Reduced Bias Semi-parametric Quantile Estimators with a Linear-type Property

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1 - Introduction

Denote by F the heavy-tailed distribution function (df) of X, the common df of the i.i.d. sample $\{X_i\}_{i=1}^{n}$, for which the extreme quantile

$$\chi_p = \chi_p(X) = F^{\leftarrow}(1-p)$$

has to be estimated. Here $F^{\leftarrow}(t) = \inf\{x : F(x) \geq t\}$.

Classical estimators based on the largest o.s. from $\underset{\sim}{X} := (X_{n:n}, X_{n-1:n}, \cdots, X_{1:n})$, will be considered here, which involve $X_{n:n} \geq X_{n-1:n} \geq \cdots \geq X_{n-k:n}$, where $X_{n-k:n}$ is an intermediate order statistic (o.s.), i.e., k is an intermediate sequence of integers,

$$k = k_n \to \infty, \quad k_n/n \to 0, \qquad \text{as } n \to \infty. \tag{1}$$

Moreover, we are mainly interested in the natural case

$$p = p_n \to 0, \ \text{as } n \to \infty, \ \text{such that } np_n \to c \geq 0. \quad \textit{extreme quantile} \tag{2}$$

Basic assumption for heavy-tailed distributions – semi-parametric approach:

$$F \in D(G_\gamma)_{\gamma>0},$$

$$G_\gamma(x) = \exp(-(1+\gamma x)^{-1/\gamma}), 1+\gamma x \geq 0, \gamma \in \mathbb{R}, G_0(x) = \exp(-e^{-x}).$$

First order condition

$$F \in D(G_\gamma)_{\gamma>0} \quad \text{iff} \quad \overline{F} \in RV_{-1/\gamma} \quad \text{iff} \quad U \in RV_\gamma,$$

$$U(t) := F^{\leftarrow}(1-1/t),\ t \geq 1;$$

that is,

$$\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = x^{-1/\gamma} \quad \text{iff} \quad \lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^\gamma, \text{ for all } x>0$$

$$\text{High Quantile–}p_n \Rrightarrow \chi_{p_n} = U\left(\frac{1}{p_n}\right). \qquad (3)$$

Second order condition

$$\lim_{t\to\infty} \frac{U(tx)/U(t) - x^{\gamma}}{A(t)} = x^{\gamma}\frac{x^{\rho}-1}{\rho}, \tag{4}$$

for all $x > 0$, where A is a suitable chosen function of constant sign near infinity, $|A| \in RV_{\rho}$, where $\rho \leq 0$ is the second order parameter.

Notation:

RV_{α} stands for positive measurable functions h: $\lim_{t\to\infty} h(tx)/h(t) = x^{\alpha}$, for all $x > 0$.

Hall's class:

$$U(t) = Ct^{\gamma}(1 + Dt^{\rho} + o(t^{\rho})), \quad \rho < 0, \quad \text{as} \quad t \to \infty. \tag{5}$$

Classical semi-parametric estimator of a high quantile χ_p

Weissman-type estimator of χ_{p_n}, (Weissman,1978)

$$\hat{\chi}^W_{p_n} = \hat{\chi}^W_{p_n}(\underset{\sim}{X}) = X_{n-k_n:n}\left(\frac{k_n}{np_n}\right)^{\hat{\gamma}_n}, \tag{6}$$

with $\hat{\gamma}_n = \hat{\gamma}_n(\underset{\sim}{X})$ some consistent estimator of the tail parameter γ.

Classical semi-parametric estimators of the tail index γ

Hill estimator (Hill,1975) $\gamma > 0$

$$\hat{\gamma}^H_n = \hat{\gamma}^H_n(\underset{\sim}{X}) = \frac{1}{k_n}\sum_{i=1}^{k_n} \log\frac{X_{n-i+1:n}}{X_{n-k_n:n}} \tag{7}$$

Moment estimator (Dekkers et al.,1989) $\gamma \in \mathbb{R}$

$$\hat{\gamma}^M_n = \hat{\gamma}^M_n(\underset{\sim}{X}) = M_n^{(1)} + 1 - \frac{1}{2}\left\{1 - \frac{(M_n^{(1)})^2}{M_n^{(2)}}\right\}^{-1}, \tag{8}$$

with $M_n^{(r)}$ the r-Moment: $M_n^{(r)} = M_n^{(r)}(\underset{\sim}{X}) = \frac{1}{k_n}\sum_{i=1}^{k_n}\left(\log\frac{X_{n-i+1:n}}{X_{n-k_n:n}}\right)^r$, for $r = 1, 2, 3$.

Both estimators (7) and (8) are scale-invariant,

$$\hat{\gamma}_n^H(\delta \underset{\sim}{X}) \stackrel{d}{=} \hat{\gamma}_n^H(\underset{\sim}{X}) \quad \text{and} \quad \hat{\gamma}_n^M(\delta \underset{\sim}{X}) \stackrel{d}{=} \hat{\gamma}_n^M(\underset{\sim}{X})$$

$\Rrightarrow$ δ-scale transformations to the data do not interfere with their stochastic behaviour.

In what concerns the quantile Weissman-type estimator (6), for $\delta > 0$,

$$\hat{\chi}_{p_n}^W(\delta \underset{\sim}{X}) = \delta X_{n-k_n:n} \left(\frac{k_n}{np_n}\right)^{\hat{\gamma}_n} \stackrel{d}{=} \delta \hat{\chi}_{p_n}^W(\underset{\sim}{X}),$$

a desirable exact property for quantile estimators, under positive **scale-transformations**.

For **λ-shift** into the data: $Y := X + \lambda$, for $\lambda \in \mathbb{R}$, we would like that

$$\hat{\chi}_{p_n}(\underset{\sim}{Y}) \stackrel{d}{=} \hat{\chi}_{p_n}(\underset{\sim}{X}) + \lambda. \tag{9}$$

$\Rrightarrow$ Theoretical **linear property for quantiles**

$$\chi_p(\delta X + \lambda) = \delta \chi_p(X) + \lambda. \tag{10}$$

Fraga Alves and Araújo Santos (2004) have studied a first approach study to this problem; therein, a simple modification of (6) has been proposed which enjoys the property (9) approximately.

Here we will present a **class of extreme quantile-estimatores** for which (9) holds **exactly**, pursuing the empirical counterpart of theoretical linear property (10).

Consider first the sample of excesses over a random threshold $X_{n_i:n}$:

$$\underset{\sim}{X}^{(i)} := \left(X_{n:n} - X_{n_i:n}, X_{n-1:n} - X_{n_i:n}, \cdots, X_{n_i+1:n} - X_{n_i:n}\right), \quad i = 1, 2.$$

- $n_1 := 1$ for df's with **finite left endpoint** $x_F := \inf\{x : F(x) > 0\}$

  – *threshold* $X_{1:n}$ *minimum*

- $n_2 := [nq] + 1$, $0 < q < 1$, for df's with **finite or infinite left endpoint** x_F

  – *threshold* $X_{[nq]+1:n}$ *empirical quantile*

- $\hat{\gamma}_n^{(i)}$ any consistent estimator of the tail parameter γ, made **location/scale invariant** by using the transformed sample $\underset{\sim}{X}^{(i)}$.

- **Modified-Weissman estimator:**

$$\tilde{\chi}_{p_n}^{(i)} = \left(X_{n-k_n:n} - X_{n_i:n}\right)\left(k_n/np_n\right)^{\hat{\gamma}_n^{(i)}} + X_{n_i:n}, \quad i = 1, 2. \tag{11}$$

2 - Tail index estimation with the sample of excesses

We propose to incorporate in the Modified-Weissman (11) quantile, estimators of tail parameter $\gamma > 0$, via the sample of excesses $\underset{\sim}{X}^{(i)}$, $i = 1, 2$, which allows to obtain exactly the linear property (9).

Denote $\hat{\gamma}_n^{H(i)}$ and $\hat{\gamma}_n^{M(i)}$, the Hill (7) and Moment (8) tail index estimators, as functions of the transformed sample $\underset{\sim}{X}^{(i)}$, $i = 1, 2$; that is,

$$\hat{\gamma}_n^{H(i)} := \hat{\gamma}_n^{H}(\underset{\sim}{X}^{(i)}) \quad \text{and} \quad \hat{\gamma}_n^{M(i)} := \hat{\gamma}_n^{M}(\underset{\sim}{X}^{(i)})$$

Asymptotic distributional representation for Hill and Moment estimators, using the sample of excesses?

BIAS? VARIANCE?

Notation: In the following χ_i , for $i = 1, 2$, denotes:

$\chi_1 := x_F$ – finite left endpoint of F $\qquad$ $\chi_2 := \chi_q^*$ – q-quantile of F: $F(\chi_q^*) = q$.

Theorem 1. (Modified-Hill) *For k an intermediate sequence as in (1) and the validity of the second order condition in (4), the asymptotic distributional representation*

$$\hat{\gamma}_n^{H(i)} \stackrel{d}{=} \gamma + \frac{\sigma_H}{\sqrt{k}} P_k + \left(d_1 A(n/k) + d_2 \frac{\chi_i}{U(n/k)} \right) (1 + o_p(1))$$

holds, where P_k is an asymptotically standard normal r.v., $\sigma_H^2 := \gamma^2$, $d_1 := \frac{1}{1-\rho}$ and $d_2 := \frac{\gamma}{\gamma+1}$.

Theorem 2. (Modified-Moment) *For k an intermediate sequence as in (1)and the validity of the second order condition in (4), the asymptotic distributional representation*

$$\hat{\gamma}_n^{M(i)} \stackrel{d}{=} \gamma + \frac{\sigma_M}{\sqrt{k}} R_k + \left(c_1 A(n/k) + c_2 \frac{\chi_i}{U(n/k)} \right) (1 + o_p(1))$$

holds, with R_k is asymptotically standard normal, $\sigma_M^2 := \gamma^2 + 1$, $c_1 := \frac{\gamma(1-\rho)+\rho}{\gamma(1-\rho)^2}$ and $c_2 := \left(\frac{\gamma}{\gamma+1}\right)^2$.

Remark 1: Notice that

$$\sigma_M^2 = \sigma_H^2 + 1, \quad c_1 = d_1 + \frac{\rho}{\gamma(1-\rho)^2} \quad \text{and} \quad c_2 = (d_2)^2;$$

consequently,

$$\sigma_M > \sigma_H, \quad c_1 < d_1 \quad \text{and} \quad c_2 < d_2.$$

Corollary 1: Since (4) holds, $|A| \in RV_\rho$, $U \in RV_\gamma$ and 3 cases must be considered: let μ_1, $\mu 2$, be finite and non-null constants.

- $\gamma > -\rho$:

$$\hat{\gamma}_n^{H(i)} \stackrel{d}{=} \gamma + \frac{\sigma_H}{\sqrt{k}} P_k + d_1 A(n/k)(1 + o_p(1))$$
$$\hat{\gamma}_n^{M(i)} \stackrel{d}{=} \gamma + \frac{\sigma_M}{\sqrt{k}} R_k + c_1 A(n/k)(1 + o_p(1))$$

  if $\sqrt{k} A(n/k) \to \mu_1$, then

  $\sqrt{k}\left(\hat{\gamma}_n^{H(i)} - \gamma\right)$ is asymptotically Normal(mean: $\mu_1 d_1$, variance: $\sigma_H^2 = \gamma^2$).
  $\sqrt{k}\left(\hat{\gamma}_n^{M(i)} - \gamma\right)$ is asymptotically Normal(mean: $\mu_1 c_1$, variance: $\sigma_M^2 = 1 + \gamma^2$).

- $\gamma = -\rho$:

  if $\sqrt{k} A(n/k) \to \mu_1$ and $\sqrt{k}/U(n/k) \to \mu_2$, then

  $\sqrt{k}\left(\hat{\gamma}_n^{H(i)} - \gamma\right)$ is asymptotically Normal(mean: $\mu_1 d_1 + \mu_2 d_2 \chi_i$, variance: $\sigma_H^2 = \gamma^2$)

  $\sqrt{k}\left(\hat{\gamma}_n^{M(i)} - \gamma\right)$ is asymptotically Normal(mean: $\mu_1 c_1 + \mu_2 c_2 \chi_i$, variance: $\sigma_M^2 = 1 + \gamma^2$).

- $\gamma < -\rho$:

$$\hat{\gamma}_n^{H(i)} \stackrel{d}{=} \gamma + \frac{\sigma_H}{\sqrt{k}} P_k + d_2 \frac{\chi_i}{U(n/k)} (1 + o_p(1))$$
$$\hat{\gamma}_n^{M(i)} \stackrel{d}{=} \gamma + \frac{\sigma_M}{\sqrt{k}} R_k + c_2 \frac{\chi_i}{U(n/k)} (1 + o_p(1))$$

  if $\sqrt{k}/U\,(n/k) \to \mu_2$, then

  $\sqrt{k}\left(\hat{\gamma}_n^{H(i)} - \gamma\right)$ is asymptotically Normal(mean: $\mu_2 d_2 \chi_i$, variance: $\sigma_H^2 = \gamma^2$)

  $\sqrt{k}\left(\hat{\gamma}_n^{M(i)} - \gamma\right)$ is asymptotically Normal(mean: $\mu_2 c_2 \chi_i$, variance: $\sigma_M^2 = 1 + \gamma^2$).

Remark 2:

- If there is evidence that the underlying F is symmetric the random threshold should be chosen as the empirical median; i.e., $X_{[nq]+1:n} = X_{[n/2]+1:n}$, since the theoretical median for the standard model $\chi_2 := \chi_q^*$, $q := 1/2$, can be chosen to be zero, $\chi_{1/2}^* = 0$.

- If there is *evidence* that the underlying F has finite left endpoint x_F, the random threshold should be chosen as the sample minimum; i.e., $X_{1:n}$, since the theoretical left endpoint for the standard model $\chi_1 := x_F$, , can be chosen to be zero, $x_F = 0$.

3 - Reduced bias tail index estimators

In Gomes and Figueiredo (2002), several *reduced bias tail index estimators* have been considered, which are based on the estimation of the second order parameter ρ and allows an asymptotic distributional representation

$$\hat{\gamma}_n \stackrel{d}{=} \gamma + \frac{\sigma_\gamma}{\sqrt{k}} Q_k + o_p\left(A\left(n/k\right)\right),$$

with Q_k standard normal.

Then it is achieved asymptotic normality for $\sqrt{k}\left(\hat{\gamma}_n - \gamma\right)$ with null mean value,

not only when $\sqrt{k}A\left(n/k\right) \to 0$,
but also when $\sqrt{k}A\left(n/k\right) \to \mu$, finite and non-null.

Other references: Gomes and Martins (2001), (2002) and Gomes and Caeiro (2002).

We have chosen the one which provides the smallest asymptotic variance for all values of ρ, the so-called **ML(maximum likelihood)** estimator.

The ML estimator:

In Hall's class (5), the scaled log-spacings, $U_i = i\Big[\ln \frac{X_{n-i+1:n}}{X_{n-k:n}}\Big]_{,1\le i\le k}$, are approximately exponential with mean $\mu_i = \gamma e^{D(\frac{i}{n})^{-\rho}}$.

Based on the joint maximization of the log-likelihood of the U_i, in γ, D and ρ, they proposed the **Estimator for the tail index** γ,

$$\hat{\gamma}_n^{ML} := \gamma_n^{ML(\hat{\rho})}(\underset{\sim}{X}) = \frac{1}{k}\sum_{i=1}^{k} U_i - \Big(\frac{1}{k}\sum_{i=1}^{k} i^{-\hat{\rho}}U_i\Big)\frac{(\sum_{i=1}^{k} i^{-\hat{\rho}})(\sum_{i=1}^{k} U_i) - k(\sum_{i=1}^{k} i^{-\hat{\rho}}U_i)}{(\sum_{i=1}^{k} i^{-\hat{\rho}})(\sum_{i=1}^{k} i^{-\hat{\rho}}U_i) - k(\sum_{i=1}^{k} i^{-2\hat{\rho}}U_i)}$$

Estimator of the second order parameter ρ

We shall consider here an estimator of ρ proposed by Fraga Alves et. al.(2003), which depend on the moments $M_n^{(i)}(k)_{,i=1,2,3}$ through the statistic:

$$T_n := T_{n,k}(\underset{\sim}{X}) = \frac{\Big(M_n^{(1)}(k)\Big) - \Big(M_n^{(2)}(k)/2\Big)^{1/2}}{\Big(M_n^{(2)}(k)/2\Big)^{1/2} - \Big(M_n^{(3)}(k)/6\Big)^{1/3}}$$

The estimator of ρ is given by

$$\hat{\rho} := \hat{\rho}_n(\underset{\sim}{X}) = \min\Big(0, \frac{3(T_n-1)}{T_n-3}\Big), \text{ with } k = \min(n-1, [2n/\log\log n])$$

4 - High quantile estimation

Define the estimators of χ_{p_n} defined in (3)

- as function of the **original sample**, $\underset{\sim}{X}$, inspired by (6):

  - $\hat{\chi}^{W}_{p_n} := X_{n-k_n:n} \left(\frac{k_n}{np_n}\right)^{\hat{\gamma}^H_n}$

  - $\hat{\chi}^{M}_{p_n} := X_{n-k_n:n} \left(\frac{k_n}{np_n}\right)^{\hat{\gamma}^M_n}$

- and as function of the **sample of excesses** over $X_{n_i:n}$, $\underset{\sim}{X}^{(i)}$, inspired by (11):

  - $\tilde{\chi}^{(i)}_{p_n,H} := (X_{n-k_n:n} - X_{n_i:n}) \left(\frac{k_n}{np_n}\right)^{\hat{\gamma}^{H(i)}_n} + X_{n_i:n}, \quad i = 1, 2.$

  - $\tilde{\chi}^{(i)}_{p_n,M} := (X_{n-k_n:n} - X_{n_i:n}) \left(\frac{k_n}{np_n}\right)^{\hat{\gamma}^{M(i)}_n} + X_{n_i:n}, \quad i = 1, 2.$

Theorem 3. *In Hall's class (5), for intermediate sequences k_n that satisfy*

$$\log(np_n)/\sqrt{k_n} \to 0, \text{ as } n \to \infty,$$

with p_n such that (2) holds, then

$$\frac{\sqrt{k_n}}{\sigma_H \log(k_n/(np_n))}\left(\frac{\tilde{\chi}^{(i)}_{p_n,H}}{\chi_{p_n}} - 1\right) = P_k + \sqrt{k_n}\Big(d_1 A(n/k) + d_2 \frac{\chi_i}{U(n/k)}\Big)(1 + o_p(1))$$

holds, where P_k is an asymptotically standard normal r.v., $\sigma_H^2 := \gamma^2$, $d_1 := \frac{1}{1-\rho}$ and $d_2 := \frac{\gamma}{\gamma+1}$.

Moreover, for:

1. $\gamma > -\rho$ *and* $\sqrt{k_n}A(n/k_n) \to \mu_1$, *finite, as* $n \to \infty$, *then the mean value is* $\mu_1 d_1$;

2. $\rho < -\gamma$ *and* $\sqrt{k_n}/U(n/k_n) \to \mu_2$, *finite, as* $n \to \infty$, *then the mean value is* $\mu_2 d_2 \chi_i$;

3. $\rho = -\gamma$, $\sqrt{k_n}A(n/k_n) \to \mu_1$, *finite, and* $\sqrt{k_n}/U(n/k_n) \to \mu_2$, *finite, as* $n \to \infty$, *then the mean value is* $\mu_1 d_1 + \mu_2 d_2 \chi_i$.

Theorem 4. *In Hall's class, for intermediate sequences k_n that satisfy*

$$\log(np_n)/\sqrt{k_n} \to 0, \ \textit{as } n \to \infty,$$

with p_n such that (2) holds, then

$$\frac{\sqrt{k_n}}{\sigma_M \log(k_n/(np_n))}\left(\frac{\tilde{\chi}^{(i)}_{p_n,M}}{\chi_{p_n}} - 1\right) = R_k + \sqrt{k_n}\Big(c_1 A(n/k) + c_2 \frac{\chi_i}{U(n/k)}\Big)(1 + o_p(1))$$

holds, R_k is asymptotically a standard normal r.v., $\sigma_M^2 := \gamma^2 + 1$, $c_1 := \frac{\gamma(1-\rho)+\rho}{\gamma(1-\rho)^2}$ and $c_2 := \left(\frac{\gamma}{\gamma+1}\right)^2$.

Moreover, for:

1. $\gamma > -\rho$*: and* $\sqrt{k_n}A(n/k_n) \to \mu_1$*, finite, as* $n \to \infty$*, then the mean value is* $\mu_1 c_1$*;*

2. $\rho < -\gamma$ *and* $\sqrt{k_n}/U(n/k_n) \to \mu_2$*, finite, as* $n \to \infty$*, then the mean value is* $\mu_2 c_2 \chi_i$*;*

3. $\rho = -\gamma$*,* $\sqrt{k_n}A(n/k_n) \to \mu_1$*, finite, and* $\sqrt{k_n}/U(n/k_n) \to \mu_2$*, finite, as* $n \to \infty$*, then the mean value is* $\mu_1 c_1 + \mu_2 c_2 \chi_i$*.*

Observation: We define, as before, the

Modified-ML estimator as a function of the sample of excesses over a random threshold $X_{n_i:n}$, i.e.,

$$\tilde{\chi}^{(i)}_{p,ML} := (X_{n-k_n:n} - X_{n_i:n}) \left(\frac{k_n}{np_n}\right)^{\hat{\gamma}_n^{ML(i)}} + X_{n_i:n}, \quad i = 1, 2,$$

$$\hat{\gamma}_n^{ML(i)} := \gamma_n^{ML(\hat{\rho})}(\underset{\sim}{X}^{(i)})$$

with

$$\underset{\sim}{X}^{(i)} := \left(X_{n:n} - X_{n_i:n}, X_{n-1:n} - X_{n_i:n}, \cdots, X_{n_i+1:n} - X_{n_i:n}\right)_{,i=1,2}.$$

Theoretical properties for $\tilde{\chi}^{(i)}_{p_n,ML}$ are still under study...

5 - Simulation study

We will compare the exact performance of the following high quantile estimators, under shift-transformations:

$$\hat{\chi}^{W}_{p_n},\ \hat{\chi}^{M}_{p_n},\ \tilde{\chi}^{(i)}_{p_n,H},\ \tilde{\chi}^{(i)}_{p_n,M},\ \tilde{\chi}^{(i)}_{p_n,ML}\quad (i=1,2)$$

number of replicas $N = 200$;

sample size of $\underset{\sim}{X}$: $n = 1000$

$\underset{\sim}{X}$: Generated random variables as $X \frown F$ and shifted as $Y_j = X + \lambda_j\ (j = 1, 2)$

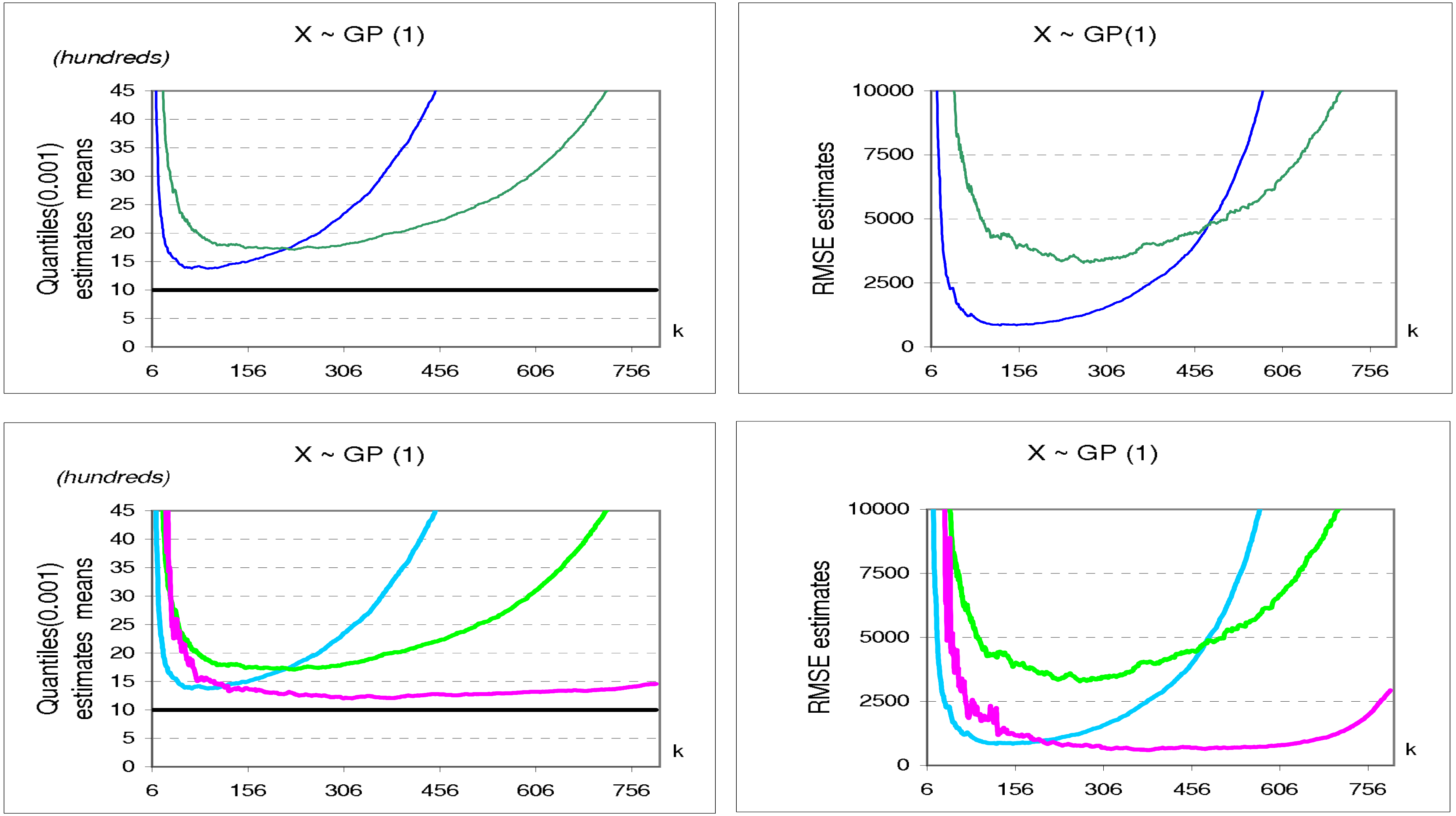
$\lambda_1 = -\chi_{.75}(X)$, if $x_F = 0$; $\lambda_1 = \chi_{.75}(X)$, if $x_F = -\infty$;

$\lambda_2 = \chi_{.01}(X)$.

Estimation of high quantile $\chi_p = F^{\leftarrow}(0.999); p = 0.001$

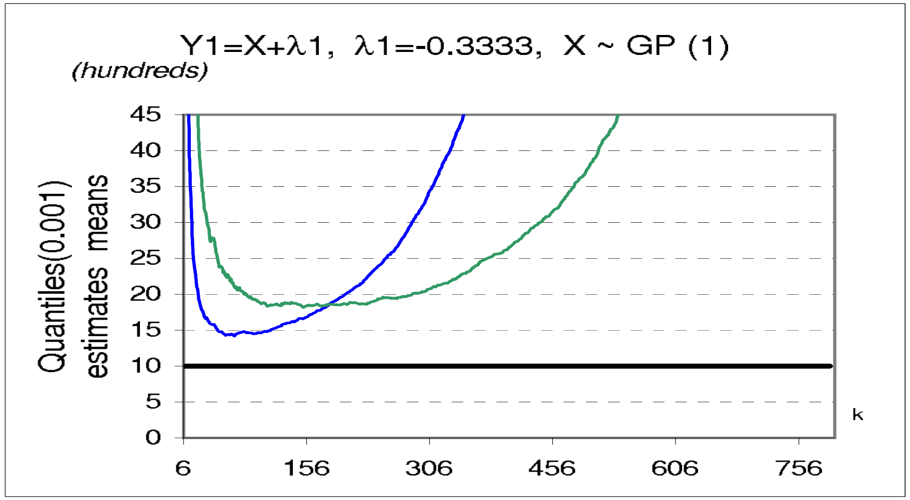
Means of $N = 200$ estimates and empirical **Root Mean Squared Error** (RMSE),
for $(k = 6, \cdots, 800)$.

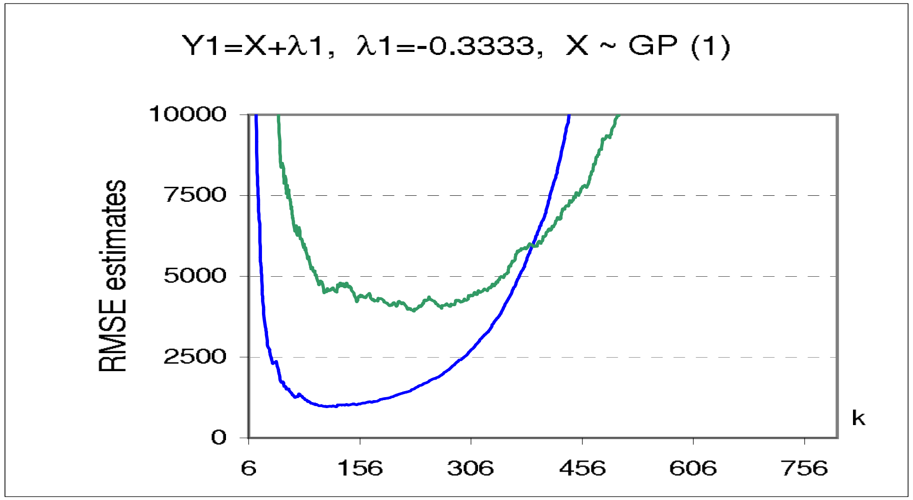
Generalized Pareto Model $(\gamma = 1; \rho = -1)$

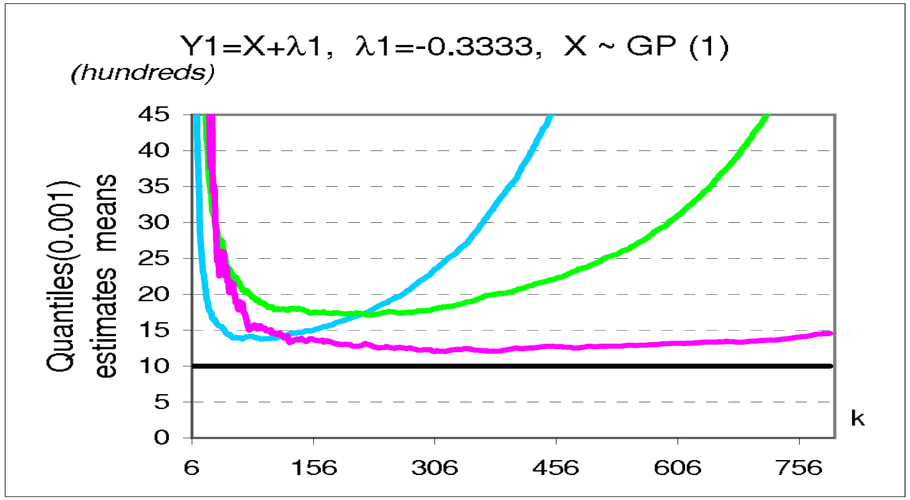


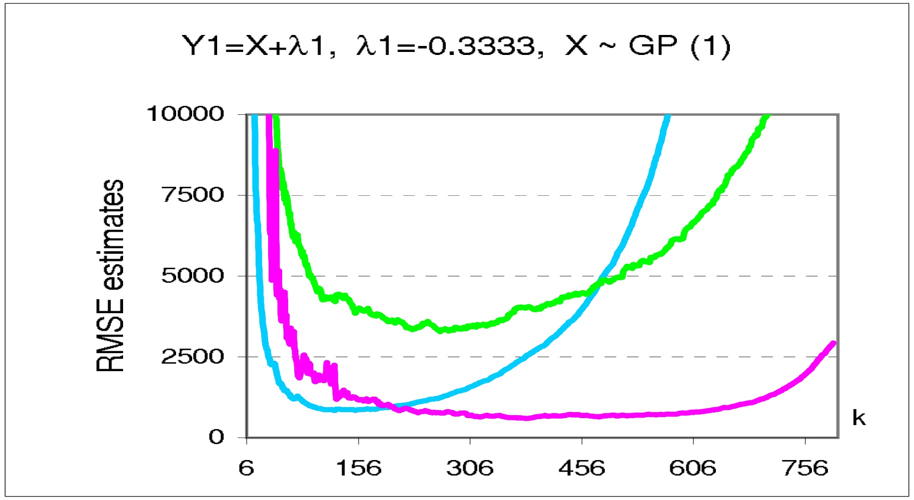
$\hat{\chi}^{W}_{p_n}$, $\hat{\chi}^{M}_{p_n}$, $\tilde{\chi}^{(1)}_{p_n,H}$, $\tilde{\chi}^{(1)}_{p_n,M}$, $\tilde{\chi}^{(1)}_{p_n,ML}$

Generalized Pareto Model $(\gamma = 1; \rho = -1)$



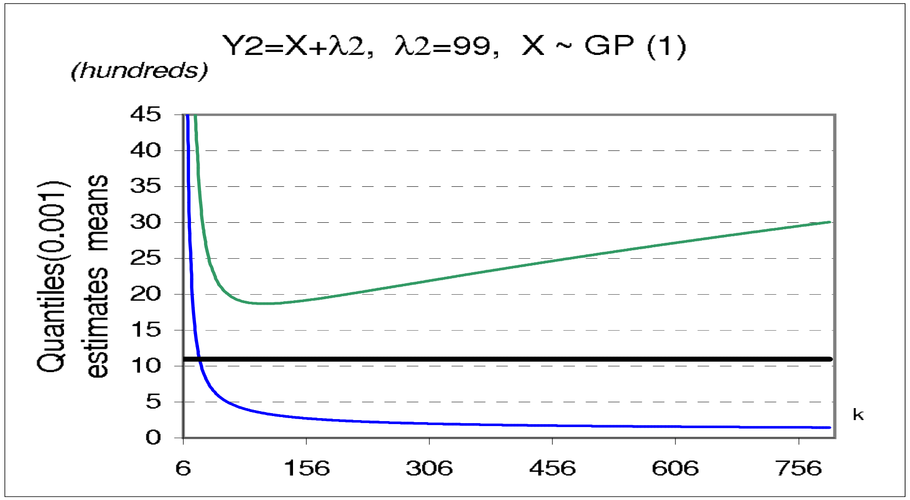


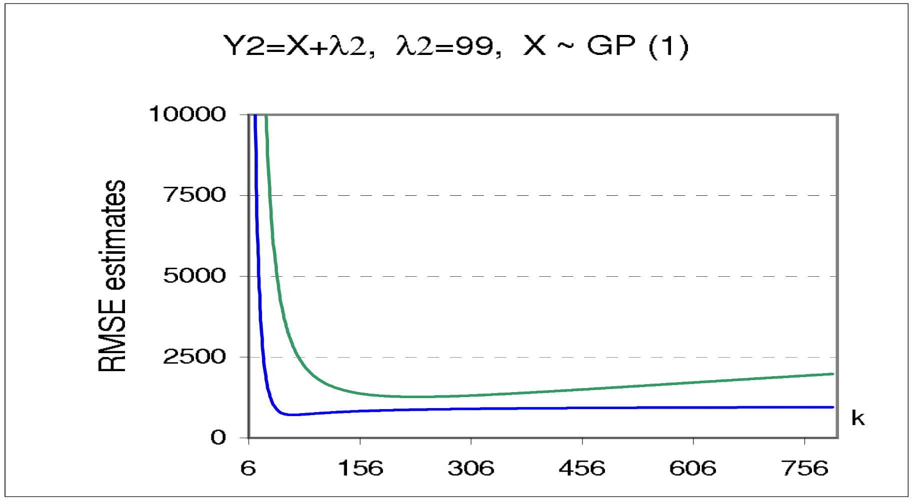


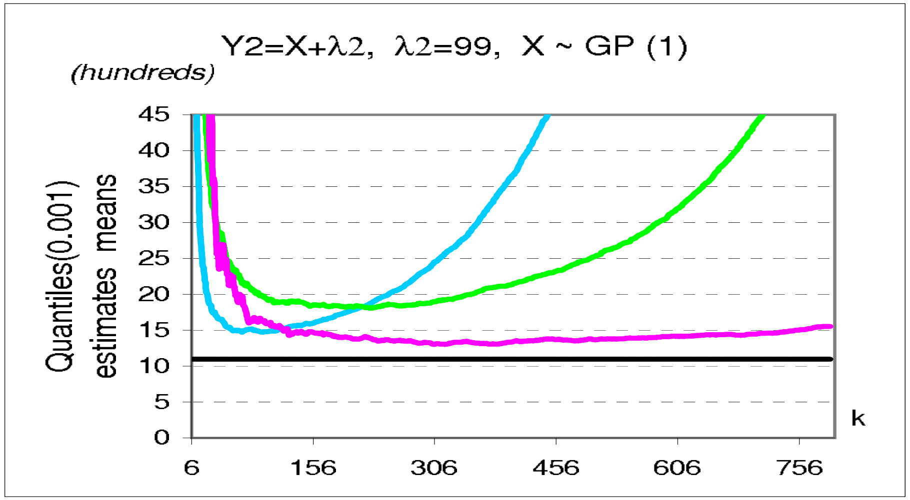


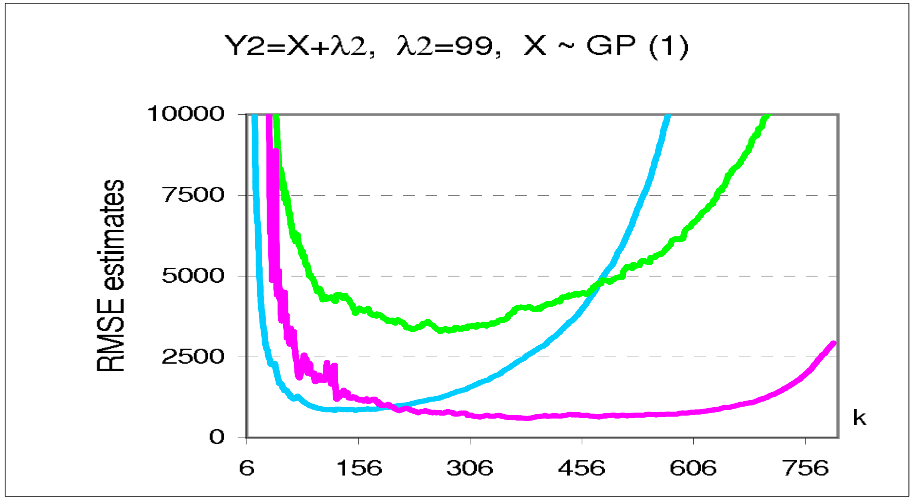
$$\hat{\chi}^{W}_{p_n},\ \hat{\chi}^{M}_{p_n},\ \tilde{\chi}^{(1)}_{p_n,H},\ \tilde{\chi}^{(1)}_{p_n,M},\ \tilde{\chi}^{(1)}_{p_n,ML}$$

Generalized Pareto Model ($\gamma = 1; \rho = -1$)



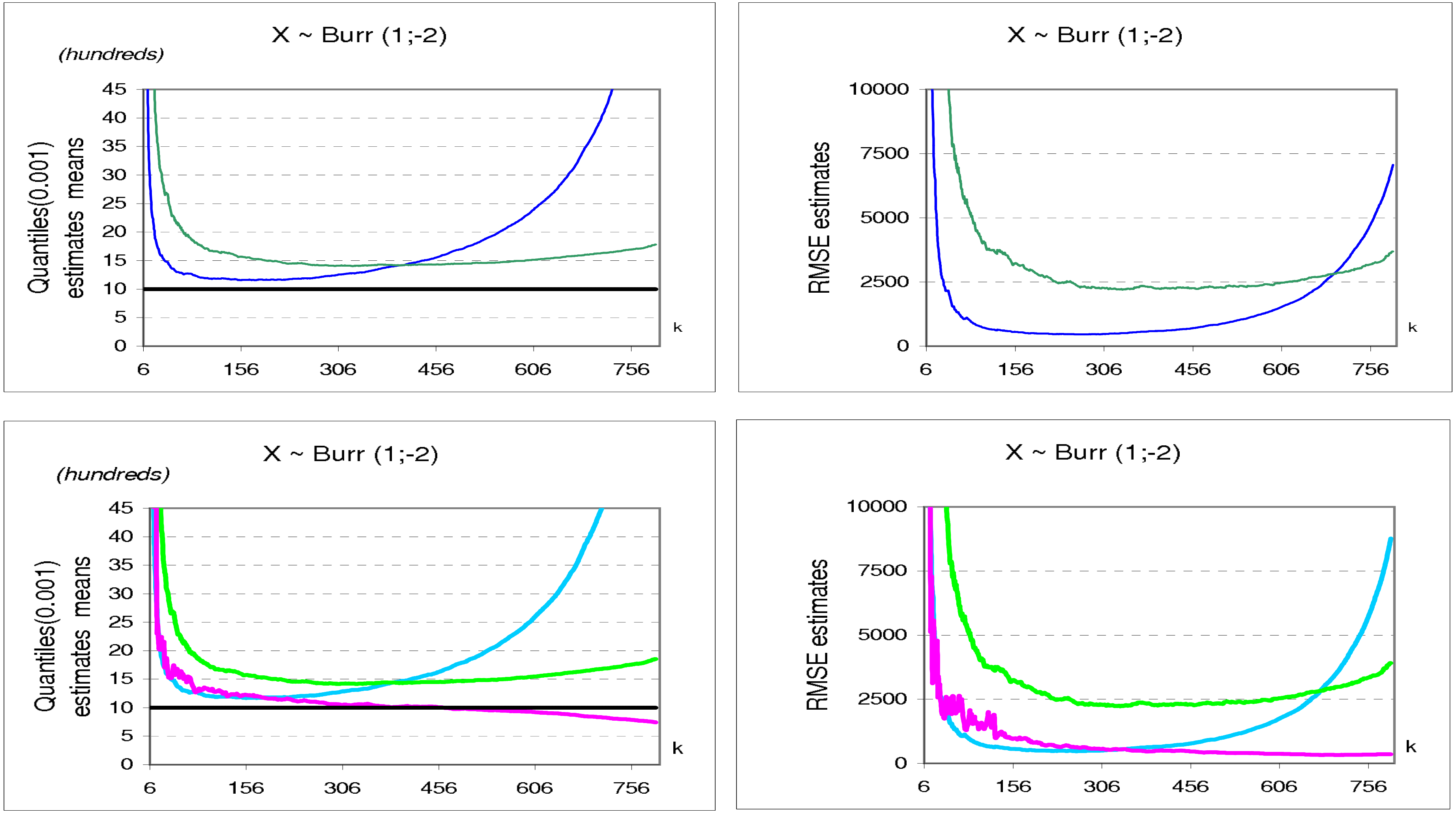






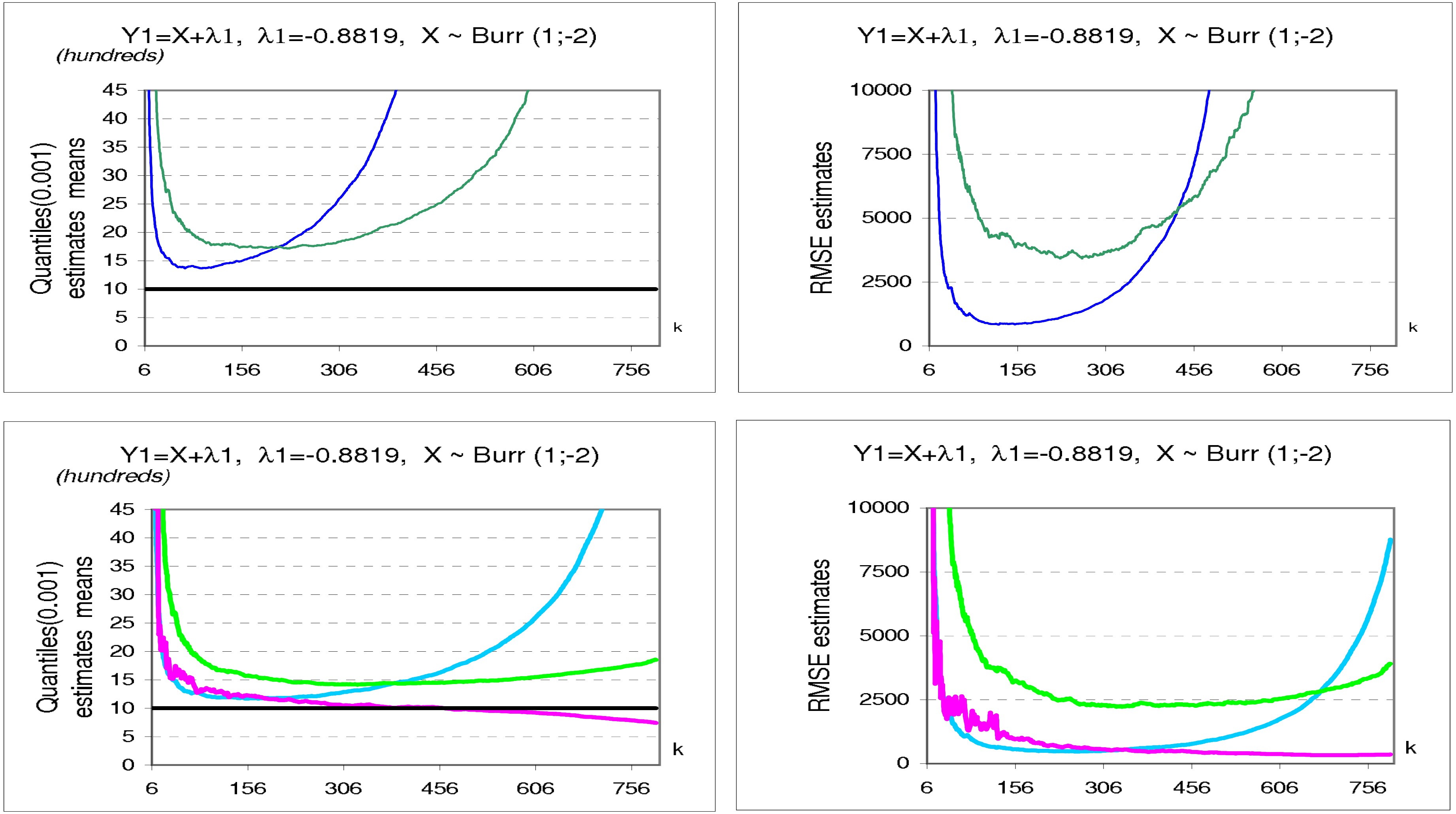
$\hat{\chi}^{W}_{p_n}$, $\hat{\chi}^{M}_{p_n}$, $\tilde{\chi}^{(1)}_{p_n,H}$, $\tilde{\chi}^{(1)}_{p_n,M}$, $\tilde{\chi}^{(1)}_{p_n,ML}$

Burr Model ($\gamma = 1; \rho = -2$)



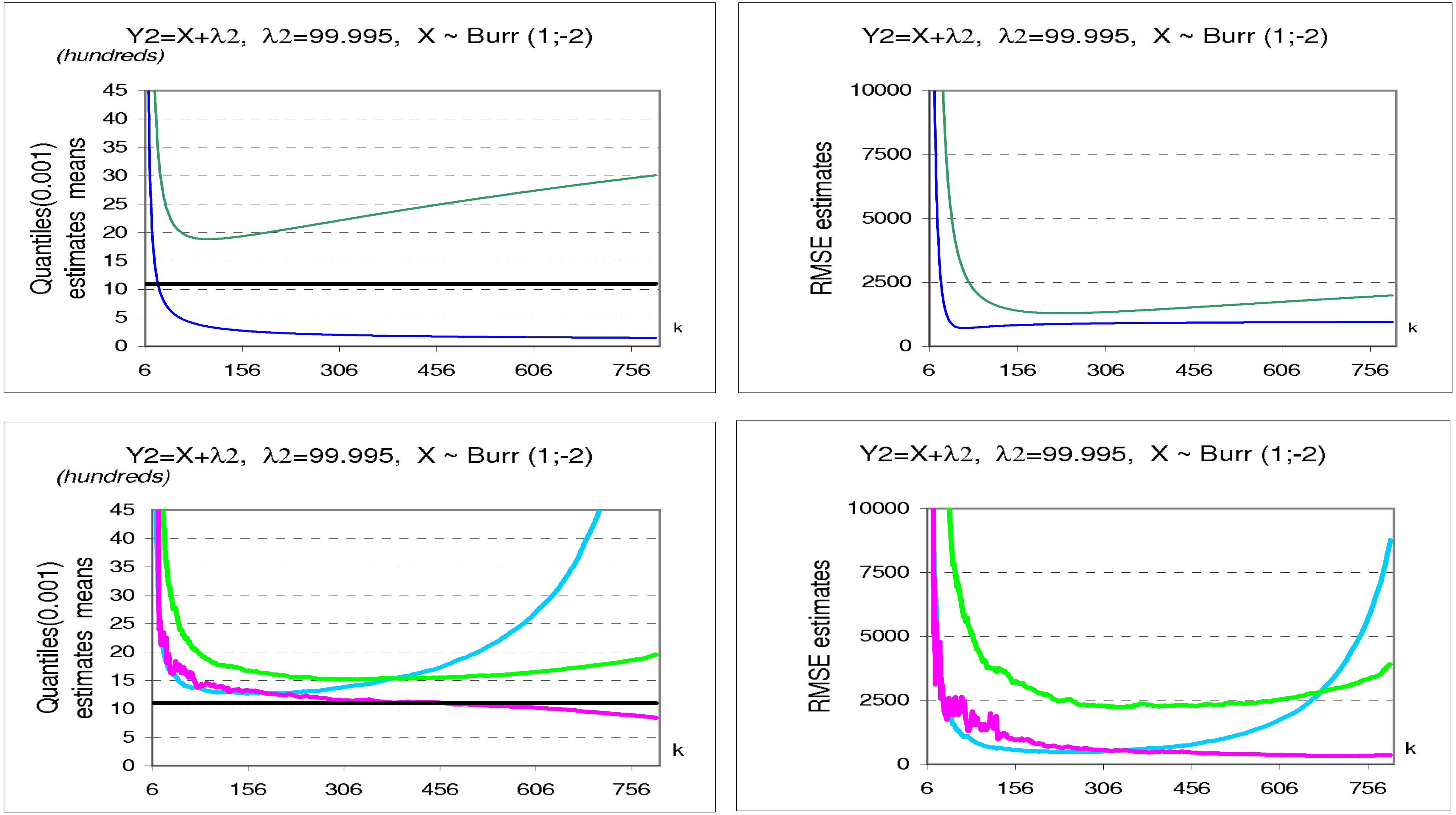
$\hat{\chi}^{W}_{p_n}$, $\hat{\chi}^{M}_{p_n}$, $\tilde{\chi}^{(1)}_{p_n,H}$, $\tilde{\chi}^{(1)}_{p_n,M}$, $\tilde{\chi}^{(1)}_{p_n,ML}$

Burr Model $(\gamma = 1; \rho = -2)$



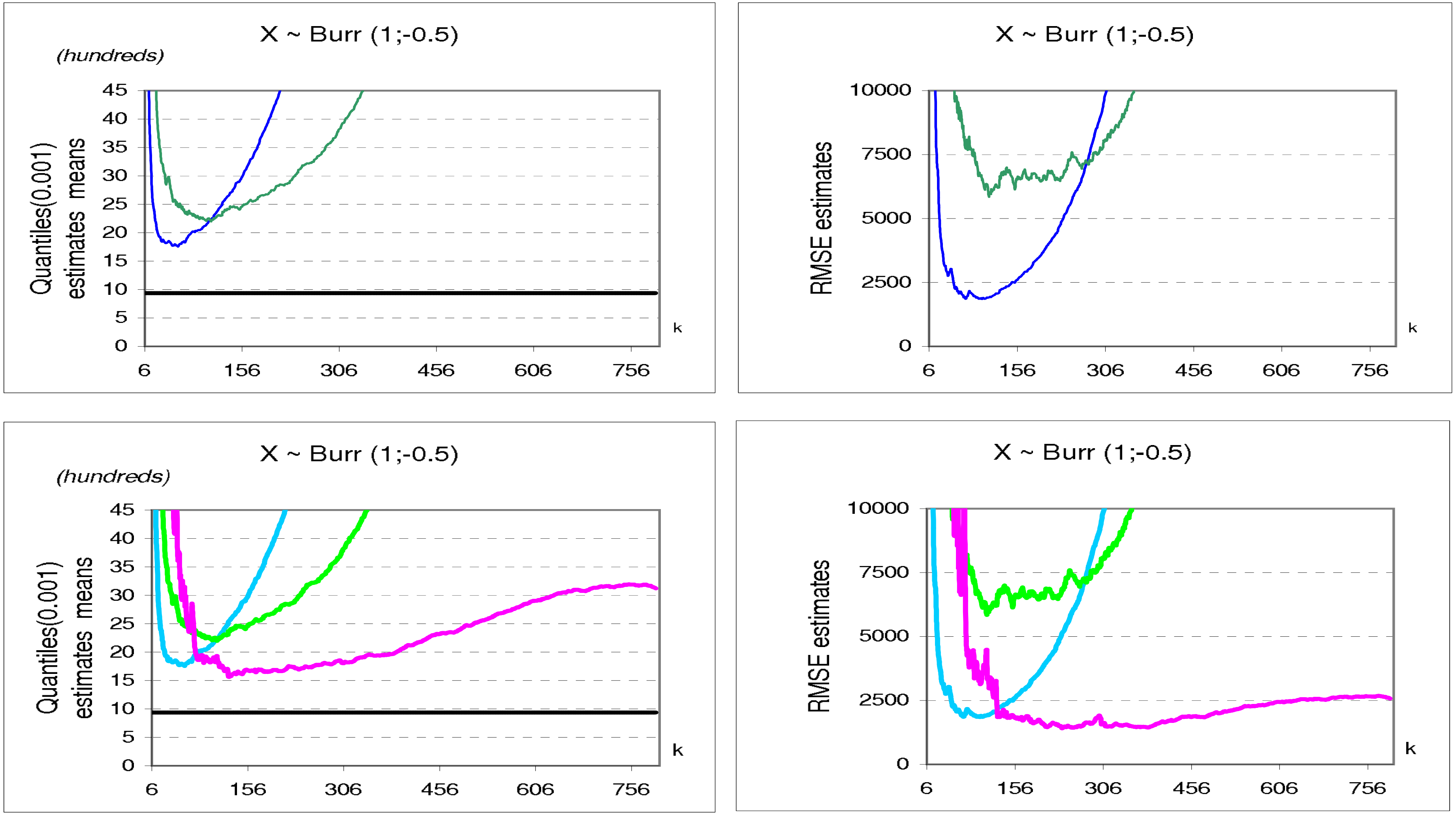
$\hat{\chi}^{W}_{p_n}$, $\hat{\chi}^{M}_{p_n}$, $\tilde{\chi}^{(1)}_{p_n,H}$, $\tilde{\chi}^{(1)}_{p_n,M}$, $\tilde{\chi}^{(1)}_{p_n,ML}$

Burr Model $(\gamma = 1; \rho = -2)$



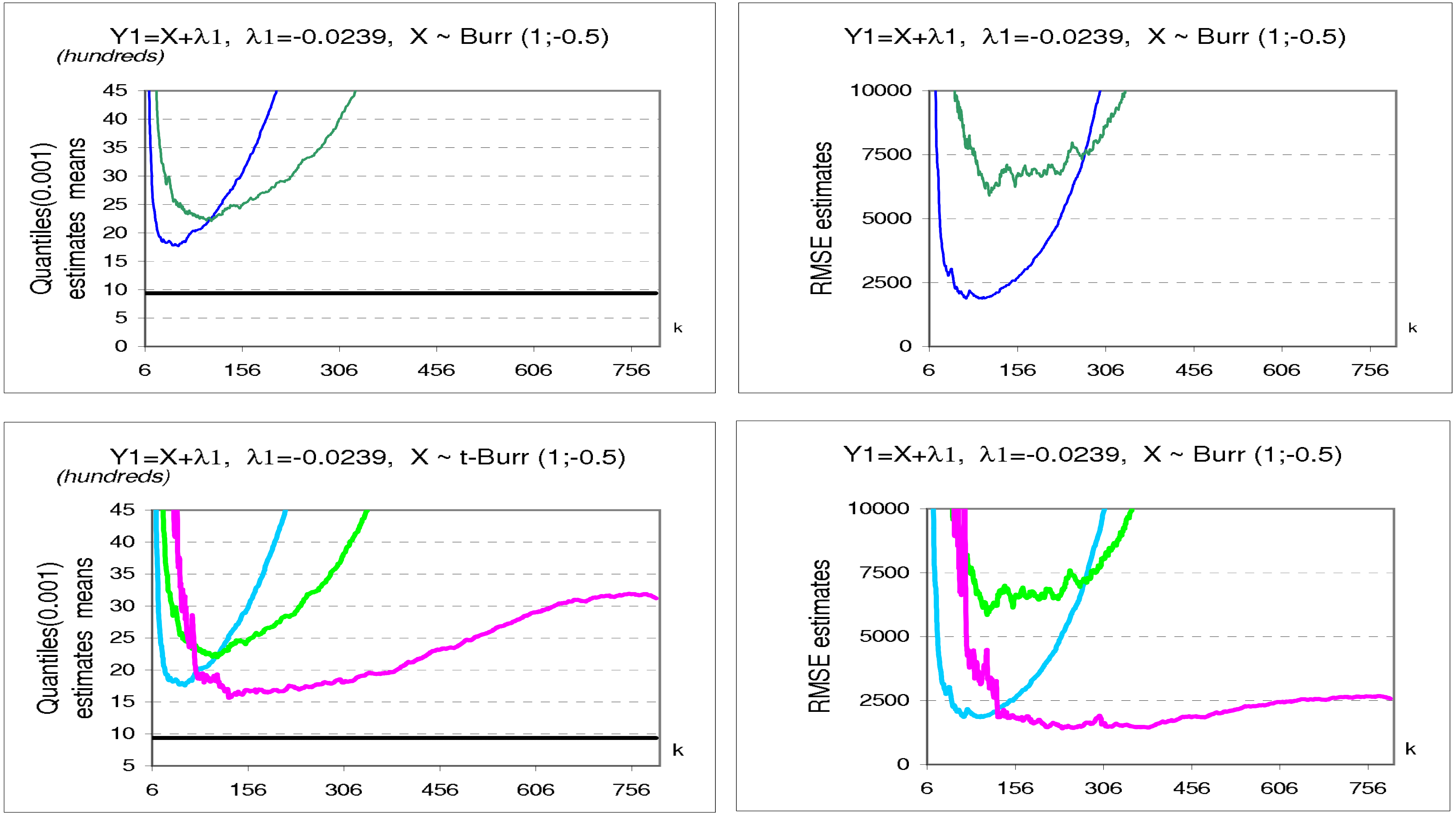
$\hat{\chi}^{W}_{p_n}$, $\hat{\chi}^{M}_{p_n}$, $\tilde{\chi}^{(1)}_{p_n,H}$, $\tilde{\chi}^{(1)}_{p_n,M}$, $\tilde{\chi}^{(1)}_{p_n,ML}$

Burr Model $(\gamma = 1; \rho = -0.5)$



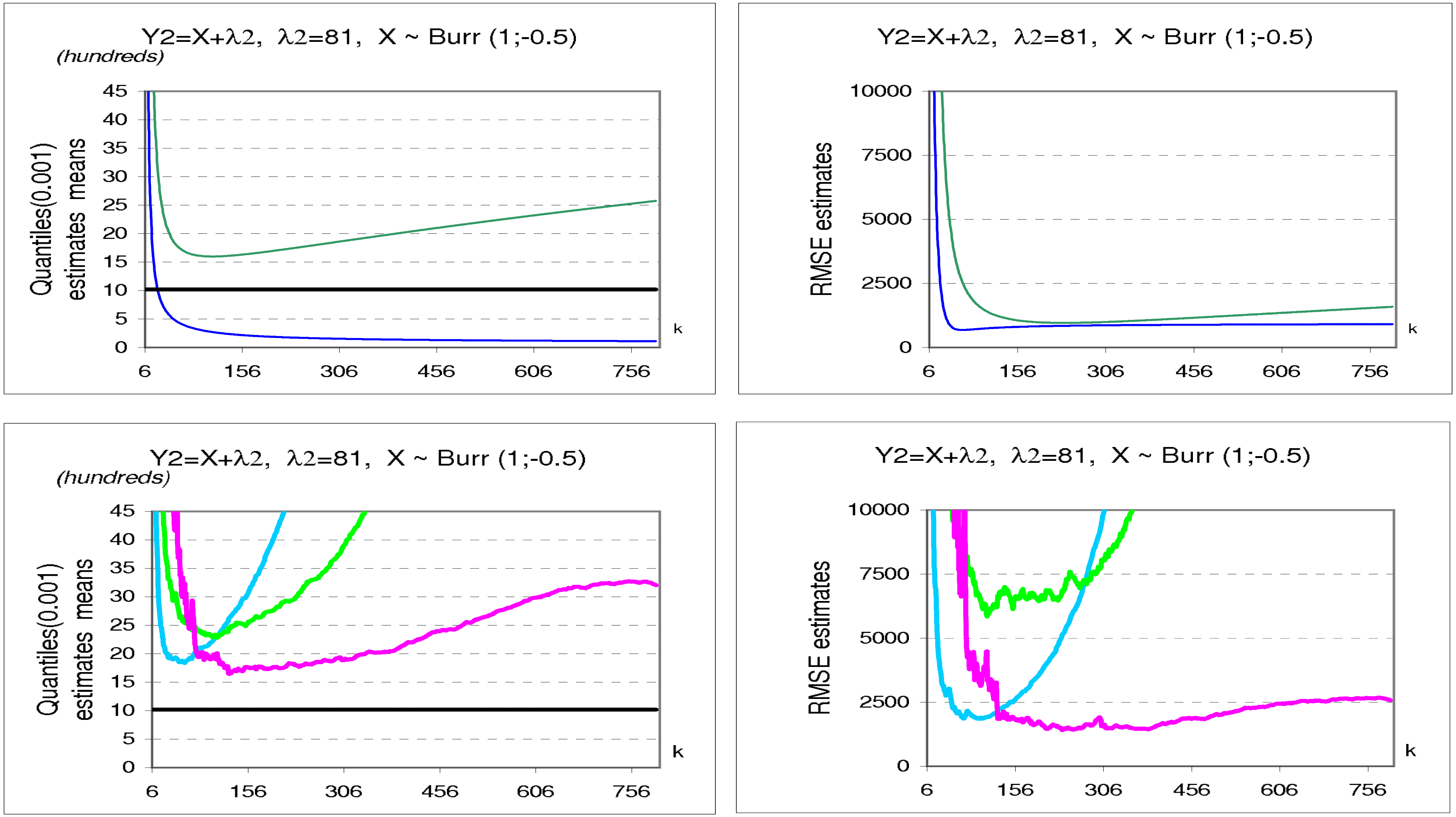
$$\hat{\chi}^{W}_{p_n},\ \hat{\chi}^{M}_{p_n},\ \tilde{\chi}^{(1)}_{p_n,H},\ \tilde{\chi}^{(1)}_{p_n,M},\ \tilde{\chi}^{(1)}_{p_n,ML}$$

Burr Model $(\gamma = 1; \rho = -0.5)$



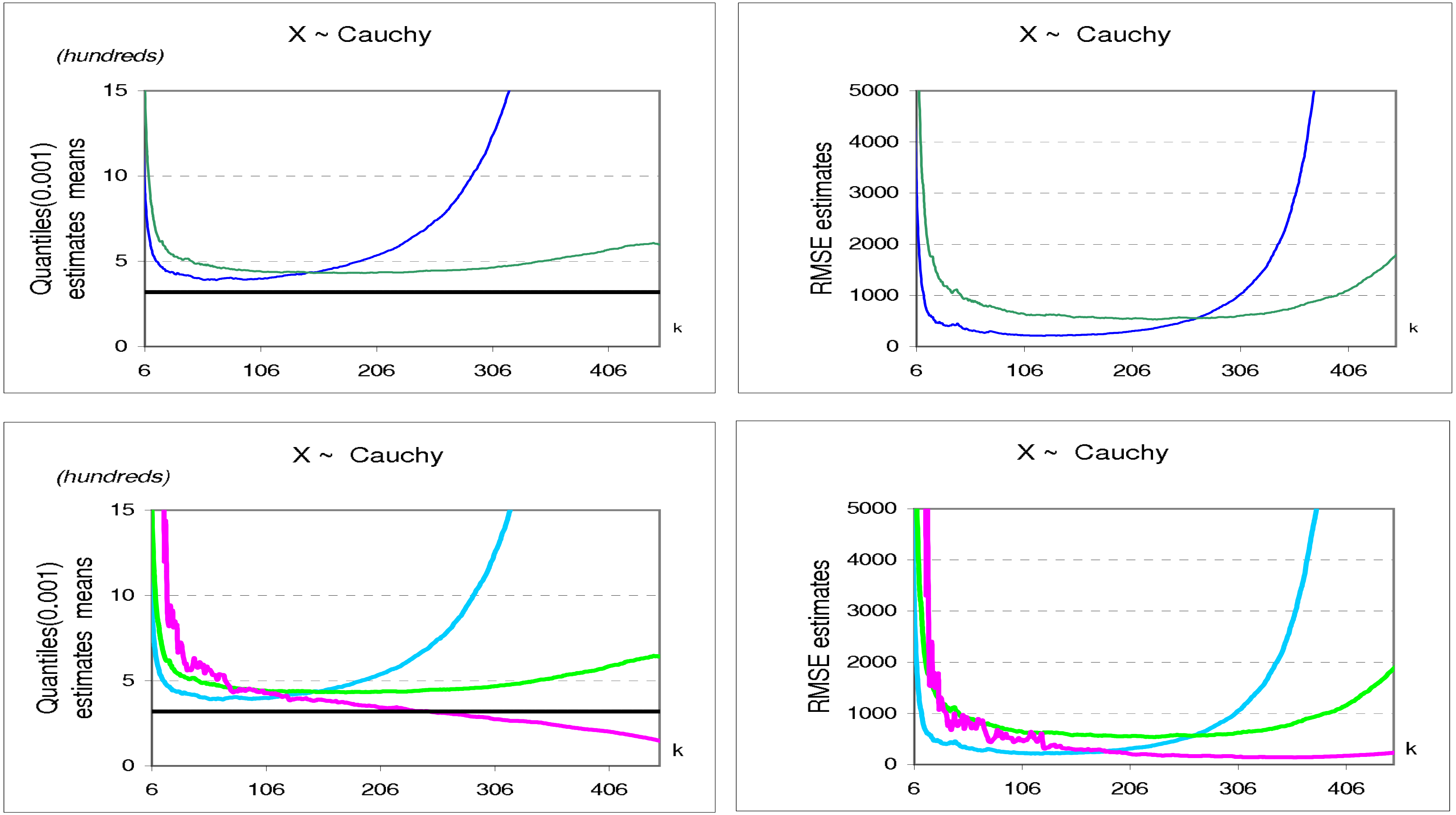
$\hat{\chi}^{W}_{p_n}$, $\hat{\chi}^{M}_{p_n}$, $\tilde{\chi}^{(1)}_{p_n,H}$, $\tilde{\chi}^{(1)}_{p_n,M}$, $\tilde{\chi}^{(1)}_{p_n,ML}$

Burr Model $(\gamma = 1; \rho = -0.5)$



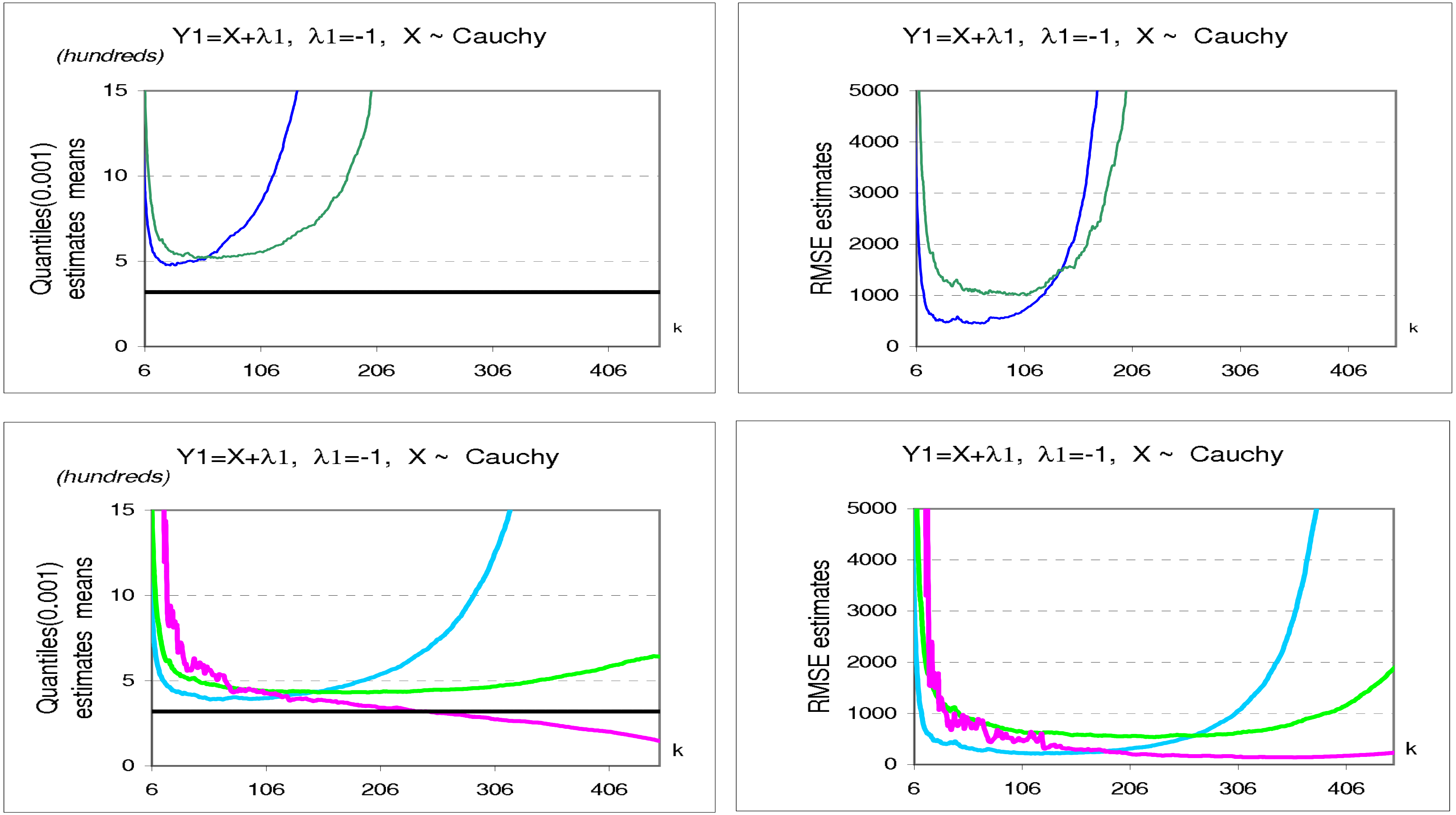
$\hat{\chi}^{W}_{p_n}$, $\hat{\chi}^{M}_{p_n}$, $\tilde{\chi}^{(1)}_{p_n,H}$, $\tilde{\chi}^{(1)}_{p_n,M}$, $\tilde{\chi}^{(1)}_{p_n,ML}$

Cauchy Model $(\gamma = 1; \rho = -2)$



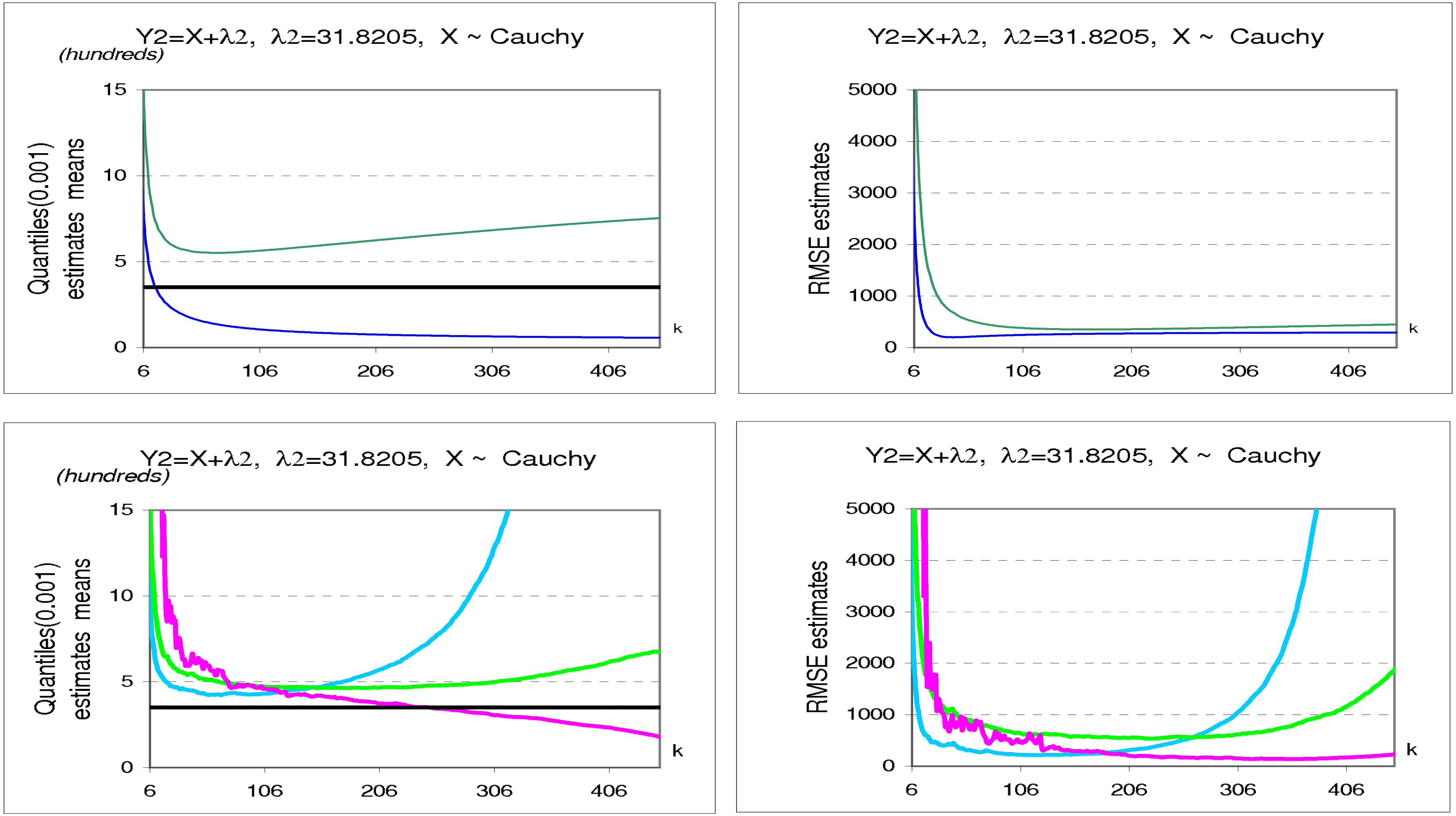
$\hat{\chi}^{W}_{p_n}$, $\hat{\chi}^{M}_{p_n}$, $\tilde{\chi}^{(2)}_{p_n,H}$, $\tilde{\chi}^{(2)}_{p_n,M}$, $\tilde{\chi}^{(2)}_{p_n,ML}$

Cauchy Model $(\gamma = 1; \rho = -2)$



$\hat{\chi}^{W}_{p_n}$, $\hat{\chi}^{M}_{p_n}$, $\tilde{\chi}^{(2)}_{p_n,H}$, $\tilde{\chi}^{(2)}_{p_n,M}$, $\tilde{\chi}^{(2)}_{p_n,ML}$

Cauchy Model $(\gamma = 1; \rho = -2)$



$\hat{\chi}^{W}_{p_n}$, $\hat{\chi}^{M}_{p_n}$, $\tilde{\chi}^{(2)}_{p_n,H}$, $\tilde{\chi}^{(2)}_{p_n,M}$, $\tilde{\chi}^{(2)}_{p_n,ML}$

We also compare the exact performance of the following high quantile estimators:

$$\hat{\chi}_{p_n}^{POTML}, \; \tilde{\chi}_{p_n,ML}^{(i)} \quad (i = 1, 2)$$

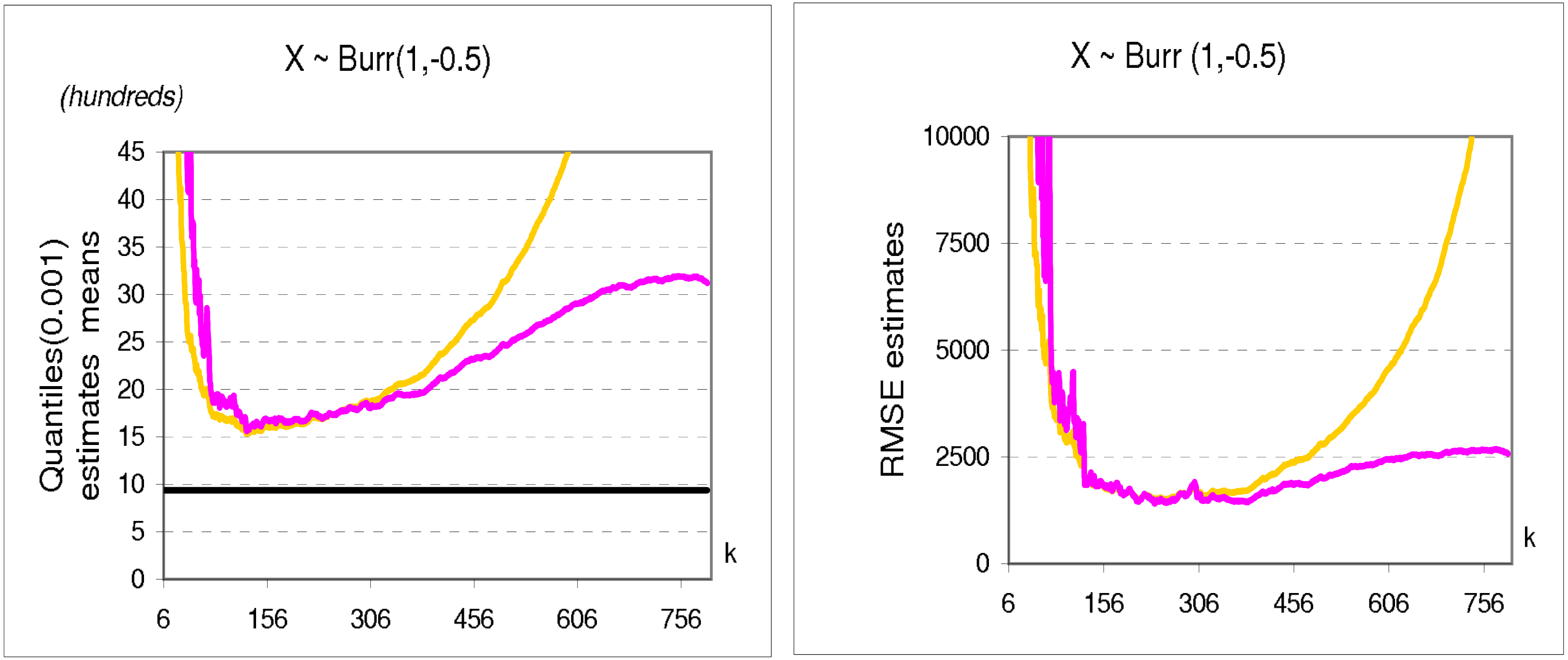
number of replicas $N = 200$;

sample size of $\underset{\sim}{X}$: $n = 1000$

Estimation of high quantile $\chi_p = F^{\leftarrow}(0.999); p = 0.001$

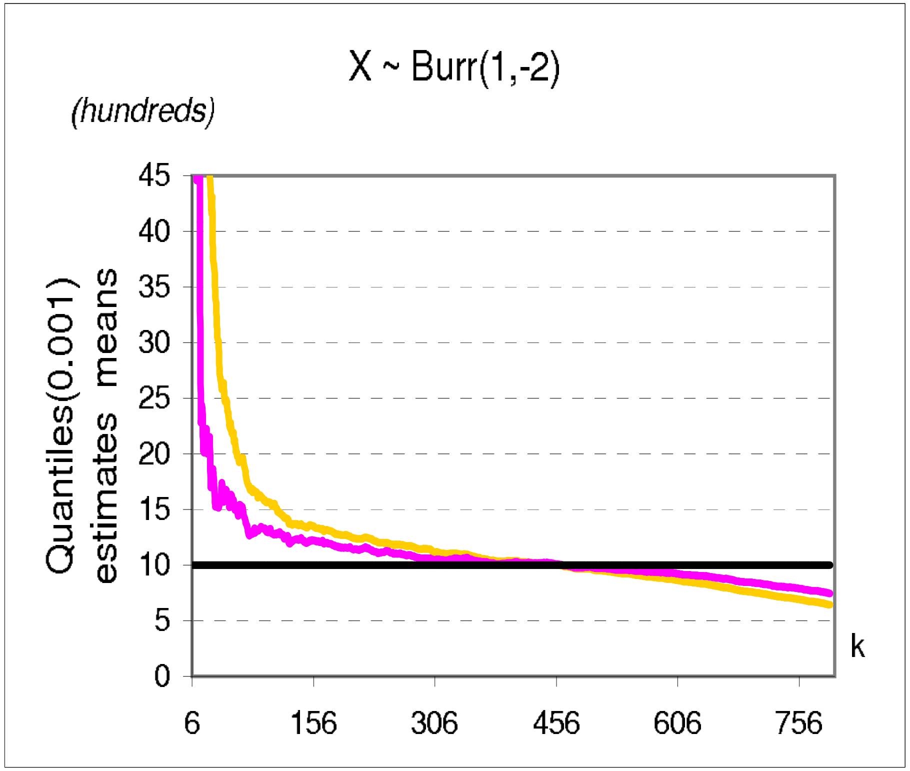
Means of $N = 200$ estimates and empirical Root Mean Squared Error (RMSE), for ($k = 6, \cdots, 800$).

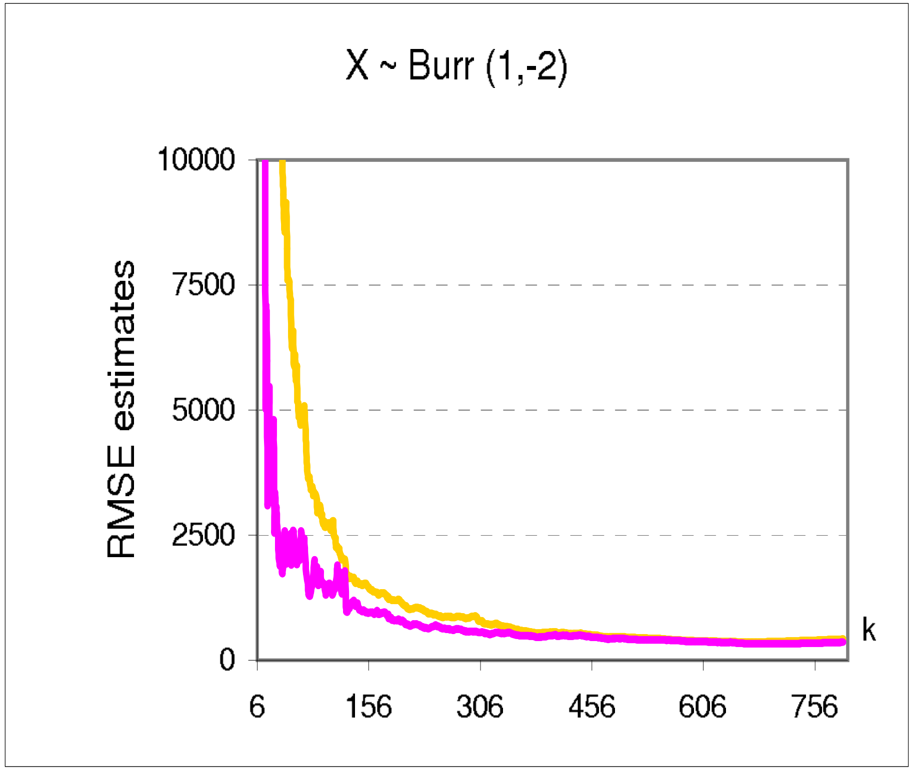
Burr Model $(\gamma = 1; \rho = -0.5)$



$\tilde{\chi}_{p_n}^{POTML}$, $\tilde{\chi}_{p_n,ML}^{(1)}$

Burr Model $(\gamma = 1; \rho = -2)$





$\tilde{\chi}_{p_n}^{POTML}$, $\tilde{\chi}_{p_n,ML}^{(1)}$

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