Extremal behavior of stochastic integrals driven by regularly varying Lévy processes

Henrik Hult

joint with Filip Lindskog

http://www.orie.cornell.edu/~hult

18th August, 2005

Preliminaries

Let $Z_1, Z_2, \ldots$ be iid random variables with d.f. F. If F is subexponential ($1 - F$ regularly varying), then

$$P\left(\sum_{i=1}^{n} Z_i > u\right) \sim nP(Z_1 > u) \qquad (\sim nu^{-\alpha}L(u)) \,.$$

Preliminaries

Let $Z_1, Z_2, \ldots$ be iid random variables with d.f. F. If F is subexponential ($1-F$ regularly varying), then

$$P\left(\sum_{i=1}^{n} Z_i > u\right) \sim nP(Z_1 > u) \qquad (\sim nu^{-\alpha}L(u)).$$

If $N \sim \text{Po}(\lambda)$ is independent of $Z_1, Z_2, \ldots$, then

$$P\left(\sum_{i=1}^{N} Z_i > u\right) \sim E(N)P(Z_1 > u) \qquad (\sim E(N)u^{-\alpha}L(u)).$$

Preliminaries

Let $Z_1, Z_2, \ldots$ be iid random variables with d.f. F. If F is subexponential ($1-F$ regularly varying), then

$$P\left(\sum_{i=1}^{n} Z_i > u\right) \sim nP(Z_1 > u) \qquad (\sim nu^{-\alpha}L(u))\,.$$

If $N \sim \mathrm{Po}(\lambda)$ is independent of $Z_1, Z_2, \ldots,$ then

$$P\left(\sum_{i=1}^{N} Z_i > u\right) \sim E(N)P(Z_1 > u) \qquad (\sim E(N)u^{-\alpha}L(u))\,.$$

"Sum is large because one term is large"

Introduction

- Consider a d-dimensional stochastic process $\mathbf{X} = (\mathbf{X}_t; t \in [0,1])$, $\mathbf{X}_t = (X_t^{(1)}, \ldots, X_t^{(d)})'$. Suppose $\mathbf{X}_0 = \mathbf{0}$.

Introduction

- Consider a d-dimensional stochastic process $\mathbf{X} = (\mathbf{X}_t; t \in [0,1])$, $\mathbf{X}_t = (X_t^{(1)}, \ldots, X_t^{(d)})'$. Suppose $\mathbf{X}_0 = \mathbf{0}$.

- A trajectory $t \mapsto \mathbf{X}_t(\omega)$ is said to be **extreme** if $\sup_{t \in [0,1]} \|\mathbf{X}_t(\omega)\|$ is **large**. That is, the process escapes from a **large** ball during $[0,1]$.

Introduction

- Consider a d-dimensional stochastic process $\mathbf{X} = (\mathbf{X}_t; t \in [0,1])$, $\mathbf{X}_t = (X_t^{(1)}, \dots, X_t^{(d)})'$. Suppose $\mathbf{X}_0 = \mathbf{0}$.

- A trajectory $t \mapsto \mathbf{X}_t(\omega)$ is said to be **extreme** if $\sup_{t\in[0,1]} \|\mathbf{X}_t(\omega)\|$ is **large**. That is, the process escapes from a **large** ball during $[0,1]$.

- We are interested in describing the extreme trajectories of $(\mathbf{X}_t; t \in [0,1])$ under the assumption of heavy tails (regular variation) of the underlying probability distributions.

Introduction

- Consider a d-dimensional stochastic process $\mathbf{X} = (\mathbf{X}_t; t \in [0,1])$, $\mathbf{X}_t = (X_t^{(1)}, \dots, X_t^{(d)})'$. Suppose $\mathbf{X}_0 = \mathbf{0}$.

- A trajectory $t \mapsto \mathbf{X}_t(\omega)$ is said to be **extreme** if $\sup_{t\in[0,1]} \|\mathbf{X}_t(\omega)\|$ is **large**. That is, the process escapes from a **large** ball during $[0,1]$.

- We are interested in describing the extreme trajectories of $(\mathbf{X}_t; t \in [0,1])$ under the assumption of heavy tails (regular variation) of the underlying probability distributions.

- Another objective is to determine the tail-behavior of some functional h of the sample path of $(\mathbf{X}_t; t \in [0,1])$. That is, to find the decay of $P(h(\mathbf{X}) > u)$ as $u \to \infty$.

Why are we interested in these questions?

- Heavy-tailed distributions appear naturally in insurance, finance, telecommunications, climate, hydrology, etc.

Why are we interested in these questions?

- Heavy-tailed distributions appear naturally in insurance, finance, telecommunications, climate, hydrology, etc.

- The probability of extreme events are often of considerable interest in applications: large insurance claims, large financial losses, long service/queueing times, wind speeds, flooding, etc.

Why are we interested in these questions?

- Heavy-tailed distributions appear naturally in insurance, finance, telecommunications, climate, hydrology, etc.

- The probability of extreme events are often of considerable interest in applications: large insurance claims, large financial losses, long service/queueing times, wind speeds, flooding, etc.

- Functionals of sample paths (suprema, average) correspond naturally to these events.

Why are we interested in these questions?

- Heavy-tailed distributions appear naturally in insurance, finance, telecommunications, climate, hydrology, etc.

- The probability of extreme events are often of considerable interest in applications: large insurance claims, large financial losses, long service/queueing times, wind speeds, flooding, etc.

- Functionals of sample paths (suprema, average) correspond naturally to these events.

- Understanding of the extreme sample paths can provide useful information about the cause of extreme events and insights in the estimation of the probability of such events.

Regular variation

- A d-dimensional random vector $\mathbf{X}$ is **regularly varying** if there is an $\alpha > 0$ and a measure μ (on $\overline{\mathbb{R}^d}_{\mathbf{0}}$) such that

$$u^{\alpha} L(u) P(u^{-1}\mathbf{X} \in \cdot\,) \xrightarrow{v} \mu(\cdot), \text{ on } \overline{\mathbb{R}^d}_{\mathbf{0}},$$

for some slowly varying function L. We write $\mathbf{X} \in \mathrm{RV}_{\alpha}(L, \mu)$

Regular variation

- A d-dimensional random vector $\mathbf{X}$ is **regularly varying** if there is an $\alpha > 0$ and a measure μ (on $\overline{\mathbb{R}^d}_{\mathbf{0}}$) such that

$$u^{\alpha} L(u) P(u^{-1}\mathbf{X} \in \cdot\,) \xrightarrow{v} \mu(\cdot), \text{ on } \overline{\mathbb{R}^d}_{\mathbf{0}},$$

for some slowly varying function L. We write $\mathbf{X} \in \mathrm{RV}_{\alpha}(L, \mu)$

- We say a measure (distribution) ν on $\mathbb{R}^d$ is regularly varying, $\nu \in \mathrm{RV}_{\alpha}(L, \mu)$ if $u^{\alpha} L(u) \nu(u\cdot\,) \xrightarrow{v} \mu(\cdot)$.

Regular variation

- A d-dimensional random vector $\mathbf{X}$ is **regularly varying** if there is an $\alpha > 0$ and a measure μ (on $\overline{\mathbb{R}}^d_{\mathbf{0}}$) such that

$$u^\alpha L(u) P(u^{-1}\mathbf{X} \in \cdot\ ) \xrightarrow{v} \mu(\cdot), \quad \text{on } \overline{\mathbb{R}}^d_{\mathbf{0}},$$

for some slowly varying function L. We write $\mathbf{X} \in \mathrm{RV}_\alpha(L, \mu)$

- We say a measure (distribution) ν on $\mathbb{R}^d$ is regularly varying, $\nu \in \mathrm{RV}_\alpha(L, \mu)$ if $u^\alpha L(u)\nu(u\cdot\ ) \xrightarrow{v} \mu(\cdot)$.

- The measure μ has the representation

$$\mu(\mathrm{d}r, \mathrm{d}\theta) = c\,\alpha r^{-\alpha-1}\mathrm{d}r\sigma(\mathrm{d}\theta).$$

σ is a probability measure on the unit sphere and is called the **spectral measure**.

Regular variation on $\mathbb{D}[0,1]$

- A stochastic process $\mathbf{X} = (\mathbf{X}_t; t \in [0,1])$ is regularly varying on $\mathbb{D}[0,1]$, if

$$u^\alpha L(u) P(u^{-1}\mathbf{X} \in A) \to m(A)$$

for $A \subset \mathbb{D}[0,1]$, bounded away from $\mathbf{0}$ with $m(\partial A) = 0$. (de Haan and Lin, 2001)

Regular variation on $\mathbb{D}[0,1]$

- A stochastic process $\mathbf{X} = (\mathbf{X}_t; t \in [0,1])$ is regularly varying on $\mathbb{D}[0,1]$, if

$$u^\alpha L(u) P(u^{-1}\mathbf{X} \in A) \to m(A)$$

for $A \subset \mathbb{D}[0,1]$, bounded away from $\mathbf{0}$ with $m(\partial A) = 0$. (de Haan and Lin, 2001)

For continuous mappings $h : \mathbb{D} \to \mathbb{E}$

$$u^\alpha L(u) P(h(u^{-1}\mathbf{X}) \in B) \to m \circ h^{-1}(B)$$

for $B \subset \mathbb{E}$.

A stochastic integral

- Let $\mathbf{X} = (\mathbf{X}_t; t \in [0,1])$, $\mathbf{X}_t = (X_t^{(1)}, \ldots, X_t^{(d)})'$ be a multivariate Lévy process (we take a càdlàg version) and $\mathbf{Y} = (\mathbf{Y}_t; t \in [0,1])$ a predictable càglàd process.

A stochastic integral

- Let $\mathbf{X} = (\mathbf{X}_t; t \in [0,1])$, $\mathbf{X}_t = (X_t^{(1)}, \dots, X_t^{(d)})'$ be a multivariate Lévy process (we take a càdlàg version) and $\mathbf{Y} = (\mathbf{Y}_t; t \in [0,1])$ a predictable càglàd process.

We consider the stochastic integral $(\mathbf{Y} \cdot \mathbf{X})$ where

$$(\mathbf{Y} \cdot \mathbf{X})_t = \Big(\int_0^t Y_s^{(1)} dX_s^{(1)}, \dots, \int_0^t Y_s^{(d)} dX_s^{(d)} \Big)'.$$

A stochastic integral

- Let $\mathbf{X} = (\mathbf{X}_t; t \in [0,1])$, $\mathbf{X}_t = (X_t^{(1)}, \dots, X_t^{(d)})'$ be a multivariate Lévy process (we take a càdlàg version) and $\mathbf{Y} = (\mathbf{Y}_t; t \in [0,1])$ a predictable càglàd process.

We consider the stochastic integral $(\mathbf{Y} \cdot \mathbf{X})$ where

$$(\mathbf{Y} \cdot \mathbf{X})_t = \Big(\int_0^t Y_s^{(1)} dX_s^{(1)}, \dots, \int_0^t Y_s^{(d)} dX_s^{(d)} \Big)'.$$

- We will approximate the extreme trajectories of $(\mathbf{Y} \cdot \mathbf{X})$ when $\mathbf{X}$ is a regularly varying Lévy process with index $\alpha > 0$ and $\mathbf{Y}$ has 'lighter tails' than $\mathbf{X}$.

Assumptions

- The Lévy process $\mathbf{X}$ is regularly varying in the sense that the Lévy measure $\nu \in \mathrm{RV}_\alpha(L, \mu)$; i.e.

$$u^\alpha L(u)\nu(u\cdot\,) \xrightarrow{v} \mu(\cdot) \text{ on } \mathcal{B}(\overline{\mathbb{R}^d}_{\mathbf{0}}).$$

Assumptions

- The Lévy process $\mathbf{X}$ is regularly varying in the sense that the Lévy measure $\nu \in \mathrm{RV}_\alpha(L, \mu)$; i.e.

$$u^\alpha L(u)\nu(u\cdot\,) \xrightarrow{v} \mu(\cdot) \text{ on } \mathcal{B}(\overline{\mathbb{R}^d}_{\mathbf{0}}).$$

(Equivalently, $\mathbf{X}_1 \in \mathrm{RV}_\alpha(L, \mu)$).

CORNELL

Assumptions

- The Lévy process $\mathbf{X}$ is regularly varying in the sense that the Lévy measure $\nu \in \mathrm{RV}_\alpha(L, \mu)$; i.e.

$$u^\alpha L(u)\nu(u\cdot\,) \xrightarrow{v} \mu(\cdot) \text{ on } \mathcal{B}(\overline{\mathbb{R}^d}_{\mathbf{0}}).$$

(Equivalently, $\mathbf{X}_1 \in \mathrm{RV}_\alpha(L, \mu)$).

- The càglàd process $\mathbf{Y}$ satisfies the moment condition

$$E(\sup_{t\in[0,1]} \|\mathbf{Y}_t\|^{\alpha+\delta}) < \infty,$$

for some $\delta > 0$.

Intuition

- By the Lévy-Itô decomposition the stochastic integral can then be written as

$$(\mathbf{Y}\cdot\mathbf{X})_t = \underbrace{(\mathbf{Y}\cdot\widetilde{\mathbf{X}})_t}_{\alpha+\delta \text{ moment finite}} + \underbrace{\sum_{k=1}^{N_t}\mathbf{Y}_{\tau_k}\mathbf{Z}_k}_{\text{tail decays like } u^{-\alpha}}, \qquad \mathbf{Y}_{\tau_k}\perp\mathbf{Z}_k.$$

$\widetilde{\mathbf{X}}$ has bounded jumps, $\mathbf{Z}_k \in \mathrm{RV}_\alpha(L,\lambda^{-1}\mu)$, $\|\mathbf{Z}_k\|\geq 1$, and N Po-process with intensity $\lambda = \nu\{\mathbf{x} : \|\mathbf{x}\|\geq 1\}$.

Intuition

- By the Lévy-Itô decomposition the stochastic integral can then be written as

$$(\mathbf{Y}\cdot\mathbf{X})_t = \underbrace{(\mathbf{Y}\cdot\widetilde{\mathbf{X}})_t}_{\alpha+\delta \text{ moment finite}} + \underbrace{\sum_{k=1}^{N_t} \mathbf{Y}_{\tau_k}\mathbf{Z}_k}_{\text{tail decays like } u^{-\alpha}}, \qquad \mathbf{Y}_{\tau_k} \perp \mathbf{Z}_k.$$

$\widetilde{\mathbf{X}}$ has bounded jumps, $\mathbf{Z}_k \in \mathrm{RV}_\alpha(L, \lambda^{-1}\mu)$, $\|\mathbf{Z}_k\| \geq 1$, and N Po-process with intensity $\lambda = \nu\{\mathbf{x} : \|\mathbf{x}\| \geq 1\}$.

- Moment condition on $\mathbf{Y}$ and independence $\Rightarrow$ $\mathbf{Y}_{\tau_k}\mathbf{Z}_k$ regularly varying with index α (Breiman)

Intuition

- By the Lévy-Itô decomposition the stochastic integral can then be written as

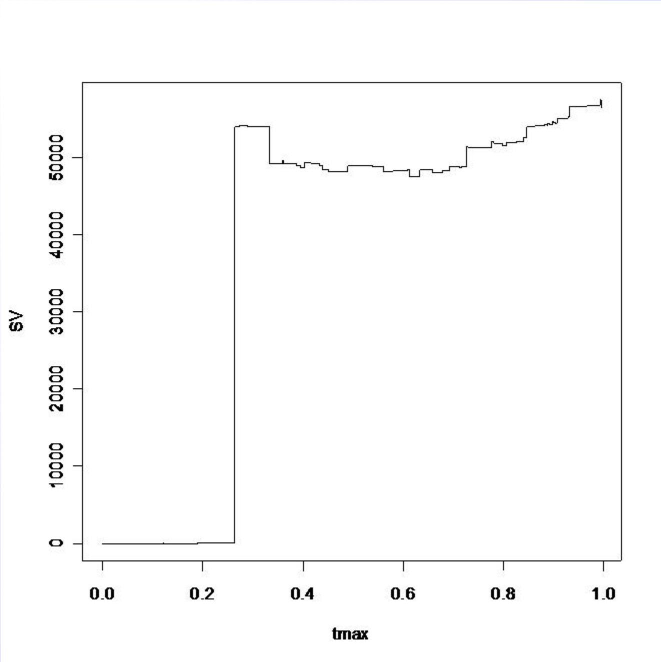
$$(\mathbf{Y}\cdot\mathbf{X})_t = \underbrace{(\mathbf{Y}\cdot\widetilde{\mathbf{X}})_t}_{\alpha+\delta \text{ moment finite}} + \underbrace{\sum_{k=1}^{N_t}\mathbf{Y}_{\tau_k}\mathbf{Z}_k}_{\text{tail decays like } u^{-\alpha}}, \qquad \mathbf{Y}_{\tau_k} \perp \mathbf{Z}_k.$$

$\widetilde{\mathbf{X}}$ has bounded jumps, $\mathbf{Z}_k \in \mathrm{RV}_\alpha(L, \lambda^{-1}\mu)$, $\|\mathbf{Z}_k\| \geq 1$, and N Po-process with intensity $\lambda = \nu\{\mathbf{x} : \|\mathbf{x}\| \geq 1\}$.

- Moment condition on $\mathbf{Y}$ and independence $\Rightarrow$ $\mathbf{Y}_{\tau_k}\mathbf{Z}_k$ regularly varying with index α (Breiman)

- Sum is large because one of the $\mathbf{Z}_k$'s is large.

Simulated stochastic integral



Simulated stochastic integral of $(Y \cdot X)$ where X is a Compound Poisson process with Cauchy-distributed jumps and intensity $\lambda = 100$ and $Y_t = \sqrt{\|X_{t-}\|}$. Out of 1000 simulations the trajectory with larges suprema is plotted.

Main results

- An extreme trajectory may be approximated by a simple step function;

Main results

- An extreme trajectory may be approximated by a simple step function; given that $\sup_t \|(\mathbf{Y}\cdot\mathbf{X})_t\|$ is large we may approximate

$$t \mapsto (\mathbf{Y}\cdot\mathbf{X})_t \approx t \mapsto \mathbf{Y}_\tau \Delta \mathbf{X}_\tau 1_{[\tau,1]}(t), \qquad t \in [0,1],$$

where τ is the time of the biggest jump of $\mathbf{X}$.

Main results

- An extreme trajectory may be approximated by a simple step function; given that $\sup_t \|(\mathbf{Y} \cdot \mathbf{X})_t\|$ is large we may approximate

$$t \mapsto (\mathbf{Y} \cdot \mathbf{X})_t \approx t \mapsto \mathbf{Y}_\tau \Delta \mathbf{X}_\tau 1_{[\tau,1]}(t), \qquad t \in [0,1],$$

where τ is the time of the biggest jump of $\mathbf{X}$.

- Formally: for all $\varepsilon > 0$,

$$P(d(u^{-1}(\mathbf{Y} \cdot \mathbf{X}), u^{-1}\mathbf{Y}_\tau \Delta \mathbf{X}_\tau 1_{[\tau,1]}) > \varepsilon \mid \|(\mathbf{Y} \cdot \mathbf{X})\|_\infty > u) \to 0,$$

as $u \to \infty$ and d is the complete J_1-metric on the space of càdlàg functions.

Regular variation on $\mathbb{D}[0,1]$

- The process $(\mathbf{Y} \cdot \mathbf{X})$ is regularly varying on $\mathbb{D}[0,1]$, i.e.

$$u^{\alpha} L(u) P(u^{-1}(\mathbf{Y} \cdot \mathbf{X}) \in A) \to m(A)$$

for $A \subset \mathbb{D}[0,1]$, bounded from $\mathbf{0}$ with $m(\partial A) = 0$.

Regular variation on $\mathbb{D}[0,1]$

- The process $(\mathbf{Y} \cdot \mathbf{X})$ is regularly varying on $\mathbb{D}[0,1]$, i.e.

$$u^\alpha L(u) P(u^{-1}(\mathbf{Y} \cdot \mathbf{X}) \in A) \to m(A)$$

for $A \subset \mathbb{D}[0,1]$, bounded from $\mathbf{0}$ with $m(\partial A) = 0$.

- The limit measure is given by

$$m(B) = E(\mu\{\mathbf{x} \in \overline{\mathbb{R}^d}_{\mathbf{0}} : \mathbf{x}\mathbf{Y}_V 1_{[V,1]} \in B\}),$$

where $V \sim \mathrm{Unif}(0,1)$ is independent of $\mathbf{Y}$.

Regular variation on $\mathbb{D}[0,1]$

- The process $(\mathbf{Y}\cdot\mathbf{X})$ is regularly varying on $\mathbb{D}[0,1]$, i.e.

$$u^{\alpha}L(u)P(u^{-1}(\mathbf{Y}\cdot\mathbf{X})\in A)\to m(A)$$

for $A\subset\mathbb{D}[0,1]$, bounded from $\mathbf{0}$ with $m(\partial A)=0$.

- The limit measure is given by

$$m(B)=E(\mu\{\mathbf{x}\in\overline{\mathbb{R}^d}_{\mathbf{0}}:\mathbf{x}\mathbf{Y}_V 1_{[V,1]}\in B\}),$$

where $V\sim\mathrm{Unif}(0,1)$ is independent of $\mathbf{Y}$.

I.o.w. we have the approximation in distribution (on $\mathbb{D}[0,1]$)

$$(\mathbf{Y}\cdot\mathbf{X})_{\cdot}\approx\mathbf{Y}_V\mathbf{Z}1_{[V,1]}(\cdot).$$

Continuous Mappings

- For a regularly varying stochastic process $\mathbf{X}$

$$u^{\alpha} L(u) P(u^{-1}\mathbf{X} \in A) \to m(A).$$

Continuous Mappings

- For a regularly varying stochastic process $\mathbf{X}$

$$u^\alpha L(u) P(u^{-1}\mathbf{X} \in A) \to m(A).$$

- The measure m describes the extreme behavior of the process $\mathbf{X}$.

Continuous Mappings

- For a regularly varying stochastic process $\mathbf{X}$

$$u^{\alpha} L(u) P(u^{-1}\mathbf{X} \in A) \to m(A).$$

- The measure m describes the extreme behavior of the process $\mathbf{X}$.
- The support of m determines the possible extreme trajectories of the process $\mathbf{X}$.

Continuous Mappings

- For a regularly varying stochastic process $\mathbf{X}$

$$u^{\alpha} L(u) P(u^{-1}\mathbf{X} \in A) \to m(A).$$

- The measure m describes the extreme behavior of the process $\mathbf{X}$.
- The support of m determines the possible extreme trajectories of the process $\mathbf{X}$.
- We can derive a mapping theorem: for a mapping $h : \mathbb{D} \to \mathbb{E}$ with $m(\mathrm{Disc}_h) = 0$ and s.t. $h^{-1}(A)$ bounded from $\mathbf{0}$ for all bounded $A \in \mathcal{B}(E)$,

$$u^{\alpha} L(u) P(h(u^{-1}\mathbf{X}) \in A) \to m \circ h^{-1}(A).$$

Consequences

For fixed $t > 0$, $d = 1$, and $Y \geq 0$ a.s.,

$$(i) \qquad P((Y \cdot X)_t > u) \sim E\Big(\int_0^t Y_s^\alpha ds\Big)\nu(u, \infty).$$

Consequences

For fixed $t > 0$, $d = 1$, and $Y \geq 0$ a.s.,

$$(i) \qquad P((Y \cdot X)_t > u) \sim E\Big(\int_0^t Y_s^\alpha ds\Big)\nu(u,\infty).$$

$$(ii) \qquad P(\sup_{s\in[0,t]} (Y \cdot X)_s > u) \sim E\Big(\int_0^t Y_s^\alpha ds\Big)\nu(u,\infty).$$

Consequences

For fixed $t > 0$, $d = 1$, and $Y \geq 0$ a.s.,

$$(i) \qquad P((Y \cdot X)_t > u) \sim E\Big(\int_0^t Y_s^\alpha ds\Big)\nu(u,\infty).$$

$$(ii) \qquad P(\sup_{s\in[0,t]} (Y \cdot X)_s > u) \sim E\Big(\int_0^t Y_s^\alpha ds\Big)\nu(u,\infty).$$

$$(iii) \qquad \lim_{u\to\infty} \frac{P(\sup_{s\in[0,t]}(Y \cdot X)_s > u)}{P((Y \cdot X)_t > u)} = 1.$$

Motivation

Consider the asymptotics of $P((Y \cdot X)_t > u)$ as $u \to \infty$ where the Lévy measure $\nu \in \mathrm{RV}(\alpha, \mu)$.

Motivation

Consider the asymptotics of $P((Y \cdot X)_t > u)$ as $u \to \infty$ where the Lévy measure $\nu \in \mathrm{RV}(\alpha, \mu)$.
We have

$$u^\alpha L(u) P((Y \cdot X)_t > u) = u^\alpha L(u) P(u^{-1}(Y \cdot X) \in \pi_t^{-1}(1, \infty))$$

Motivation

Consider the asymptotics of $P((Y \cdot X)_t > u)$ as $u \to \infty$ where the Lévy measure $\nu \in \mathrm{RV}(\alpha, \mu)$.
We have

$$u^\alpha L(u) P((Y \cdot X)_t > u) = u^\alpha L(u) P(u^{-1}(Y \cdot X) \in \pi_t^{-1}(1, \infty))$$
$$\to E\Big(\mu\{x \in (0, \infty] : Y_V x 1_{[V,1]}(t) \in (1, \infty)\}\Big)$$

Motivation

Consider the asymptotics of $P((Y \cdot X)_t > u)$ as $u \to \infty$ where the Lévy measure $\nu \in \mathrm{RV}(\alpha, \mu)$.
We have

$$
\begin{aligned}
u^\alpha L(u) P((Y \cdot X)_t > u) &= u^\alpha L(u) P(u^{-1}(Y \cdot X) \in \pi_t^{-1}(1,\infty)) \\
&\to E\Big(\mu\{x \in (0,\infty] : Y_V x 1_{[V,1]}(t) \in (1,\infty)\}\Big) \\
&= E[(Y_V 1_{[V,1]}(t))^\alpha]\mu(1,\infty)
\end{aligned}
$$

Motivation

Consider the asymptotics of $P((Y \cdot X)_t > u)$ as $u \to \infty$ where the Lévy measure $\nu \in \mathrm{RV}(\alpha, \mu)$.
We have

$$
\begin{aligned}
u^\alpha L(u) P((Y \cdot X)_t > u) &= u^\alpha L(u) P(u^{-1}(Y \cdot X) \in \pi_t^{-1}(1, \infty)) \\
&\to E\Big(\mu\{x \in (0, \infty] : Y_V x 1_{[V,1]}(t) \in (1, \infty)\}\Big) \\
&= E[(Y_V 1_{[V,1]}(t))^\alpha] \mu(1, \infty) \\
&= \int_0^t E(Y_s^\alpha) ds \mu(1, \infty).
\end{aligned}
$$

Motivation

Consider the asymptotics of $P((Y \cdot X)_t > u)$ as $u \to \infty$ where the Lévy measure $\nu \in \mathrm{RV}(\alpha, \mu)$.
We have

$$
\begin{aligned}
u^\alpha L(u) P((Y \cdot X)_t > u) &= u^\alpha L(u) P(u^{-1}(Y \cdot X) \in \pi_t^{-1}(1, \infty)) \\
&\to E\Big(\mu\{x \in (0, \infty] : Y_V x 1_{[V,1]}(t) \in (1, \infty)\}\Big) \\
&= E[(Y_V 1_{[V,1]}(t))^\alpha] \mu(1, \infty) \\
&= \int_0^t E(Y_s^\alpha) ds \mu(1, \infty).
\end{aligned}
$$

Since $u^\alpha L(u) \nu(u, \infty) \to \mu(1, \infty)$ we obtain

$$
P((Y \cdot X)_t > u) \sim \int_0^t E(Y_s^\alpha) \nu(u, \infty), \quad \text{as } u \to \infty.
$$