Shock models *

Allan Gut

Göteborg
August 16, 2005

*Much of this is joint work with Jürg Hüsler

Two main cases

Systems break down because of

- *cumulative* effect of shocks;
- *extreme* individual shock.

Notation

- $\{X_k\}$ magnitude of shocks;
- $\{Y_k\}$ time between shocks;
- $\{(X_k, Y_k)\}$ i.i.d.;
- $S_n = \sum_{k=1}^{n} X_k$, $T_n = \sum_{k=1}^{n} Y_k$;
- Means, variances: $\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \ldots$

Models

The cumulative case

$$\nu(t) = \text{min}\{n : S_n > t\}, \quad t \geq 0.$$

Lifetime/failure time: $T_{\nu(t)}$.

The extreme case

$$\tau(t) = \text{min}\{n : X_n > t\}, \quad t \geq 0.$$

Lifetime/failure time: $T_{\tau(t)}$.

Stopping times behave differently

— however...

Failure times are Stopped Random Walks.

But first ... a general problem

Suppose

- $\{Y_n,\, n \geq 1\}$ arbitrary;

- $Y_n \to Y$ as $n \to \infty$;

- $\{N(t),\, t \geq 0\}$ positive integer valued;

- $N(t) \to \infty$ as $t \to \infty$;

- in some sense, $\stackrel{a.s.}{\longrightarrow}$ $\stackrel{p}{\longrightarrow}$ $\stackrel{r}{\longrightarrow}$ $\stackrel{d}{\longrightarrow}$.

What about

$$Y_{N(t)} \stackrel{a.s.}{\longrightarrow} ? \quad \stackrel{p}{\longrightarrow} ? \quad \stackrel{r}{\longrightarrow} ? \quad \stackrel{d}{\longrightarrow} ???$$

Almost sure convergence

Proposition 1

Suppose

- $\{Y_n, \, n \geq 1\}$ arbitrary;
- $Y_n \stackrel{a.s.}{\to} Y$ as $n \to \infty$;
- $\{N(t), \, t \geq 0\}$ positive integer valued;
- $N(t) \stackrel{a.s.}{\to} \infty$ as $t \to \infty$.

Then

$$Y_{N(t)} \stackrel{a.s.}{\to} Y \quad \text{as} \quad t \to \infty.$$

Proof

Union of two nullsets.

The central limit theorem

Proposition 2 — Anscombe (Rényi)

Suppose

- $\{X_k,\ k \geq 1\}$ i.i.d.;
- $E\,X = 0$, $\quad \text{Var}\,X = \sigma^2 < \infty$;
- $S_n = \sum_{k=1}^n X_k,\ n \geq 1$;
- $\frac{N(t)}{t} \xrightarrow{p} \theta \quad \text{as} \quad t \to \infty \qquad (0 < \theta < \infty).$

Then

$$\frac{S_{N(t)}}{\sigma\sqrt{t\theta}} \xrightarrow{d} N(0,1) \quad \text{as} \quad t \to \infty.$$

Proof

CLT + Kolmogorov's inequality.

Two dimensions (with Svante Janson)

♣ $\{(U_n^{(x)}, U_n^{(y)}), n \geq 1\}$ r.w.,

♣ i.i.d. increments $\{(X_k, Y_k), k \geq 1\}$,

♣ $\mu_y = E\,Y_1 > 0$, $\mu_x = E\,X_1$ exists.

♣ First passage time process:

$$\tau(t) = \min\{n : U_n^{(y)} > t\}, \quad t \geq 0.$$

Problem:

What about

$$\{U_{\tau(t)}^{(x)}, \ t \geq 0\}\,?$$

LLN for $U^{(x)}_{\tau(t)}$

$$\frac{U^{(x)}_{\tau(t)}}{t} \xrightarrow{a.s.} \frac{\mu_x}{\mu_y} \quad \text{as} \quad t \to \infty$$

CLT for $U^{(x)}_{\tau(t)}$

If $\gamma^2 = \text{Var}\,(\mu_y X_1 - \mu_x Y_1) > 0$, then

$$\frac{U^{(x)}_{\tau(t)} - \frac{\mu_x}{\mu_y} t}{\sqrt{\mu_y^{-3} \gamma^2\, t}} \xrightarrow{d} N(0,1) \quad \text{as} \quad t \to \infty.$$

Many applications   Typically:

- $\{Y_k\}$ times,

- $\{X_k\}$ marks / rewards .....

Stopped random walks. Springer (1988).

Back to shocks ...

Cumulative shocks

♯ $\{X_k\}$ magnitude of shocks,

♯ $\{Y_k\}$ time between shocks,

♯ $S_n = \sum_{k=1}^{n} X_k$, $T_n = \sum_{k=1}^{n} Y_k$,

♯ $\nu(t) = \min\{n : S_n > t\}$.

Theorem 1

(i) If $\mu_x > 0$, and $|\mu_y| < \infty$, then

$$\frac{T_{\nu(t)}}{t} \xrightarrow{a.s.} \frac{\mu_y}{\mu_x} \quad \text{as} \quad t \to \infty.$$

(ii) If, in addition, $\gamma^2 > 0$, then

$$\frac{T_{\nu(t)} - \frac{\mu_y}{\mu_x} t}{\sqrt{\mu_x^{-3} \gamma^2 t}} \xrightarrow{d} N(0,1) \quad \text{as} \quad t \to \infty.$$

Extreme shocks

- ♭ $x_F := \sup\{x : F(x) < 1\}$,
- ♭ $p_t = P(X_1 > t)$.
- ♭ Stopping times:

$$\tau(t) = \min\{n : X_n > t\}, \quad t \geq 0.$$

Then, $\tau(t)$ geometric, mean $1/p_t$.

Theorem 2

If $p_t \to 0$ as $t \to x_F$, then

(i) $p_t \tau(t) \xrightarrow{d} \text{Exp}(1)$ as $t \to x_F$.

(ii) Suppose $|\mu_y| < \infty$. Then

$$p_t T_{\tau(t)} \xrightarrow{d} \mu_y \, \text{Exp}(1) \quad \text{as} \quad t \to x_F.$$

Cont'd

Proof

$$p_t T_{\tau(t)} \;=\; \frac{T_{\tau(t)}}{\tau(t)} \cdot p_t \tau(t) \xrightarrow{d} \mu_y \,\text{Exp}(1)$$
$$(= \quad \text{Exp}(\mu_y) \quad \text{if} \quad \mu_y > 0).$$

Note:

No LLN for $\tau(t)$; no Anscombe.

Also

Weak convergence in $D[0, \infty)$.

Mixed shock models

The system breaks down when

- the cumulative shocks reach "some high" level or

- a single "large" shock appears

whichever comes first, viz.,

the system breaks down at

$$\min\{\nu(t), \tau(t)\}.$$

However,

$$\nu(t) = O(t) \qquad \tau(t) = O(1/p_t),$$

so that, necessary for nontrivial results:

- $\nu(t) \sim \tau(t)$,
- $x_F = \infty$.

Define λ_t, the θ/t-quantile:

$$P(X_1 > \lambda_t) = \theta/t,$$

and set

$$\tau_\lambda(t) = \min\{n : X_n > \lambda_t\}, \quad t \geq 0.$$

The system breaks down at time

$$\kappa(t) = \min\{\nu(t), \tau_\lambda(t)\}.$$

Applications/examples ......

- Boxing. A knock-out may be caused by many small punches or a real big one.

- Rain in Uppsala. On August 17, 1997, Uppsala had extreme rain during one hour; the basement at home was flooded. A year later again, but due to rain, on and off, for some days.
  More generally: Flooding in rivers or dams.

- Fatigue, tenacity. A rope, a wire.
  Less generally: A coat hanger.

- Environmental damage. A factory may on and off leak poisonous waste products into a river killing the vegetation and the fish. Or: some catastrophy.

- Radioactivity. A variation on the previous example; many minor emissions or a sudden melt-down.

Theorem

If $\mu_x > 0$, $|\mu_y| < \infty$, then

(a) $\frac{\kappa(t)}{t} \xrightarrow{d} Z$ as $t \to \infty$, where

$$\begin{cases} f_Z(y) = \theta e^{-\theta y},\ 0 < y < 1/\mu_x, \\ P(Z = 1/\mu_x) = e^{-\theta/\mu_x}, \end{cases}$$

or, equivalently,

$$F_Z(y) = \begin{cases} 1 - e^{-\theta y}, & \text{for} \quad 0 < y < 1/\mu_x, \\ 1, & \text{for} \quad y \geq 1/\mu_x, \end{cases}$$

(b) $\frac{T_{\kappa(t)}}{t} \xrightarrow{d} \mu_y Z$ as $t \to \infty$,

(c) $\frac{S_{\kappa(t)}}{t} \xrightarrow{d} \mu_x Z$ as $t \to \infty$,

(d) $\frac{X_{\kappa(t)}}{t} \xrightarrow{p} 0$ as $t \to \infty$.

Results for moments also exist.

Basic tool

Proposition 3

$\{U_t,\, t \geq 0\}$ and $\{V_t,\, t \geq 0\}$.

Suppose that

$$U_t \xrightarrow{p} a \in \mathbb{R} \quad \text{and} \quad V_t \xrightarrow{d} V \quad \text{as} \quad t \to \infty.$$

Then, as $t \to \infty$,

$$P(\min\{U_t, V_t\} > y) \to \begin{cases} P(V > y), & \text{for} \quad y < a, \\ 0, & \text{for} \quad y > a, \end{cases}$$

and

$$P(\max\{U_t, V_t\} \leq y) \to \begin{cases} 0, & \text{for} \quad y < a, \\ P(V \leq y), & \text{for} \quad y > a. \end{cases}$$

Note: Point masses at $y = a$.

Comparing stopping times

For example, as $t \to \infty$:

$$\frac{E\,\nu(t)}{t} \to \frac{1}{\mu_x},$$

$$\frac{E\,\tau_\lambda(t)}{t} \to \frac{1}{\theta},$$

$$\frac{E\,\kappa(t)}{t} \to \frac{1}{\theta}\Big(1 - e^{-\theta/\mu_x}\Big) \leq \begin{cases} \frac{1}{\mu_x}, \\ \frac{1}{\theta}. \end{cases}$$

Note

$\lim_{t\to\infty} E\,\kappa(t)/t$ smallest (of course).

In particular

$\theta = \mu_x$:

$$\frac{E\,\nu(t)}{t} \sim \frac{E\,\tau_\lambda(t)}{t} \sim \frac{1}{\mu_x},$$

$$\frac{E\,\kappa(t)}{t} \sim \frac{1}{\mu_x}\Big(1 - e^{-1}\Big) \sim 0.632\frac{1}{\mu_x}.$$

Comparing failure times

$$\frac{E\,T_{\nu(t)}}{t} \longrightarrow \frac{\mu_y}{\mu_x} \quad \text{as} \quad t \to \infty,$$

$$\frac{E\,T_{\tau_\lambda(t)}}{t} \longrightarrow \frac{\mu_y}{\theta} \quad \text{as} \quad t \to \infty,$$

$$\frac{E\,T_{\kappa(t)}}{t} \longrightarrow \frac{\mu_y}{\theta}(1 - e^{-\mu_y/\theta}) \quad \text{as} \quad t \to \infty.$$

In particular

$\theta = \mu_x$:

$$\frac{E\,T_{\nu(t)}}{t} \sim \frac{E\,T_{\tau_\lambda(t)}}{t} \sim \frac{\mu_y}{\mu_x},$$

$$\frac{E\,T_{\kappa(t)}}{t} \sim (1 - e^{-1})\,\frac{\mu_y}{\mu_x} \approx 0.632\,\frac{\mu_y}{\mu_x}.$$

More realistically

- ♡ "Minor shocks" have no long time effect;
- ♡ "Discount" of earlier shocks;
- ♡ Level varies as $t \nearrow$.

Which necessitates limit theorems for

- ♡ Delayed sums;
- ♡ Windows;
- ♡ with/without random size.

More general setup

- Delayed sums, lag sums, windows

$$S_{k,n} = \sum_{j=n-k+1}^{n} X_j, \quad 1 \le k \le n.$$

- Let $k_n \sim cn^\gamma$, $0 < \gamma < 1$. Consider

$$S_{k_n,n} = \sum_{j=n-k_n+1}^{n} X_j, \quad n \ge 1.$$

- $\nu(t) = \min\{n : S_{k_n,n} > t\}, \quad t \ge 0.$

Note

$T_{\nu(t)}$ = time until failure,

$T_{k_{\nu(t)},\nu(t)}$ = duration of the fatal window.

Strong laws as $t \to \infty$

$$\frac{k_{\nu(t)}}{t} \xrightarrow{a.s.} \frac{1}{\mu_x},$$

$$\frac{\nu(t)}{t^{1/\gamma}} \xrightarrow{a.s.} \frac{1}{(c\mu_x)^{1/\gamma}},$$

$$\frac{S_{k_{\nu(t)},\nu(t)}}{t} \xrightarrow{a.s.} 1,$$

$$\frac{S_{\nu(t)}}{t^{1/\gamma}} \xrightarrow{a.s.} \frac{\mu_x^{1-(1/\gamma)}}{c^{1/\gamma}},$$

$$\frac{T_{k_{\nu(t)},\nu(t)}}{t} \xrightarrow{a.s.} \frac{\mu_y}{\mu_x},$$

$$\frac{T_{\nu(t)}}{t^{1/\gamma}} \xrightarrow{a.s.} \frac{\mu_y}{(c\mu_x)^{1/\gamma}}$$

Interpretation

- Size of the fatal window $= \mathcal{O}(t)$;
- Total number of shocks at failure $= \mathcal{O}(t^{1/\gamma})$;
- Shock load of fatal window is $= \mathcal{O}(t)$;
- Complete shock load at failure $= \mathcal{O}(t^{1/\gamma})$;
- Duration of fatal window $= \mathcal{O}(t)$;
- Total lifetime $= \mathcal{O}(t^{1/\gamma})$.

Proofs follow the same technique
Asymptotic normality also provable.

A further extension

Recall the *Extreme shock model*:
One "very" large shock is fatal.
What about "some" rather large shocks?

In addition to fatal/nonfatal shocks,
introduce harmful shocks
that weaken the system.

Let

$$t = \alpha_t(0) \geq \alpha_t(1) \geq \alpha_t(2) \geq \cdots \geq \beta_t.$$

A shock X is

$$\begin{cases} \text{fatal,} & \text{if} \quad X > t, \\ \text{harmful, nonfatal,} & \text{if} \quad \beta_t < X < t, \\ \text{innocent,} & \text{if} \quad X < \beta_t. \end{cases}$$

If X_i is harmful, nonfatal, then X_j, $j > i$ is

$$\begin{cases} \text{fatal,} & \text{if} \quad X_j > \alpha_t(1), \\ \text{harmful, nonfatal,} & \text{if} \quad \beta_t < X_j < \alpha_t(1), \\ \text{innocent,} & \text{if} \quad X < \beta_t. \end{cases}$$

And so on.

Results

With

$$L_t(n) = \#\{i \le n : X_i \ge \beta_t\} \quad (L_t(0) = 0),$$

and

$$\tau(t) = \min\{n : X_n \ge \alpha_t(L_t(n-1))\},$$

one obtains

$$P(\tau(t) > m) \;=\; \sum_{j=0}^{m} \binom{m}{j} F^{m-j}(\beta_t) \times \prod_{k=0}^{j-1} \Big(F(\alpha_t(k)) - F(\beta_t)\Big).$$

Special cases

$$\alpha_t(k) = t \quad \longleftrightarrow \quad \text{nonfatal=harmless,}$$

$$\beta_t = t \quad \longleftrightarrow \quad \text{extreme model}$$

Stopping asymptotics

Theorem 3 If $1-F(\beta_t)\to 0$, and

$$\frac{1-F(\alpha_t(k))}{1-F(\beta_t)}\to c_k \quad \text{for} \quad k=1,2,\ldots,$$

then

$$P\Big((1-F(\beta_t))\tau(t)>z\Big)\to\sum_{j\ge 0}e^{-z}\frac{z^j}{j!}\prod_{k=0}^{j-1}(1-c_k).$$

Theorem 4 If $1-F(\alpha_t(\infty))\to 0$, and

$$\frac{1-F(\alpha_t(k))}{1-F(\alpha_t(\infty))}\to_{(t\to\infty)} a_k \nearrow_{(k\to\infty)} 1,$$

$$\frac{1-F(\alpha_t(\infty))}{1-F(\beta_t)}\to 0,$$

then

$$P\Big((1-F(\alpha_t(\infty))\tau(t)>z\Big)\to e^{-z}.$$

Lifetime asymptotics

Theorem 5 Suppose that $|\mu_y| < \infty$.
Under conditions of Theorems 1 or 2,

$$p_t \, T_{\tau(t)} \xrightarrow{d} \mu_y \, Z \quad \text{as} \quad t \to \infty,$$

where Z is as in Theorem 3 or 4.

Mixing again

Set

$$\kappa(t) = \min(\nu(t), \tau(t)),$$
$$\kappa^*(t) = \max(\nu(t), \tau(t)).$$

Quantiles via

$$P(X_1 > u_t) = \theta t^{-1/\gamma},$$

for $u_t = \beta(t)$ and $\alpha_t(\infty)$, respectively.

Theorem 6

Under earlier conditions,

$$\kappa(t)/t^{1/\gamma} \xrightarrow{d} \min\{(1/c\mu_x)^{1/\gamma}, Z/\theta\},$$

and

$$\kappa^*(t)/t^{1/\gamma} \xrightarrow{d} \max\{(1/c\mu_x)^{1/\gamma}, Z/\theta\}.$$

Note

Thus, $\mathcal{O}(t^{1/\gamma})$ instead of $\mathcal{O}(t)$.