

Comparison of semi-parametric reduced bias' quantile estimators

Maria Ivette Gomes (CEAUL and DEIO, Universidade de Lisboa)

Fernanda Figueiredo (Universidade do Porto (FEP) and CEAUL)

Bjorn Vandewalle (Katholieke Universiteit Leuven and CEAUL)

EVA, OSLO 2005

Plan

- Motivation and Introduction
- Second Order Reduced Bias' Tail Index Estimators
- Asymptotic Behaviour of Reduced Bias' High Quantile Estimators
- Simulated Behaviour of High Quantile Estimators
- Case-study
- Some Overall Conclusions

Motivation

- Heavy-tailed models are quite useful in the most diversified areas (insurance, economics, finance, telecommunications, biostatistics,...) and the classical semi-parametric estimators of extreme events' parameters usually exhibit a reasonably high bias for low thresholds, i.e., for large values of k , the number of top o.s. used for the estimation.
- Recently, new classes of reduced bias' tail index estimators have been introduced in the literature. The estimation of the second order parameters in the bias at a level k_1 larger than k , the level at which we compute the tail index estimators, enables keeping the asymptotic variance of the new estimators equal to the asymptotic variance of the Hill estimator,

$$H(k) := \sum_{i=1}^k U_i/k, \quad U_i := i(\ln X_{n-i+1:n} - \ln X_{n-i:n}), \quad 1 \leq i \leq k.$$

1. Motivation and Introduction

- Here we deal with bias reduction techniques for heavy tails, trying to improve the performance of the classical high quantile estimators, strongly dependent on the tail index estimation.

Main objectives of this presentation:

- 1 Introduce new classes of high quantiles' estimators in the lines of Gomes and Figueiredo (2003) and Matthys and Beirlant (2003)
- 2 Prove their consistency and asymptotic normality under appropriate conditions
- 3 Compare them with alternative ones through Monte Carlo simulations

Introduction

Definition 1: A model F is said to have a heavy right tail whenever the maximum, linearly normalized, of an i.i.d. sample of size n , converges weakly, as $n \rightarrow \infty$, towards the *Extreme Value* d.f.,

$$EV_{\gamma}(x) = \begin{cases} \exp(-(1+\gamma x)^{-1/\gamma}), & 1+\gamma x > 0 & \text{if } \gamma \neq 0 \\ \exp(-\exp(-x)), & x \in \mathbb{R} & \text{if } \gamma = 0 \end{cases},$$

with $\gamma > 0$. We write $F \in \mathcal{D}_{\mathcal{M}}(EV_{\gamma})$, with $\mathcal{D}_{\mathcal{M}}$ denoting domain of attraction for maxima.

- Let RV_{α} denote the class of regularly varying functions with index α , i.e., positive measurable functions g such that $\lim_{t \rightarrow \infty} g(tx)/g(t) = x^{\alpha}$, for all $x > 0$.
- For $\gamma > 0$, $U(t) := F^{\leftarrow}(1 - 1/t) = \inf \{x : F(x) \geq 1 - 1/t\}$ and F^{\leftarrow} the generalized inverse of the underlying model F ,

$$F \in \mathcal{D}_{\mathcal{M}}(EV_{\gamma}) \Leftrightarrow 1 - F \in RV_{-1/\gamma} \Leftrightarrow U \in RV_{\gamma}.$$

Main objective of this paper: Estimate a value χ_p such that $1 - F(\chi_p) = p$, with p small, more specifically,

$$\chi_p = U(1/p), \quad p = p_n \rightarrow 0, \quad n p_n \rightarrow K, \text{ as } n \rightarrow \infty, \quad 0 \leq K \leq 1.$$

- We shall assume to be working in Hall's class of models, where there exist $\gamma > 0$, $\rho < 0$, $C > 0$ and $\beta \neq 0$ such that $U(t) = Ct^\gamma \left(1 + \frac{\gamma\beta t^\rho}{\rho} + o(t^\rho)\right)$, as $t \rightarrow \infty$.
- We are going to base inference on the largest k top o.s., i.e., we shall assume k to be *intermediate*,

$$k = k_n \rightarrow \infty, \quad k = o(n) \quad \text{as } n \rightarrow \infty.$$

Possible semi-parametric quantile estimator:

$$Q_{\hat{\gamma}}^{(p)}(k) := X_{n-k:n} \left(\frac{k}{np} \right)^{\hat{\gamma}} \quad (\text{Weissman, 1978}).$$

Classical quantile estimator:

$$Q_H^{(p)}(k) := X_{n-k:n} \left(\frac{k}{np} \right)^{\sum_{i=1}^k U_i/k} =: X_{n-k:n} \left(\frac{k}{np} \right)^{H(k)}.$$

- To derive the asymptotic non-degenerate behaviour of the semi-parametric estimators, we assume

$$\lim_{t \rightarrow \infty} \frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{A(t)} = \frac{x^\rho - 1}{\rho}, \text{ for all } x > 0,$$

where $A(\cdot)$ is a function of constant sign near infinity, $\rho \leq 0$ is the second order parameter and $|A| \in RV_\rho$ (Geluk and de Haan, 1987).

- We assume $\rho < 0$, and since we are working with models in Hall's class, the previous second order condition holds true with $A(t) = \gamma \beta t^\rho$, and for an adequate k , we may guarantee the asymptotic normality of the Hill estimator.

Proposition 1 [de Haan and Peng, 1998]: We may write the asymptotic distributional representation

$$H(k) := \frac{1}{k} \sum_{i=1}^k U_i \stackrel{d}{=} \gamma + \frac{\gamma P_k}{\sqrt{k}} + \frac{A(n/k)}{1-\rho} (1 + o_p(1)),$$

$P_k = \sqrt{k} \left(\sum_{i=1}^k E_i/k - 1 \right)$, with $\{E_i\}$ standard exponential i.i.d. r.v.'s. Consequently, if we choose a level k such that $\sqrt{k} A(n/k) \rightarrow \lambda \neq 0$, finite, as $n \rightarrow \infty$, $\sqrt{k} (H(k) - \gamma)$ is asymptotically normal, with a non-null bias given by $\lambda/(1-\rho)$.

- Most of the times, this type of estimates exhibit a strong bias for moderate k and sample paths with very short stability regions around the target value γ .
- This problem has been recently addressed by several authors, who consider the possibility of dealing with the bias term in an appropriate way, building different new estimators, $\hat{\gamma}_R(k)$ say, the so-called second order reduced bias' estimators.

Second Order Reduced Bias' Tail Index Estimators

Definition 2: A tail index estimator $\hat{\gamma}_R(k)$ is said to be a second order reduced bias' estimator if, for k intermediate, and under the second order framework, we may write

$$\hat{\gamma}_R(k) \stackrel{d}{=} \gamma + \frac{\sigma_R P_k^R}{\sqrt{k}} + o_p(A(n/k)),$$

with P_k^R an asymptotically standard normal r.v., $\sigma_R > 0$ and $A(\cdot)$ being again the function controlling the speed of convergence of maximum values, linearly normalized, towards a non-degenerate r.v. with d.f. EV_γ .

Remark 1: $\sqrt{k}(\hat{\gamma}_R(k) - \gamma)$ is asymptotically normal with a null mean value even when $\sqrt{k} A(n/k) \rightarrow \lambda$, finite, possibly non-null, as $n \rightarrow \infty$.

2. Second Order Reduced Bias' Tail Index Estimators

- Gomes and Figueiredo (2003) suggest the use of reduced bias' tail index estimators in the quantile estimator functional expression, in order to reduce also the dominant component of the classical quantile estimator's asymptotic bias.
- Mathys and Beirlant (2003) try also to reduce the bias of the classical quantile estimators, going directly into the second order framework. With $Y_{i:n}, 1 \leq i \leq n$, denoting the set of ascending o.s. associated to a standard Pareto i.i.d. sample,

$$\frac{\chi_p}{X_{n-k:n}} = \frac{U(1/p)}{U(Y_{n-k:n})} \underset{p}{\sim} a_n^\gamma \left(1 + A(n/k) \frac{a_n^\rho - 1}{\rho} \right), \quad a_n = k/(n p_n).$$

- For $A(t) = \gamma \beta t^\rho$, $(\hat{\gamma}, \hat{\beta}, \hat{\rho})$ a suitable estimator of (γ, β, ρ) , they get

$$\overline{Q}_{\hat{\gamma}}^{(p)}(k) := X_{n-k:n} \left(\frac{k}{np} \right)^{\hat{\gamma}} \exp \left(\hat{\gamma} \hat{\beta} \left(\frac{n}{k} \right)^{\hat{\rho}} \frac{(k/(np))^{\hat{\rho}} - 1}{\hat{\rho}} \right).$$

2. Second Order Reduced Bias' Tail Index Estimators

- It is known (Gomes and Figueiredo, 2003) that the use of a reduced bias' tail index estimator $\hat{\gamma}_R$ provides better results than the use of the classical Hill estimator H .
- The obvious question that we shall try to answer both theoretically and computationally, is the following:
Is it better to work with
 - ① the estimator $Q_{\hat{\gamma}}^{(p)}$ and a reduced bias estimator $\hat{\gamma} \equiv \hat{\gamma}_R$ of γ ,
 - ② the estimator $\overline{Q}_{\hat{\gamma}}^{(p)}$ and a classical estimator of γ , like the Hill estimator $H(k)$,
 - ③ or the estimator $\overline{Q}_{\hat{\gamma}}^{(p)}$ and a reduced bias estimator $\hat{\gamma}_R$ of γ ?

2. Second Order Reduced Bias' Tail Index Estimators

- We shall use the second order reduced bias' tail index estimator from Gomes and Martins (2002). With the notation

$$s_{\rho}(k) = \frac{1}{k} \sum_{i=1}^k \left(\frac{i}{k}\right)^{-\rho} \quad \text{and} \quad S_{\rho}(k) = \frac{1}{k} \sum_{i=1}^k \left(\frac{i}{k}\right)^{-\rho} U_i,$$

we may write the “maximum likelihood” estimator for the tail index γ in the form

$$M(k) \equiv M_{\hat{\rho}}(k) := S_0(k) - S_{\hat{\rho}}(k) \times \frac{s_{\hat{\rho}}(k) S_0(k) - S_{\hat{\rho}}(k)}{s_{\hat{\rho}}(k) S_{\hat{\rho}}(k) - S_{2\hat{\rho}}(k)}.$$

Remark 2: This estimator attains the minimal asymptotic variance in Drees' class of functionals (Drees, 1998), given by $(\gamma(1 - \rho)/\rho)^2$.

2. Second Order Reduced Bias' Tail Index Estimators

- Notice that we may also write,

$$M(k) = S_0(k) - \hat{\beta}_{\hat{\rho}}(k) \left(\frac{n}{k}\right)^{\hat{\rho}} S_{\hat{\rho}}(k)$$

with

$$\hat{\beta}_{\hat{\rho}}(k) := \left(\frac{k}{n}\right)^{\hat{\rho}} \frac{s_{\hat{\rho}}(k) S_0(k) - S_{\hat{\rho}}(k)}{s_{\hat{\rho}}(k) S_{\hat{\rho}}(k) - S_{2\hat{\rho}}(k)}.$$

- In the lines of the paper by Gomes *et al.* (2005), we shall consider, for a suitable consistent estimator of the second order parameter ρ -estimator, $\hat{\rho}$, the β -estimator, $\hat{\beta} \equiv \hat{\beta}_{\hat{\rho}}(k_1)$, computed at

$$k_1 = \min(n - 1, [2n / \ln \ln n]).$$

- The estimate $\hat{\beta}$ is then incorporated in $M(k)$, and we get

$$\overline{M}(k) \equiv \overline{M}_{\hat{\beta}, \hat{\rho}}(k) := S_0(k) - \hat{\beta} \left(\frac{n}{k}\right)^{\hat{\rho}} S_{\hat{\rho}}(k).$$

2. Second Order Reduced Bias' Tail Index Estimators

Proposition 2 [Gomes *et al.*, 2005]: If the second order condition holds, if $k = k_n$ is a sequence of intermediate positive integers, and if $\sqrt{k} A(n/k) \rightarrow \lambda$, finite and non necessarily null, as $n \rightarrow \infty$, then

$$\sqrt{k} (\overline{M}_{\beta, \rho}(k) - \gamma) \xrightarrow[n \rightarrow \infty]{d} \text{Normal}(0, \gamma^2).$$

This same limiting behaviour holds true if we replace $\overline{M}_{\beta, \rho}$ by $\overline{M}_{\hat{\beta}, \hat{\rho}}$, provided that $\hat{\rho} - \rho = o_p(1)$ for every k -value on which we base the tail index estimation, and we choose $\hat{\beta} := \hat{\beta}_{\hat{\rho}}(k_1)$.

- The M and \overline{M} estimators have been plugged in Q and \overline{Q} , providing us with the estimators $Q_M^{(p)}$, $Q_{\overline{M}}^{(p)}$, $\overline{Q}_M^{(p)}$ and $\overline{Q}_{\overline{M}}^{(p)}$, respectively. These estimators require the estimation of the shape second order parameter ρ .

Estimation of the Shape Second Order Parameters

- We shall consider again particular members of the class of estimators of ρ proposed in Fraga Alves *et al.* (2003). They depend on the statistics:

$$T_n^{(\tau)}(k) := \begin{cases} \frac{\left(M_n^{(1)}(k)\right)^\tau - \left(M_n^{(2)}(k)/2\right)^{\tau/2}}{\left(M_n^{(2)}(k)/2\right)^{\tau/2} - \left(M_n^{(3)}(k)/6\right)^{\tau/3}} & \text{if } \tau \neq 0 \\ \frac{\ln\left(M_n^{(1)}(k)\right) - \frac{1}{2}\ln\left(M_n^{(2)}(k)/2\right)}{\frac{1}{2}\ln\left(M_n^{(2)}(k)/2\right) - \frac{1}{3}\ln\left(M_n^{(3)}(k)/6\right)} & \text{if } \tau = 0 \end{cases},$$

which converge towards $3(1 - \rho)/(3 - \rho)$, for any τ , whenever the second order condition holds, k is intermediate and $\sqrt{k} A(n/k) \rightarrow \infty$, as $n \rightarrow \infty$.

- The ρ -estimators may thus be written as

$$\hat{\rho}_\tau(k) \equiv \hat{\rho}_n^{(\tau)}(k) := \min \left(0, \frac{3(T_n^{(\tau)}(k) - 1)}{T_n^{(\tau)}(k) - 3} \right).$$

Remark 3: Different theoretical and simulated results, together with the use of these estimators in different reduced bias' statistics, have led us to advise in practice the drawing of a few sample paths of $\hat{\rho}_\tau(k)$, as functions of k , electing the value of τ which provides higher stability for large k , by means of any stability criterion, like the ones in Gomes and Figueiredo (2003) and in Gomes and Pestana (2004).

- The consideration of the level k_1 seems to be an adequate choice for the level k . The choice between the tuning parameters $\tau = 0$ and $\tau = 1$, leads us to advise the consideration of $\tau = 0$ whenever $\rho \in [-1, 0)$ and $\tau = 1$ for the region $\rho \in (-\infty, -1)$.
- We shall denote generically $\hat{\rho}$ any of these estimators. We use $\hat{\beta} = \hat{\beta}_{\hat{\rho}}(k_1)$. We shall use a subscript j to specify the value of $\tau \equiv j$, $j = 0$ or 1 , writing $\hat{\rho}_j$, $\hat{\beta}_j$, M_j and \overline{M}_j .

Asymptotic Behaviour

For intermediate k , and with $a_n := k/(np_n)$, which, given the conditions, goes to infinity as $n \rightarrow \infty$, we are here dealing with semi-parametric p -quantile estimators of the type

$$Q_{\hat{\gamma}}^{(p)}(k) := X_{n-k:n} a_n^{\hat{\gamma}(k)}$$

and

$$\overline{Q}_{\hat{\gamma}}^{(p)}(k) := X_{n-k:n} a_n^{\hat{\gamma}(k)} \exp \left(\hat{\gamma}(k) \hat{\beta} \left(\frac{n}{k} \right)^{\hat{\rho}} \frac{a_n^{\hat{\rho}} - 1}{\hat{\rho}} \right),$$

where $\hat{\gamma}(k)$ is any semi-parametric estimator of the tail index γ .

- Directly from Theorem 3.1 in Gomes and Figueiredo (2003), we may state:

Theorem 1: Under the second order framework, for intermediate k , in Hall's class of models, and whenever $\ln(n p_n) = o(\sqrt{k})$, we may write, with P_k , W_k and \overline{W}_k asymptotically standard normal r.v.'s, and with \tilde{Q} denoting either Q or \overline{Q} ,

$$\frac{\sqrt{k}}{\ln a_n} \left(\frac{\tilde{Q}_H^{(p)}(k)}{\chi_p} - 1 \right) \stackrel{d}{=} \gamma P_k + \frac{\sqrt{k} A(n/k)}{1 - \rho} + o_p(\sqrt{k} A(n/k)),$$

$$\frac{\sqrt{k}}{\ln a_n} \left(\frac{\tilde{Q}_M^{(p)}(k)}{\chi_p} - 1 \right) \stackrel{d}{=} \frac{\gamma(1 - \rho)}{\rho} W_k + o_p(\sqrt{k} A(n/k))$$

$$\frac{\sqrt{k}}{\ln a_n} \left(\frac{\tilde{Q}_{\overline{M}}^{(p)}(k)}{\chi_p} - 1 \right) \stackrel{d}{=} \gamma \overline{W}_k + o_p(\sqrt{k} A(n/k)).$$

3. Asymptotic Behaviour of Reduced Bias' High Quantile Estimators

- In the proof, we get:

$$\frac{Q_{\hat{\gamma}(k)}^{(p)}(k)}{\chi_p} - 1 \stackrel{d}{=} (\hat{\gamma}(k) - \gamma) \ln a_n + \frac{\gamma B_k}{\sqrt{k}} + \frac{A(n/k)}{\rho} + o_p(A(n/k)) \quad (*)$$

$$\frac{\overline{Q}_{\hat{\gamma}(k)}^{(p)}(k)}{\chi_p} - 1 \stackrel{d}{=} (\hat{\gamma}(k) - \gamma) \ln a_n + \frac{\gamma B_k}{\sqrt{k}} + o_p(A(n/k)). \quad (**)$$

Remark 4: If we compare (*) and (**) we notice that the main contribution in terms of bias is provided by a possible bias of $\hat{\gamma}(k)$.

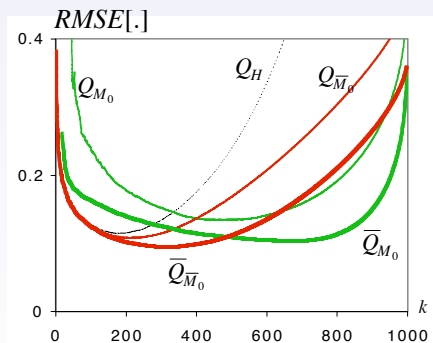
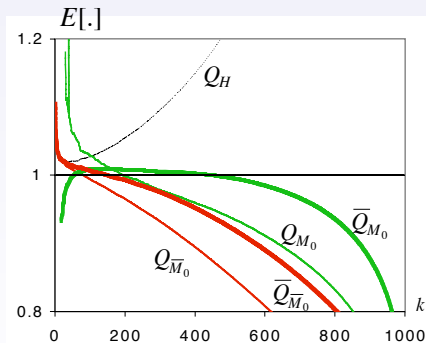
- It thus seems obvious that, for the semi-parametric estimation of a high quantile, it is better to use a reduced bias' tail index estimator than a classical tail index estimator.
- Moreover, although the main contribution comes from the first summand, we expect to get slightly better results when we use $\overline{Q}_{\hat{\gamma}}^{(p)}$ instead of $Q_{\hat{\gamma}}^{(p)}$, due to that remaining summand of the order of $A(n/k)$ in (*).

Simulated Behaviour

- For the estimation of the second order parameter ρ , and as mentioned before, we have here used the value $\tau = 0$ or $\tau = 1$, according as $|\rho| \leq 1$ or $|\rho| > 1$, together with the level k_1 .
- In Figures 1 and 2, we show, for $p = 1/n$ and $j = 0$ or 1 , the simulated patterns of mean value, $E[\cdot]$, and root mean squared error, $RMSE[\cdot]$, of $Q_H^{(p)}(k)/\chi_p$, $Q_{M_j}^{(p)}(k)/\chi_p$, $Q_{\overline{M}_j}^{(p)}(k)/\chi_p$, based thus on the Hill and the “maximum likelihood” reduced bias’ estimators M and \overline{M} . These quotients will be denoted Q_H , Q_{M_j} and $Q_{\overline{M}_j}$, respectively.
- The notations \overline{Q}_{M_j} and $\overline{Q}_{\overline{M}_j}$ hold for the same quotients associated with the estimator \overline{Q} , dependent on $M_j = M_{\hat{\rho}_j}$ and $\overline{M}_j = \overline{M}_{\hat{\beta}_j, \hat{\rho}_j}$, respectively.

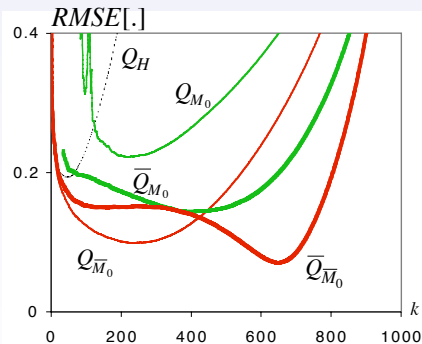
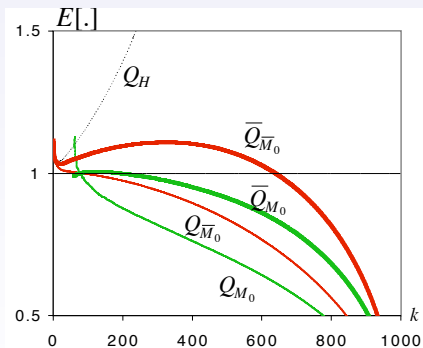
4. Simulated Behaviour of High Quantile Estimators

Figure 1: Simulated distributional behaviour of the estimators under study for an underlying *Frechet* parent with $\gamma = 1$ ($\rho = -1$).



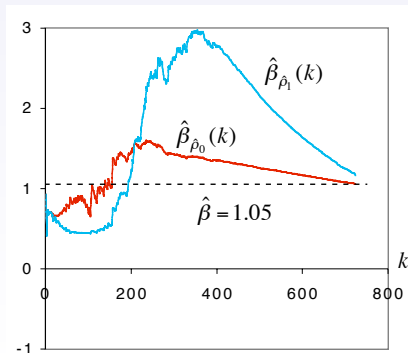
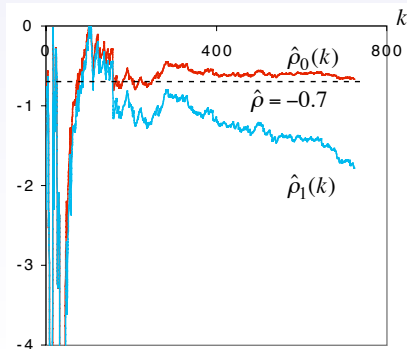
4. Simulated Behaviour of High Quantile Estimators

Figure 2: Simulated distributional behaviour of the estimators under study for an underlying *Burr* parent with $\gamma = 1$ and $\rho = -0.5$.



Case-study: The analysis of the Euro-USA Dollar daily exchange rates from 01/04/1999 till 12/14/2004.

Figure 3: The sample path of the $\hat{\rho}_\tau$ (left), as function of k , together with the sample paths of the β -estimators, for $\tau = 0$ and $\tau = 1$ (right), for the $n^+ = 748$ positive log-returns.

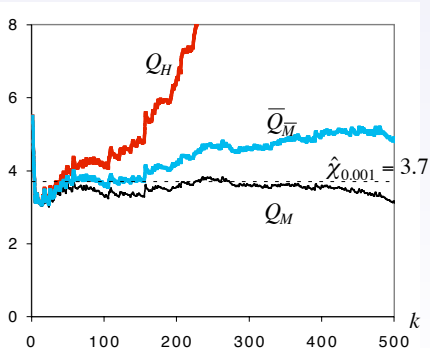
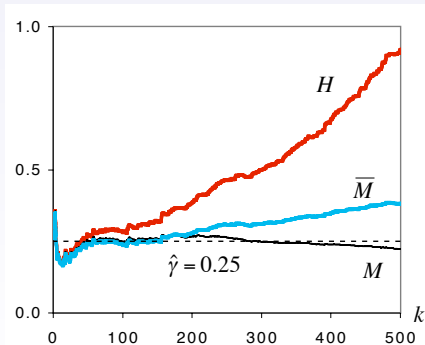


5. A Case-study

- Note that the sample paths of the ρ -estimates associated to $\tau = 0$ and $\tau = 1$ lead us to choose, on the basis of any stability criterion for large k , the estimate associated to $\tau = 0$.
- From previous experience with this type of estimates, we conclude that the underlying ρ -value is larger or equal to -1 , and the consideration of $\tau = 0$ is then advisable.
- The estimate of ρ is in this case $\hat{\rho}_0 = -0.7$, obtained at the level $k_1 = 748$. The associated β -estimator is $\hat{\beta}_0 = 1.05$.

5. A Case-study

Figure 4: Estimates of the tail index γ provided by the estimators H , M and \bar{M} (left), and the corresponding quantile estimators χ_p , associated to $p = 0.001$ (right), for the Daily Log-Returns of the Euro-USA Dollar.



5. A Case-study

- Regarding the tail index estimation, note that whereas the Hill estimator is unbiased for the estimation of the tail index γ when the underlying model is a strict Pareto model, it exhibits a relevant bias when we have only Pareto-like tails, as happens here.
- The other estimators, which are “asymptotically unbiased” reveal a smaller bias, and enable us to take a decision upon the estimate of γ to be used, with the help of any stability criterion or any heuristic procedure, like a largest run method as the one described in the sequel.

Some Overall Conclusions

- The obtained results lead us to strongly advise the use of the quantile estimator \overline{Q} , with the tail index estimator $\overline{M}(k)$, for models with $\rho = -1$.
- Anyway, the estimator Q , with the tail index estimator $\overline{M}(k)$, does also exhibit an interesting performance, particularly for all the simulated models with $\rho \neq -1$.

Remark 5: Note that, similarly to what has happened before with the tail index estimation, the computation of both second order parameters' estimators, at the high value k_1 , enables us to work with high quantiles' estimators, with a mean squared error smaller than the mean squared error of the classical estimator, for all k . Those high quantile estimators are provided by the use in Q or \overline{Q} of the tail index estimator \overline{M} .

- Drees, H. (1998). A general class of estimators of the extreme value index. *J. Statist. Planning and Inference* **98**, 95-112.
- Fraga Alves, M. I., Gomes M. I. and de Haan, L. (2003). A new class of semi-parametric estimators of the second order parameter. *Portugaliae Mathematica* **60**:2, 194-213.
- Geluk, J. and L. de Haan (1987). *Regular Variation, Extensions and Tauberian Theorems*. CWI Tract 40, Center for Mathematics and Computer Science, Amsterdam, Netherlands.
- Gomes, M. I. and Figueiredo, F. (2003). Bias reduction in risk modelling: semi-parametric quantile estimation. To appear at *Test*.
- Gomes, M. I. and M. J. Martins (2002). "Asymptotically unbiased" estimators of the tail index based on external estimation of the second order parameter. *Extremes* **5**:1, 5-31.
- Gomes, M. I., Martins, M. J. and Neves, M. (2005). "Optimal" second order reduced bias' maximum likelihood tail index estimators. *Notas e Comunicações CEAUL /2005*.
- Gomes, M. I. and Pestana (2004). A simple second order reduced bias' tail index estimator. To appear at *J. Statist. Comp. and Simul.*
- Haan, L. de and Peng, L. (1998). Comparison of tail index estimators. *Statistica Neerlandica* **52**, 60-70.
- Matthys, G. and Beirlant, J. (2003). Estimating the extreme value index and high quantiles with exponential regression models. *Statistica Sinica* **13**, 853-880.
- Weissman, I. (1978). Estimation of parameters and large quantiles based on the k largest observations. *J. Amer. Statist. Assoc.* **73**, 812-815.