Extremes of supOU processes

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Overview

- Introduction
  - i. d. i. s. r.m
  - SupOU process
  - Class of convolution equivalent tails
  - Model assumptions of this talk
- Extremal behavior
- Conclusion

Definition

A stochastic process $\boldsymbol{\Lambda = \{\Lambda(A) : A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})\}}$ is called an **i. d. i. s. r. m. (infinitely divisible independently scattered random measure)** on $\mathbb{R}_+ \times \mathbb{R}$, if for disjoint sets $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$,

- $\Lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \Lambda(A_n)$ a. s. **(r. m.)**
- $(\Lambda(A_n))_{n\in\mathbb{N}}$ is an independent sequence **(i. s.)**
- $\Lambda(A)$ is infinitely divisible for every $A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ **(i. d.)**

i. d. i. s. r. m

We consider only i. d. i. s. r. m. with characteristic function

$$\mathbb{E}[\exp(iu\Lambda(A))] = \exp(\Pi(A)\psi(u))$$

for $u \in \mathbb{R}$, $A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_+)$, where

- ψ is the **cumulant generating function** of a Lévy process with generating triplet (m, σ^2, ν)

$$\mathbb{E}[\exp(iuL(t))] = \exp(t\psi(u))$$

- $\Pi(d\omega) = \pi(dr) \times \lambda(dt)$ for $\omega = (r,t) \in \mathbb{R}_+ \times \mathbb{R}$, where λ is the **Lebesgue measure** and π is a **probability measure** on $\mathbb{R}_+$

(m, σ^2, ν, π) are called the **generating quadruple** of Λ

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Λ is called **Lévy random field**

Compound Poisson random measure

Let ν be finite. Then

$$\boldsymbol{N = \sum_{k=1}^{\infty} \varepsilon_{(R_k, \Gamma_k, Z_k)}}$$

where

- (R_k) i. i. d. with d. f. π
- (Γ_k) jump times of a Poisson process with intensity $\mu = \nu(\mathbb{R})$
- (Z_k) i. i. d. with d. f. ν/μ

is a **Poisson random measure** with intensity $\pi(dr) \times dt \times \nu(dx)$

$$\boldsymbol{\Lambda(A) = \int_{\mathbb{R}} x \, dN(A, x) = \sum_{k=1}^{\infty} Z_k \, 1_{\{(R_k, \Gamma_k) \in A\}}}$$

is a **compound Poisson random measure**

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$$\Lambda(\mathbb{R}_+ \times [0, t]) = \sum_{k=1}^{\infty} Z_k \, 1_{\{(R_k, \Gamma_k) \in \mathbb{R}_+ \times [0,t]\}} = \sum_{k=1}^{N(t)} Z_k$$

Underlying Lévy process

Let Λ be an i. d. i. s. r. m. We denote by $\boldsymbol{L = (L(t))_{t\in\mathbb{R}}}$ the **underlying driving Lévy process** with

$$\boldsymbol{L(t) = \Lambda(\mathbb{R}_+ \times [0, t])}$$

L has the characteristic triplet (m, σ^2, ν)

supOU process

Definition

The **supOU process** (superposition of Ornstein-Uhlenbeck processes) Y is defined by

$$Y(t) = \int_{\mathbb{R}_+ \times \mathbb{R}} e^{-r(t-s)} 1_{[0,\infty)}(t-s)\, d\Lambda(r,s)$$

where

- $\int_{|x|\geq 2} \log|x|\, \nu(dx) < \infty$
- $\lambda^{-1} = \int_{\mathbb{R}_+} r^{-1}\, \pi(dr) < \infty$

Special cases

- Λ a compound Poisson random measure $(\Lambda(A) = \sum_{k=1}^{\infty} Z_k \, \mathbf{1}_{\{(R_k,\Gamma_k)\in A\}})$:

$$Y(t) = \int_{\mathbb{R}_+\times\mathbb{R}} \mathrm{e}^{-r(t-s)} \, 1_{[0,\infty)}(t-s) \, d\Lambda(r,s) = \sum_{k=-\infty}^{N(t)} \mathrm{e}^{-R_k(t-\Gamma_k)} Z_k$$

- **OU-process**: $\pi(\lambda) = 1$: $Y(t) = \int_{-\infty}^{t} \mathrm{e}^{-\lambda(t-s)} \, dL(s)$
- π discrete with $\pi(\lambda_k) = p_k$ and $\sum_{k=1}^{\infty} p_k = 1$. Then

$$Y(t) = \sum_{k=1}^{\infty} \int_{-\infty}^{t} \mathrm{e}^{-\lambda_k(t-s)} \, dL_k(s)$$

where (L_k) are independent Lévy processes with characteristic triplet $(p_k m, p_k \sigma^2, p_k \nu)$

Properties of a supOU process

- (m, σ^2, ν) determines the **marginal distribution**:

$$m_Y = \frac{1}{\lambda}\left[m + \int_{|y|>1} \frac{y}{|y|}\,\nu(dy)\right] \qquad \sigma_Y^2 = \frac{\sigma^2}{2\lambda}$$

$$\nu_Y\,[x,\infty) = \frac{1}{\lambda}\int_x^\infty \frac{\nu\,[y,\infty)}{y}\,dy$$

- π determines the **correlation function** ρ:

$$\rho(h) = \lambda \int_0^\infty r^{-1} \mathrm{e}^{-hr}\,\pi(dr)$$

e. g. $\pi(dr) = \Gamma(2H+1)^{-1} r^{2H} \mathrm{e}^{-r}\,dr$ for $r > 0$, $H > 0$, then

$$\rho(h) = (h+1)^{-2H} \quad \text{for } h \geq 0$$

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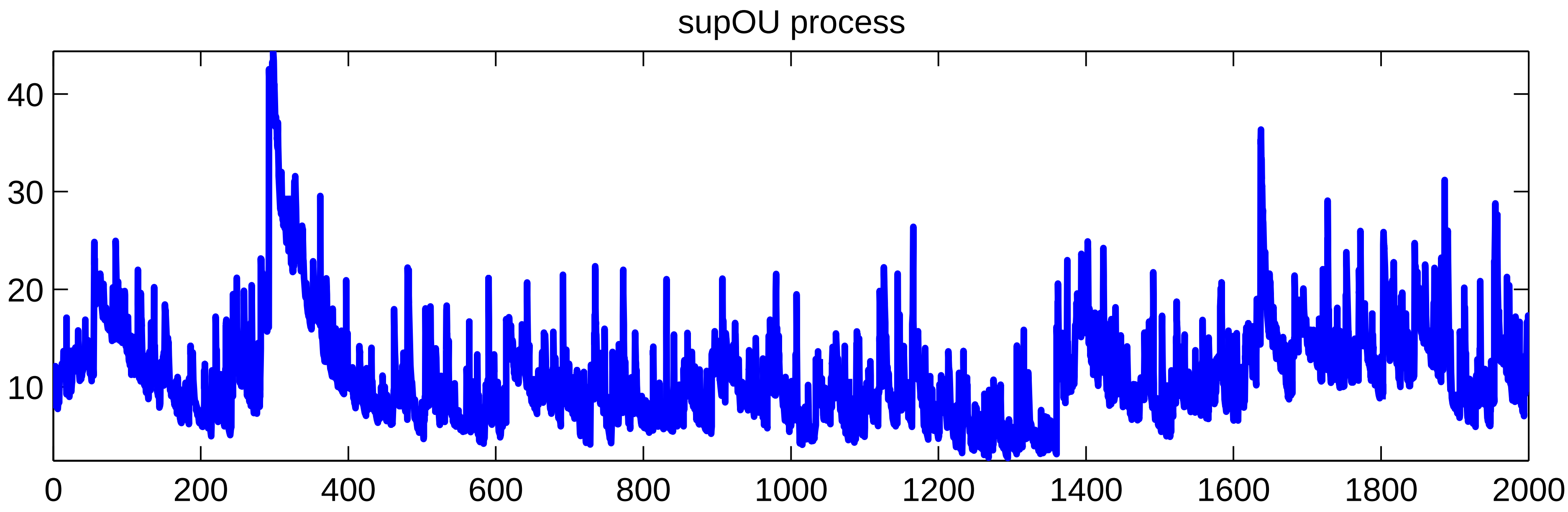
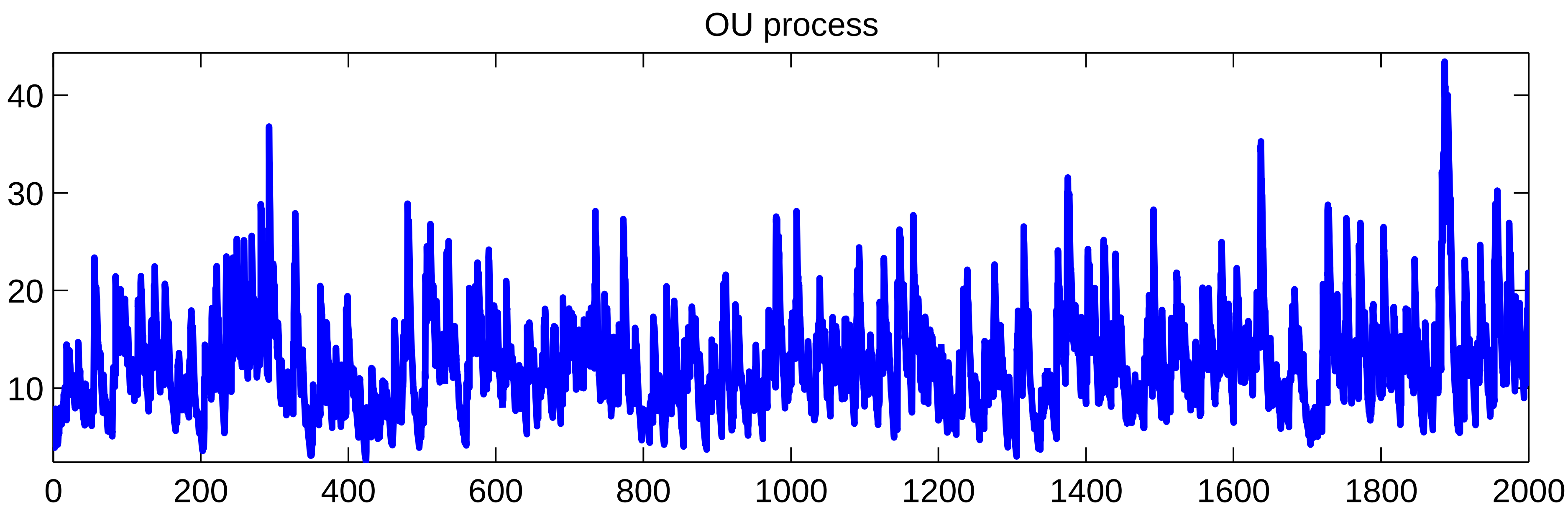
Reference: Barndorff-Nielsen (2001), Barndorff-Nielsen and Shephard (2001)

Examples


supOU process
40
30
20
10
0
200
400
600
800
1000
1200
1400
1600
1800
2000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Class of convolution equivalent tails $\mathcal{S}(\gamma)$

Let F be a d. f. on $\mathbb{R}$ with $F(x) < 1$ for every $x \in \mathbb{R}$. F belongs to the class $\mathcal{S}(\gamma)$, $\gamma \geq 0$, if

(i) F belongs to the class $\mathcal{L}(\gamma)$, $\gamma \geq 0$, i. e. for all $y \in \mathbb{R}$ locally uniformly

$$\lim_{x\to\infty} \overline{F}(x+y)/\overline{F}(x) = \mathrm{e}^{-\gamma y}$$

(ii)
$$\lim_{x\to\infty} \overline{F^{2*}}(x)/\overline{F}(x) = 2\int_{\mathbb{R}} \mathrm{e}^{\gamma x}\, dF(x) < \infty$$

The class $\gamma = 0$ is called **subexponential** d. f. s denoted by $\mathcal{S}$

Examples:

- $\gamma = 0$: stable-, Weibull-, loggamma-, Pareto distribution
- $\gamma > 0$: generalized inverse Gaussian distribution

Properties of $\mathcal{S}(\gamma)$

Let F be infinitely divisible with Lévy measure ν and $\gamma \geq 0$. Then

$$\boldsymbol{F} \in \mathcal{S}(\gamma) \quad \Leftrightarrow \quad \frac{\nu\,(1,\cdot]}{\nu(1,\infty)} \in \mathcal{S}(\gamma)$$

$$\Leftrightarrow \quad \lim_{x\to\infty} \frac{\overline{F}(x)}{\nu(x,\infty)} = \int_{\mathbb{R}} \mathrm{e}^{\gamma x}\, F(dx)$$

Model

In this talk we restrict our attention to a supOU process driven by a **positive compound Poisson random measure**, i. e. Z_k is positive and $\nu(\mathbb{R}) < \infty$.

$$Y(t) = \sum_{k=-\infty}^{N(t)} \mathrm{e}^{-R_k(t-\Gamma_k)} Z_k$$

a) $L(1) \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$:

$$\lim_{n\to\infty} n\mathbb{P}(L(1) > a_n x + b_n) = \mathrm{e}^{-x}$$

b) $L(1) \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Phi_\alpha) = \mathcal{R}_\alpha$:

$$\lim_{n\to\infty} n\mathbb{P}(L(1) > a_n x) = x^{-\alpha}$$

Overview

- Introduction
- Extremal behavior
  - Tail behavior of Y
  - Tail behavior of $M(h)$
  - Point process behavior
  - Running maxima
- Conclusion

Representation

If $L(1) \in \mathcal{L}(\gamma)$ then

$$\mathbb{P}(L(1) > x) = c(x) \exp\left[-\int_0^x \frac{1}{a(y)}\, dy\right], \quad x > 0,$$

where

$$a, c : \mathbb{R}_+ \to \mathbb{R}_+$$

$$\lim_{x\to\infty} c(x) = c > 0$$

a is absolutely continuous

$$\lim_{x\to\infty} a(x) = \frac{1}{\gamma}$$

$$\lim_{x\to\infty} a'(x) = 0$$

Tail behavior of $Y(t)$

$$\lim_{x\to\infty} a(x)/x = 0 \qquad\qquad \lim_{x\to\infty} a(x) = 1/\gamma$$

a) $L(1) \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$:

$$\mathbb{P}(Y(t) > x) \sim \frac{1}{\lambda} \frac{a(x)}{x} \frac{\mathbb{E}e^{\gamma Y(t)}}{\mathbb{E}e^{\gamma L(1)}} \mathbb{P}(L(1) > x) \quad \text{for } x \to \infty$$

b) $L(1) \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Phi_\alpha)$:

$$\mathbb{P}(Y(t) > x) \sim \frac{1}{\lambda\alpha} \mathbb{P}(L(1) > x) \quad \text{for } x \to \infty$$

Tail behavior of $M(h)$

Let $h > 0$ and $M(h) = \sup_{0 \le t \le h} Y(t)$

a) $L(1) \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$:

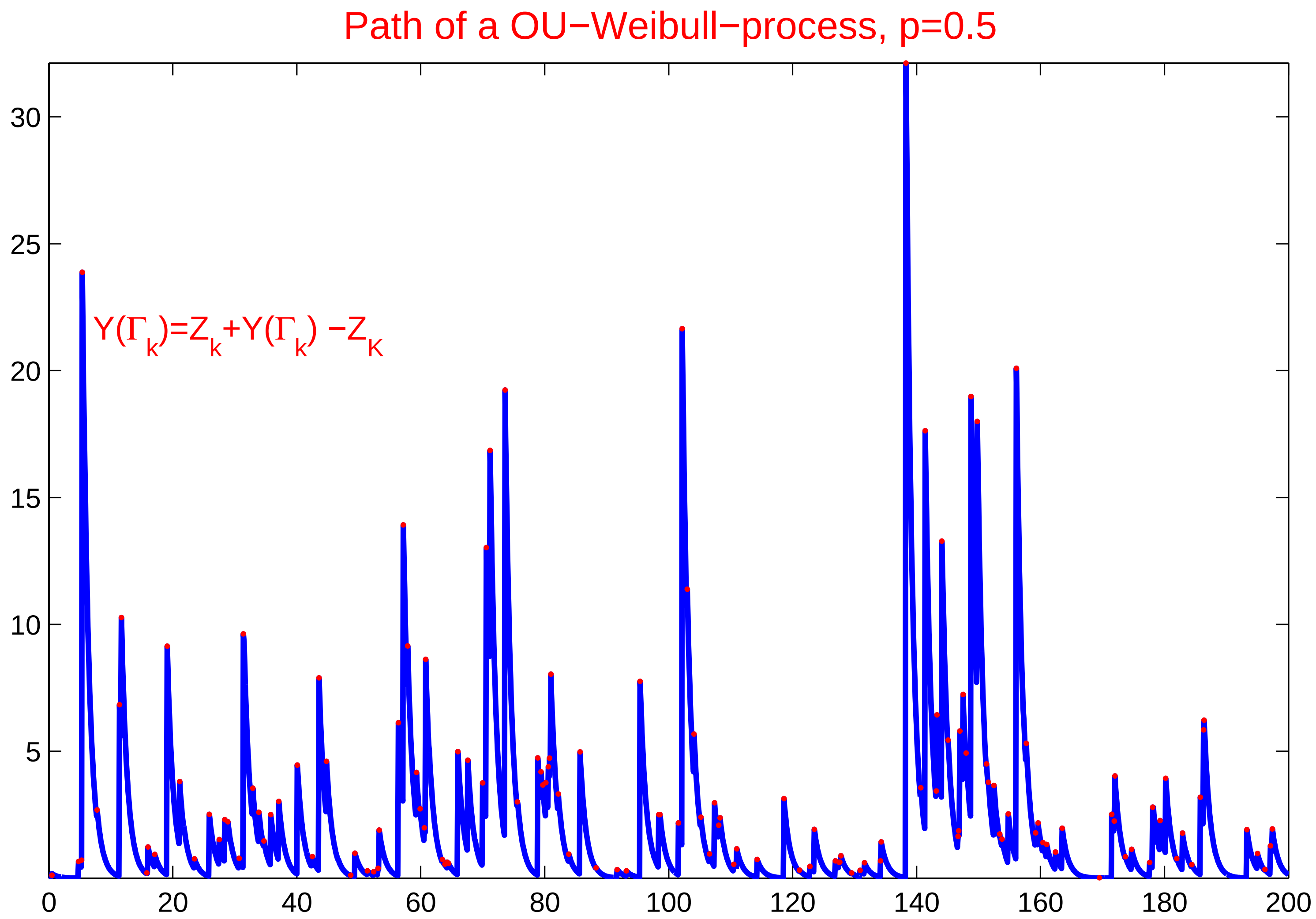
$$\mathbb{P}(M(h) > x) \sim h \frac{\mathbb{E}e^{\gamma Y(t)}}{\mathbb{E}e^{\gamma L(1)}} \mathbb{P}(L(1) > x) \quad \text{for } x \to \infty$$

b) $L(1) \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Phi_\alpha)$:

$$\mathbb{P}(M(h) > x) \sim [h + (\lambda\alpha)^{-1}] \mathbb{P}(L(1) > x) \quad \text{for } x \to \infty$$

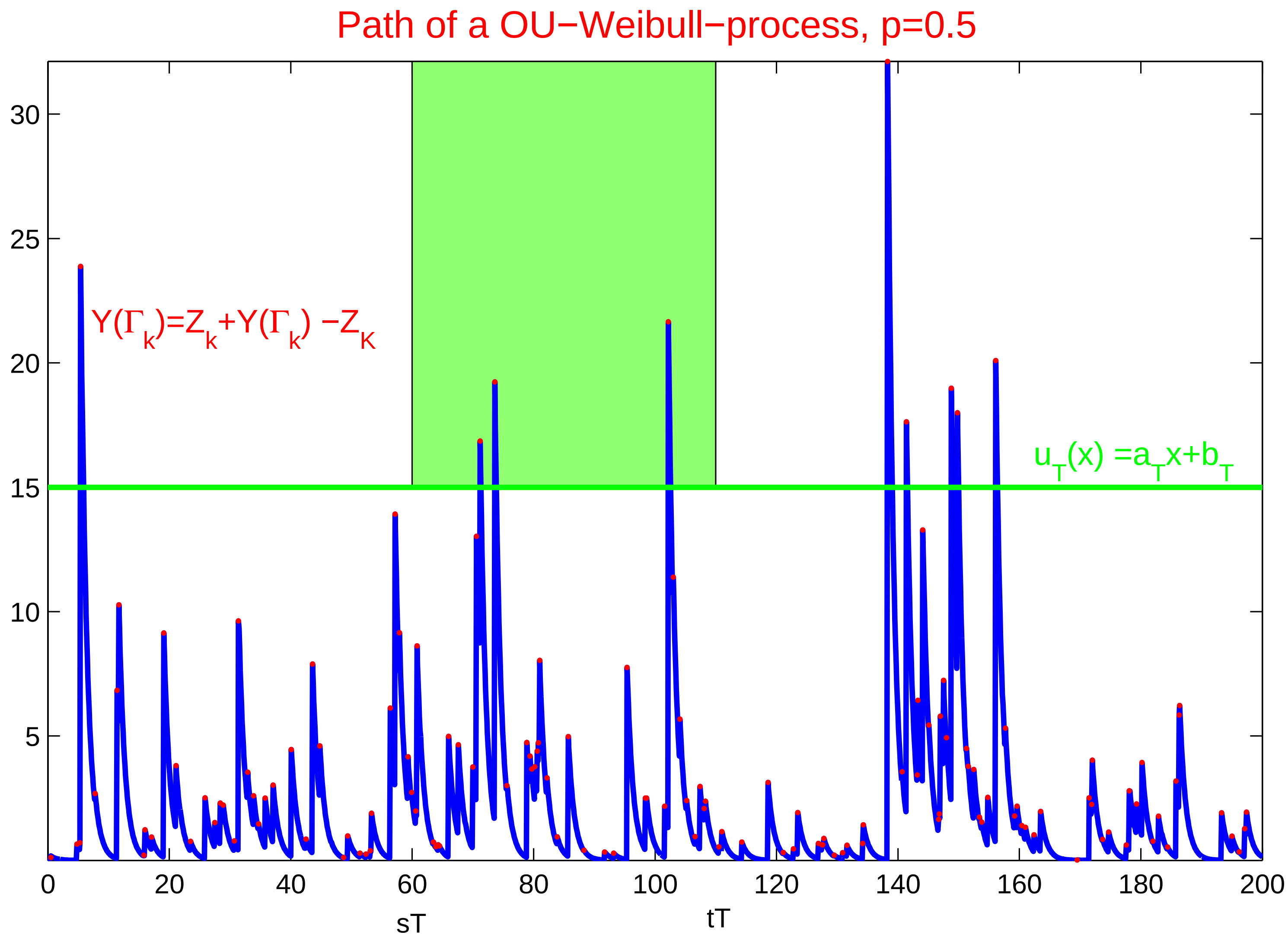
Example: OU-Weibull Process

$$L(t) = \sum_{k=1}^{N(t)} Z_k, \qquad Y(t) = \sum_{k=-\infty}^{N(t)} e^{-\lambda(t-\Gamma_k)} Z_k$$



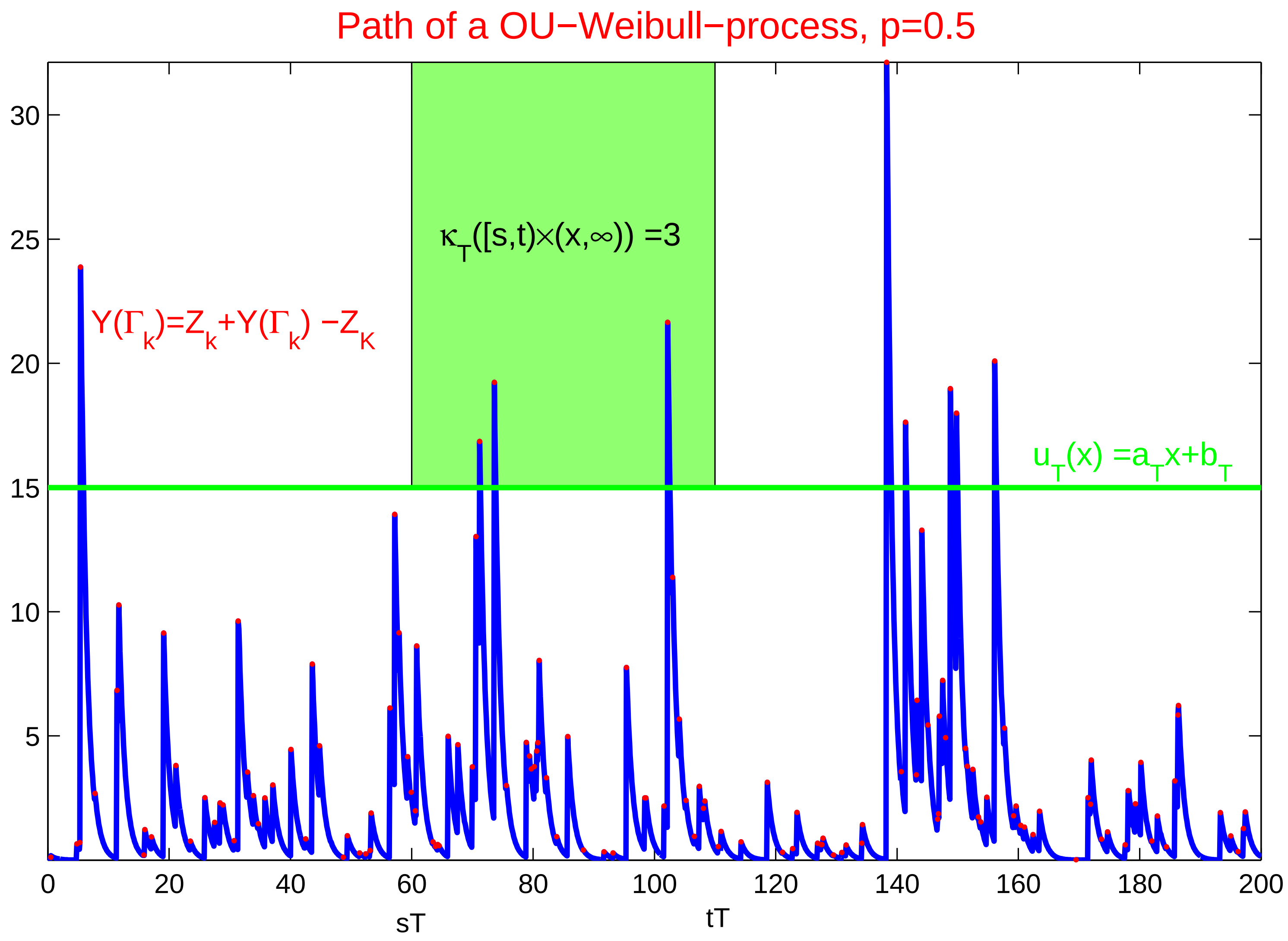
Example: OU-Weibull Process

$$Y(\Gamma_k) > a_T x + b_T \quad \Longleftrightarrow \quad a_T^{-1}(Y(\Gamma_k) - b_T) > x$$



Example: OU-Weibull Process

$$\kappa_T = \sum_{k=1}^{\infty} \varepsilon_{\left(\frac{\Gamma_k}{T}, a_T^{-1}(Y(\Gamma_k)-b_T)\right)}, \qquad I = [s,t) \times (x, \infty)$$



Point process behavior

a) $L(1) \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$: $\sum_{k=1}^{\infty} \varepsilon_{(s_k, P_k)} \sim \mathrm{PRM}(dt \times \mathrm{e}^{-x}\, dx)$

$$\sum_{k=1}^{\infty} \varepsilon_{(T^{-1}\Gamma_k, a_T^{-1}(Y(\Gamma_k)-b_T))} \stackrel{T\to\infty}{\Longrightarrow} \sum_{k=1}^{\infty} \varepsilon_{(s_k, P_k+\gamma Y_k)}$$

b) $L(1) \in \mathcal{R}_\alpha$: $\sum_{k=1}^{\infty} \varepsilon_{(s_k, P_k)} \sim \mathrm{PRM}(dt \times \alpha x^{-\alpha-1}\, dx)$

$$\sum_{k=1}^{\infty} \varepsilon_{(T^{-1}\Gamma_k, a_T^{-1}Y(\Gamma_k))} \stackrel{T\to\infty}{\Longrightarrow} \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \varepsilon_{(s_k, \exp(-R_k \Gamma_{k,j}) P_k)}$$

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b) $L(1) \in \mathcal{R}_\alpha$:

Motivation:

$Y(\Gamma_k + t) = \sum_{i=-\infty}^{N(\Gamma_k+t)} e^{-R_i(\Gamma_k + t - \Gamma_i)} Z_i$

$Y(\Gamma_k + t) \approx e^{-R_k t} Z_k$

$Y(\Gamma_{k+j}) \approx e^{-R_k(\Gamma_{j+k} - \Gamma_k)} Z_k$

$$\sum_{k=1}^{\infty} \varepsilon_{(T^{-1}\Gamma_k, a_T^{-1} Y(\Gamma_k))} \overset{T\to\infty}{\Longrightarrow} \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \varepsilon_{(s_k, \exp(-R_k \Gamma_{k,j}) P_k)}$$

Running maxima

Let $M(T) = \sup_{0 \le t \le T} Y(t)$.

a) $\boldsymbol{L(1) \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)}$:

$$\lim_{T \to \infty} \mathbb{P}(a_T^{-1}(M(T) - b_T) \le x) = \exp(-\mathbb{E}\mathrm{e}^{-\gamma Y(t)} \mathrm{e}^{-x})$$

b) $\boldsymbol{L(1) \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Phi_\alpha)}$:

$$\lim_{T \to \infty} \mathbb{P}(a_T^{-1} M(T) \le x) = \exp(-x^{-\alpha})$$

Conclusion

Extension

$$\boldsymbol{Y(t) = \int_{\mathbb{R}_+ \times \mathbb{R}} f(r, t-s)\, d\Lambda(r,s)}$$

- Kernel functions $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ with $f(r,0) = f^+$ and $f(r,s) < f^+$ for $r \in \text{supp}(\pi), s > 0$, which are non-increasing in both coordinates and $\lim_{s\to\infty} f(r,s) = 0$ for $r \in \text{supp}(\pi)$
- $L(1) \in \mathcal{S}$:
  - any kernel function with a finite number of extremes
  - neglecting the assumption, that Λ is a positive compound Poisson random measure

Remark

- The long memory property has no influence on the extremal behavior

References

Fasen, V. (2005) *Extremes of regularly varying Lévy driven mixed moving average processes*, to appear in J. Appl. Probab.

Fasen, V. (2005) *Extremes of subexponential Lévy driven moving average processes*, submitted for publication.

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