Ruin problem for integrated stationary Gaussian process

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Consider a random process

$$Y_t = \int_0^t X_s\, ds - ct^\theta,$$

where $X_t, t \geq 0$ is a stationary real-valued zero-mean Gaussian process with continuous trajectories and twice differentiable covariation function $R(t)$, $c > 0$, $\theta > 1/2$. Such a model arises, for example, in ruin financial problems, telecommunications, and information storage problems [6,7].

Define the ruin probability

$$P(u) = P\{\exists\, t \geq 0 : Y_t \geq u\} = P\{\max_{t\geq 0} Y_t \geq u\}.$$

The random process $\int_0^t X_s\, ds$ is a process with stationary increments. The ruin probability $P(u)$ has been studied in a number of papers for various models of processes with stationary increments. In [3,7] an exact asymptotic of $P(u)$ as $u \to \infty$ has been found for $Y_t = B_{\alpha/2}(t) - ct$, where $B_{\alpha/2}$ is the fractional Brownian motion. The most similar in problem formulation work is the paper [5]. There, an asymptotic of the ruin probability for $\theta = 1$ was found in the following form:

$$P(u) = \frac{H_\eta G}{c^2}\, e^{-Hc^2/G^2} e^{-uc/G}(1 + o(1))$$

as $u \to \infty$. Here H_η is a generalized Pickands constant for the process $\eta = c(G\sqrt{2})^{-1} \int_0^t X_t\, dt$. But the constant H_η has not been calculated, and its dependance on characteristics of the original process X_t remains unclear.

Application of the Rice's method allows to obtain the exact asymptotic of the $P(u)$ as $u \to \infty$ under some conditions, thus we are able to calculate the value of H_η.

Intuitively it is clear that for large u the event that the process Y_t crosses the level u more than once is rare, so the probability $P(u)$ approximately equals the mean number of crossings. The Rice's method allows to formalize this idea.

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Let a random variable $N_u([0,T])$ be equal to the number of crossings of the level u by the process Y_t on the segment $[0,T]$. In [1,2] it was shown that for a random process with continuously differentiable trajectories and for any segment S the following relation holds

$$0 \le EN_u(S) - P(\max_{t\in S} Y_t \ge u) \le EN_u(S)(N_u(S)-1).$$

Application of this method gives us

Theorem 1 *Suppose that* $G = \int_0^\infty R(s)\,ds > 0$, $H = \int_0^\infty sR(s)\,ds$ *is finite, and* $u^{2-2/\theta}\int_{u^{1/\theta}}^\infty sR(s)\,ds \to 0, u\to\infty$. *Then*

$$P(u) = \frac{\sqrt{R(0)}}{\sqrt{2\pi}} u^{-1+1/\theta}(2\theta-1)^{1/2-1/\theta}c^{-1/\theta}\times$$
$$\times \exp\left\{-u^{2-1/\theta}\frac{(1+\tau_{\min}^\theta(2\theta-1)^{-1}))^2}{4G(2\theta-1)^{-1/\theta}c^{-1/\theta}\tau_{\min}-4Hu^{-1/\theta}}\right\}(1+o(1))$$

as $u\to\infty$, *where* $\tau_{\min}=\tau_{\min}(u)$ *is a point of minimum of the function*

$$v(\tau) = \frac{(1+\tau^\theta(2\theta-1)^{-1}))^2}{4G(2\theta-1)^{-1/\theta}c^{-1/\theta}\tau-4Hu^{-1/\theta}}.$$

It turns out that main part of the probability $P(u)$ compose events such that the level crossing occurres for t in some neighborhood of maximum point of variance of the process Y_t. To prove this, rewrite the probability $P(u)$ in the following form:

$$P(u) = \mathbf{P}\left(\max_{t>0}\frac{1}{(1+ct^\theta/u)}\int_0^t X_s ds > u\right).$$

Then, the variance of the process $V_t = \frac{1}{(1+ct^\theta/u)}\int_0^t X_s ds$ can be represented as the sum

$$\mathbf{Var}\,V_t = \frac{2Gt-2H}{(1+ct^\theta/u)^2} - \frac{2Gt\int_t^\infty R(s)ds - 2\int_t^\infty sR(s)ds}{(1+ct^\theta/u)^2} = S_1(t)+R_1(t). \quad (1)$$

The second term in (1) is negligible due to the assumptions set, and the first term has a unique point of maximum for large enough u. If we denote this point by $t_{\max} = t_{\max}(u)$, then with the help of Piterbarg inequality [1] we can choose a segment $I = [t_{\max} - \Delta, t_{\max} + \Delta]$ with $\Delta = \Delta(u) \to 0$, such that

$$P\left(\sup_{t \notin I} Y_t > u\right) = o\left(\exp\left\{-\frac{u^2}{2S_1(t_{\max})}\right\}\right).$$

Thus, it is sufficient to estimate the values $EN_u(I)$ and $EN_u(I)(N_u(I)-1)$. The first term can be evaluated using the Rice formula [1]

$$\mathbf{E}N_u(I) = \int_I \int_0^\infty |y| p_t(u,y)\, dydt, \tag{2}$$

where $p_t(u,y)$ is a joint density of the random variables Y_t, Y_t'. Performing change of the variable $t = \left(u(2\theta-1)^{-1}/c\right)^{1/\theta} \tau$ and applying the generalization of the Laplace method to the integral (2), we obtain that

$$\mathbf{E}N_u(I) = \frac{\sqrt{R(0)}}{\sqrt{2\pi}} u^{-1+1/\theta} (2\theta-1)^{1/2-1/\theta} c^{-1/\theta} \times$$
$$\times \exp\left\{-u^{2-1/\theta} \frac{(1+\tau_{\min}^\theta (2\theta-1)^{-1}))^2}{4G(2\theta-1)^{-1/\theta} c^{-1/\theta} \tau_{\min} - 4Hu^{-1/\theta}}\right\} (1+o(1)),$$

where $\tau_{\min} = \tau_{\min}(u)$ is defined in the statement of the theorem.

The estimation of the $EN_u(I)(N_u(I)-1)$ is based on the application of the explicit formula for the second moment

$$EN_u(I)(N_u(I)-1) = \int_I \int_I \int_0^\infty \int_0^\infty y_1 y_2 \varphi_{t,s,t,s}(u,u,y_1,y_2)\, dy_1\, dy_2\, ds\, dt,$$

where $\varphi_{t,s,t,s}(u,u,y_1,y_2)$ is a joint density of the variables Y_t, Y_s, Y_t', Y_s'. Proceeding to conditional densities and applying Taylor formula, we prove that $EN_u(I)(N_u(I)-1) = o\left(EN_u(I)\right)$ as $u \to \infty$. It turns out that the requirement of twice differentiability of the covariance function $R(t)$ is significant in this method.

Consideration of the case $\theta = 1$ gives us the asymptotic

$$P(u) = \frac{\sqrt{R(0)}}{\sqrt{2\pi}} c^{-1} \exp\{-\frac{Hc^2}{G^2}\} \exp\{-uc/G\} (1+o(1)),$$

thus, comparing it with the result of Debicki, we obtain that the Pickands constant for the process $\eta(t) = \dfrac{c}{G\sqrt{2}} \displaystyle\int_0^t X_t\, dt$ equals $\sqrt{R(0)}/(\sqrt{2\pi} G c)$.

Denote the time of ruin $\tau_u = \inf\{t \geq 0 : u - Y_t \leq 0\}$. The Rice's method allows to obtain the asymptotic of the conditional distribution of τ_u as $u \to \infty$ given the ruin condition $\max_{t\geq 0} Y_t \geq u$. It is worth to mention that τ_u takes values mostly in some neighborhood of $t_{\max}$.

Theorem 2 *Let the conditions of the Theorem 1 be fulfilled. Then*

$$\mathbf{P}\left(\tau_u < f(x) | \, \tau_u < \infty\right) \to \Phi(x), \quad u \to \infty,$$

where $\Phi(x)$ *is a distribution function of the standard normal random variable, and* $f(x) = u^{3/(2\theta)-1}\sqrt{2G}(2\theta-1)^{-3/(2\theta)}c^{-3/(2\theta)}\theta^{-2}x + \left(u(2\theta-1)^{-1}/c\right)^{1/\theta}$.

Let us start with the estimation of the probability $\mathbf{P}(\tau_u < f(x))$. To prove the result we, as above, restrict consideration to the neighborhood of the point $t_{\max}$: $I = [t_{\max} - c^{-1/\theta}(2\theta-1)^{-1/\theta}u^{3/(2\theta)-1}\ln u, t_{\max} + (f(x) - t_{\max})]$. Substituting $t = t(\tau) = \left(u(2\theta-1)^{-1}/c\right)^{1/\theta}\tau + t_{\max}$, we obtain $I = (I_1 + I_2)(1 + o(1))$, where

$$\begin{aligned}
I_1 &= \frac{\left(u(2\theta-1)^{-1}/c\right)^{1/\theta}}{\sqrt{2\pi}}\int_{-u^{1/(2\theta)-1}\ln u}^{0}\frac{\sigma(t(\tau))}{a(\tau)}g(t(\tau))\exp\left\{-S_3(\tau)\right\}\,d\tau,\\
I_2 &= \frac{\left(u(2\theta-1)^{-1}/c\right)^{1/\theta}}{\sqrt{2\pi}}\int_{0}^{u^{1/(2\theta)-1}h(x)}\frac{\sigma(t(\tau))}{a(\tau)}g(t(\tau))\exp\left\{-S_3(\tau)\right\}\,d\tau,\\
S_3(\tau) &= \frac{u^2}{2S_1(t(\tau))}.
\end{aligned} \tag{3}$$

$a(\tau), g(t)$, and $\sigma(t)$ are known functions with not more than polynomial growth in u.

The first integral in (3) is estimated with the help of the Laplace method, which gives us

$$I_1 = \frac{\sqrt{R(0)}}{2\sqrt{2\pi}}u^{-1+1/\theta}(2\theta-1)^{1/2-1/\theta}c^{-1/\theta}\exp\left\{-S_3(0)\right\}(1+o(1)).$$

In the second integral we can perform the change of variable $y^2 = 2(S_3(u^{1/(2\theta)-1}\tau) - S_3(0))$, thus obtaining

$$I_2 = \frac{\sqrt{R(0)}}{\sqrt{2\pi}}u^{-1+1/\theta}(2\theta-1)^{1/2-1/\theta}c^{-1/\theta}\exp\left\{-S_3(0)\right\}\frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{y^2}{2}}dy(1+o(1)).$$

The estimation of the $EN_u(I)(N_u(I)-1)$ repeats the corresponding part of the proof of the former theorem.

To complete the proof, it remains to sum the obtained estimates and to divide it by the probability $P(\sup_{t>0} Y_t > u)$.

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