Anomalous Diffusion Index for Lévy Motions

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Diffusion Process : Infinitesimal Coefficients

$$\lim_{h\downarrow 0}\frac{1}{h}P\left(|X(t+h)-X(t)|>\epsilon\,|X(t)=x\right)=0$$

$$\lim_{h\downarrow 0}\frac{1}{h}E\left\{X(t+h)-X(t)\,|X(t)=x\right\}=\mu_X(x,t)$$

$$\lim_{h\downarrow 0}\frac{1}{h}E\left\{(X(t+h)-X(t))^2\,|X(t)=x\right\}=\sigma_X^2(x,t)$$

$\mu(\cdot)=0, \sigma^2(\cdot)=1$: Brownian Motion

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Scaled Random Walks

$$S_n = \xi_1 + \xi_2 + \ldots + \xi_n$$

$$E(\xi_j) = 0, E(\xi_j^2) = \sigma^2 > 0$$

$$X^{(n)}(t) = \frac{1}{\sigma\sqrt{n}}\{S_{[nt]} + (nt - [nt])\xi_{[nt]+1}\} \to B(t)$$

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Critical Plenomena / Lévy Flights

Anomalous diffusive regime : $X(t)$ non-Gaussian

$n^{-1}E(S_n^2) \to \infty$

Lévy Flights : Muralidhar et al. (1990), Metzler and Klafte (2000)

Ferrari et al. (2001), Costa et al. (2003) and Morgado et al. (2004),..

$$D_X = \lim_{t \to \infty} \frac{E(X^2(t))}{2t} \quad \text{(Diffusion Constant)}$$

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$D_X = \infty \quad \Rightarrow \quad$ superdiffusion

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Centered process : $P(X(t) \leq x) = P(X(t) \geq -x)$

$$\gamma_X^0 = \sup\{\gamma : \gamma > 0,\ \liminf_{t\to\infty} \frac{E|X(t)|^{1/\gamma}}{t} > 0\}$$

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Examples

- $X(t)$ zero-mean and finite moments :

$$\gamma_X = D_X = 0 \text{ (subdiffusion)}$$

- $B_\sigma(t)$ Brownian motion with variance $\sigma^2/2$:

$$\sigma^2_{B_\sigma}(x,t) = \sigma^2 \quad ; D_{B_\sigma} = \frac{\sigma^2}{2} \quad ; \gamma_{B_\sigma} = \frac{1}{2}$$

- Langevin Equation

$$dV(t) = -\mu V(t)dt + \sigma dB(t) \quad , V(0) = 0$$

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Examples

Ornstein-Uhlenbeck : $V(t) = \sigma \int_0^t e^{-\mu(t-x)} dB(s)$

$\mu_V(x,t) = -\mu x$; $\sigma_V^2(x,t) = \sigma^2$ (infinitesimal coefficients/local diffusion)

$$E|V(t)|^{1/\gamma} = e^{-\mu t/\gamma}[\frac{\sigma^2}{2\mu}(e^{2\mu t} - 1)]^{1/2\gamma} E|B(1)|^{1/\gamma}$$

$$\lim_{t\to\infty} \frac{E|V(t)|^{1/\gamma}}{t} = 0, \ \forall \gamma > 0$$

$\Rightarrow \quad \gamma_V = 0$ (subdiffusion : long-range behavior)

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Examples

- $B^H(t)$: fractional Brownian motion, $0 < H < 1$
  $\mathrm{cov}(B^H(t), B^H(s)) = \frac{\sigma^2}{2}[t^{2H} + s^{2H} - |t-s|^{2H}]$, $\sigma^2 = \mathrm{var}(B^H(1))$

Examples

$$D_{B^H} = \lim_{t\to\infty} \frac{t^{2H}\sigma^2}{2t} = \begin{cases} 0 & H < \frac{1}{2} \\ \frac{\sigma^2}{2} & H = \frac{1}{2} \\ \infty & H > \frac{1}{2}. \end{cases}$$

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Results :

Lemma 1

Let $X(\cdot)$ be a zero-mean stochastic process with stationary and independent increments. Then, if $0 < \sigma_X^2(x,t) < \infty$ we have $\gamma_X = 1/2$.

Lemma 2

If the diffusion constant $0 < D_X < \infty$ then $\gamma_X = 1/2$.

Lemma 3

Let $X(\cdot)$ and $Y(\cdot)$ be centered stochastic processes. If $a \neq 0$ and $b > 0$ are constants, then $\gamma_{|X|} = \gamma_X$, $\gamma_{aX} = \gamma_X$, $\gamma_{|X|^b} = b\gamma_X$ and $\gamma_{X+Y} \leq \max\{\gamma_X, \gamma_Y\}$.

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- $L_{\alpha,\sigma}(t)$: symmetric Lévy stable process

Lemma 4

For the Lévy stable process $L = L_{\alpha,\sigma}$,

- *if $\alpha = 2$, then $\sigma_L^2(x,t) = \sigma^2$, $D_L = \dfrac{\sigma^2}{2}$ and $\gamma_L = \dfrac{1}{2}$.*
- *If $0 < \alpha < 2$ (superdiffusion), we have $\sigma_L^2(x,t) = D_L = \infty$ and $\gamma_L = \dfrac{1}{\alpha} > \dfrac{1}{2}$.*
- *Moreover, for $\varphi_t(x) = P(L(t) > x)$, we have $\varphi_t(\cdot)$ is slowly varying if $\alpha = 2$ and $\varphi_t(\cdot)$ is $(-\dfrac{1}{\gamma_L})$ regularly varying if $0 < \alpha < 2$*

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Numerical Illustrations

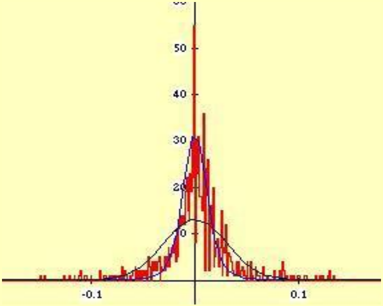
Daily Foreign Exchange Rates (http://www.federalreserve.gov)
January/2000 to January/2005 : Brazil, Switzerland, Mexico

- $\gamma_L = \dfrac{1}{\alpha}$

Table1

Country	$\hat{\alpha}_n$	$\hat{\sigma}_n$	value(k)	value(m)	value(ϵ)
Brazil(Real)	1.62424198	0.008866886626	180	15	0.005
Switzerland(Franc)	1.4737571	.49918254e-2	180	6	0.002
Mexico(Peso)	1.2723789	0.20784845e-1	200	5	0.01

Brazil : real vs US dollar



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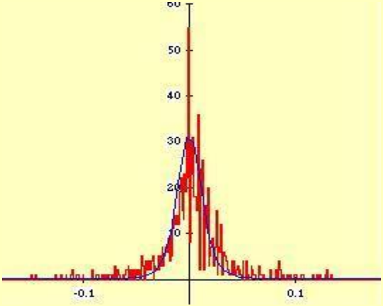
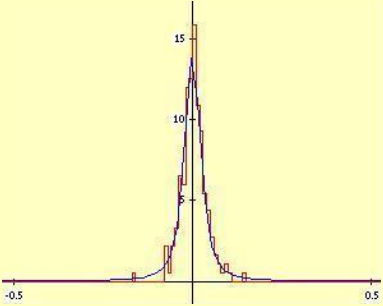
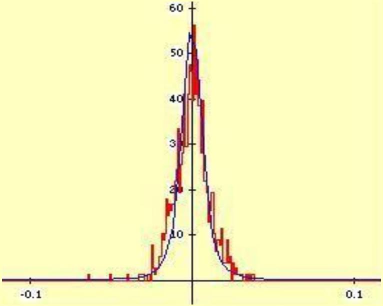


Mexico : peso vs US dollar



Switzerland : swiss franc vs US dollar



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