_2} \cdot \frac{\theta^{-1}(x(r_1 - r_2))}{x - \theta^{-1}(x(r_1 - r_2))}.$$

Remark 1. Theorem 3 can be considered as an analogue of the result of Zolotarev who obtained the Laplace transform of $\mathbb{P}(Q_1 < u)$ for $J(\cdot)$ being a spectrally positive Lévy process.

Pollaczek-Khintchine representation

Theorem 4. *Let $\{J(t), t \in \mathbb{R}\}$ be a spectrally positive Lévy process with $\mu := \mathbb{E}J(1) < r_2$. Then*

$$\mathbb{P}(Q_2 \leq u) = (1 - \varrho) \sum_{i=1}^{\infty} \varrho^{i-1} H^{\star i}(u),$$

where

•$\varrho := (r_1 - r_2)/(r_1 - \mu)$

• *$H(\cdot)$ is a distribution function such that $H(x) = 0$ for $x < 0$*

and

$$\int_0^\infty e^{-xv} \mathrm{d}H(v) = \frac{\theta^{-1}(x)}{\varrho x}$$

for $x \geq 0$.

Examples: exact distributions

- $J(t)$ is a standard Brownian motion.

$$\mathbb{P}(Q_2 > u) = \frac{r_1 - 2r_2}{r_1 - r_2} e^{-2r_2 u} \left(1 - \Psi\left(\frac{r_1 - 2r_2}{\sqrt{r_1 - r_2}}\sqrt{u}\right)\right) + \frac{r_1}{r_1 - r_2}\Psi\left(\frac{r_1}{\sqrt{r_1 - r_2}}\sqrt{u}\right),$$

where $\Psi(x) = \mathbb{P}(N > x)$.

✓ If $c_1 > 2c_2$, then

$$\mathbb{P}(Q_2 > u) \sim \frac{r_1 - 2r_2}{r_1 - r_2} e^{-2r_2 u}.$$

✓ If $c_1 \leq 2c_2$, then

$$\mathbb{P}(Q_2 > u) \sim \frac{1}{\sqrt{2\pi(r_1 - r_2)}} \frac{1}{\sqrt{u}} \exp\left(-\frac{r_1^2}{2(r_1 - r_2)} u\right).$$

Also we can get the exact distribution if

- $J(t)$ is a Poisson process.

Examples: asymptotic results

- $J(t) = X_{\alpha,1,\beta}(t)$ with $\alpha \in (1,2)$ and $\beta \in (-1,1]$.

  Then

  $$C_1 u^{1-\alpha} \leq \mathbb{P}(Q_2 > u) \leq C_2 u^{1-\alpha} \quad \text{as } u \to \infty.$$

- $J(t) = X_{\alpha,1,1}(\cdot)$ with $\alpha \in (1,2)$.

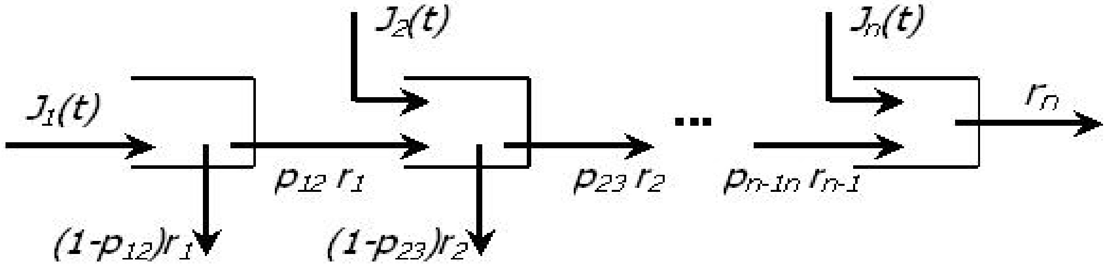
  Then

  $$\mathbb{P}(Q_2 > u) \sim$$
  $$\sim \frac{1}{\Gamma(2-\alpha)\cos(\pi(\alpha-2)/2)} \frac{1}{r_2} \left(\frac{r_1}{r_1 - r_2}\right)^{1-\alpha} u^{1-\alpha}.$$

Also we can get the asymptotic for

- $J(t)$ compound Poisson input, with regularly varying jumps

n-node tandem network



- $J(t) = (J_1(t), ..., J_n(t))'$ - n-*dimensional Lévy process with mutually independent components and* $J_1(t)$ *is a spectrally positive Lévy process,* $J_2(t), \dots, J_n(t)$ *are subordinators*
- $r = (r_1, ..., r_n)'$ - *output rates*
- $P = (p_{ij})_{i,j=1}^n$ - *routing matrix;*
  $0 < p_{ii+1} \leq 1$ *and* $p_{ij} = 0$ *if* $j \neq i + 1$

Moreover, we tacitly assume that

N1 (Work-conserving) $\qquad p_{ii+1} > \frac{r_{i+1}}{r_i}$,

N2 (Stability) $\qquad (I - P')^{-1}\mathbb{E}J(1) < r$.

n-node tandem network, ctd.

We are interested in the transient joint distribution of

- $Q(t) = (Q_1(t), ..., Q_n(t))'$ - storage process
- $B(t) = (B_1(t), ..., B_n(t))'$ - *running busy period* process,

where

$$B_i(t) = t - \sup\{0 \leq s \leq t : Q_i(s) = 0\}.$$

$Q(t)$ is the unique solution of the Skorokhod problem of $J(t) - (I - P')rt$ with reflection matrix $I - P'$, that is

S1 $Q(t) = w + J(t) - (I - P')rt + (I - P')L(t), t \geq 0$,

S2 $Q(t) \geq 0, t \geq 0$ and $Q(0) = w$,

S3 $L(0) = 0$ and L is nondecreasing, and

S4 $\sum_{i=1}^{n} \int_0^\infty Q_i(t)\, dL_i(t) = 0$.

n-node tandem network, ctd.

Proposition 1. *Suppose that $Q(t)$ is the storage process associated to the stochastic network (J, r, P). Then*

$$(I - P')^{-1} Q(t)$$

is the solution to the Skorokhod problem of

$$X(t) = (I - P')^{-1} J(t) - rt$$

with reflection matrix I.

Theorem 5. *The stationary distribution$(W(\infty), B(\infty))$ is the same as the distribution of $((I - P')\overline{X}, G)$, where $\overline{X} = (\overline{X_1}, ..., \overline{X_n})'$ and $G = (G_1, ..., G_n)'$ with*

$$\begin{aligned} \overline{X_i} &= \sup_{t \geq 0} \left(\sum_{k=1}^{i} \left(\prod_{j=1}^{k-1} p_{jj+1} \right) J_k(t) - r_i t \right) \\ G_k &= \sup\{t \geq 0 : X_k(t) = \overline{X}_k(t)\}. \end{aligned}$$

n-node tandem network, ctd.

Theorem 6. *Consider a tandem stochastic network* (J, r, P) *that* **N1-N2** *hold. Then for* $\alpha = (\alpha_1, ..., \alpha_n)' > 0$, $\beta = (\beta_1, ..., \beta_n)' > 0$

$$\mathbb{E}e^{-<\alpha,Q(\infty)>-<\beta,B(\infty)>} =$$
$$= \mathbb{E}e^{-\alpha_n \overline{X}_n - \beta_n G_n} \times$$
$$\times \prod_{j=1}^{n-1} \frac{\mathbb{E}e^{-\alpha_j \overline{X}_j - \left[\sum_{\ell=j+1}^{n} \Psi_\ell^J(\alpha_\ell) + \sum_{\ell=j+1}^{n}(p_{\ell-1\ell} r_{\ell-1} - r_\ell)\alpha_\ell + \sum_{p=j}^{n} \beta_p\right] G_j}}{\mathbb{E}e^{-p_{jj+1}\alpha_{j+1}\overline{X}_j - \left[\sum_{\ell=j+1}^{n} \Psi_\ell^J(\alpha_\ell) + \sum_{\ell=j+1}^{n}(p_{\ell-1\ell} r_{\ell-1} - r_\ell)\alpha_\ell + \sum_{p=j+1}^{n} \beta_p\right] G_j}},$$

where

$$X(t) = (I - P')^{-1} J(t) - rt$$
$$\Psi_i^J(\lambda) = -\log\left(\mathbb{E}e^{-\lambda J_i(1)}\right).$$

The formula can be made explicit by the use of fluctuation identities. But is a bit long...