Tail Approximations to the Density Function in EVT

Deyuan Li

(*joint work with Jürg Hüsler*)

University of Bern

EVA 2005, Gothenburg, August 15–19.

Outline of the talk:

1 Introduction

Assume $F \in D(G_\gamma)$ and let $U(t) = F^{\leftarrow}(1 - 1/t)$, $t \geq 1$.

Drees, de Haan and Li (2003, 2005):

$$\frac{\frac{U(tx)-U(t)}{a(t)} - \frac{x^\gamma - 1}{\gamma}}{A(t)} \to K_{\gamma,\rho}(x) \tag{1}$$

$$\Downarrow$$

$$\sup_{x \in D_{t,\rho}} w_{F}(t,x) \left| \frac{t\bar{F}(a_0(t)x + b_0(t)) - (1+\gamma x)^{-1/\gamma}}{A_0(t)} - d_{F}(x) \right| = o(1).$$

Here,

1. **a_0, b_0 and A_0 are some special chosen functions:**

$$a_0 \sim a, \quad b_0 - U = o(a_0), \quad A_0 \in \mathrm{RV}(\rho).$$

2. **$D_{t,\rho}$ is the set of x such that $-\frac{1}{\gamma \vee 0} \overset{\text{slow}}{\longleftarrow} x_t \leq x < \frac{1}{(-\gamma)\vee 0}$.**

3. **w_{F} is a power function of $(1+\gamma x)^{-1/\gamma}$, depending on F only if $\gamma = \rho = 0$.**

$$\Downarrow$$

$$\frac{t\bar{F}(a_0(t)x + b_0(t))}{(1+\gamma x)^{-1/\gamma}} \to 1, \quad \frac{1 - F^n(a_0(n)x + b_0(n))}{1 - G_\gamma(x)} \to 1$$

uniform for $-\frac{1}{\gamma \vee 0} \overset{\text{slower}}{\longleftarrow} x_t \leq x < \frac{1}{(-\gamma)\vee 0}$, but excluding $\gamma = \rho = 0$.

<u>Question:</u> can we find "some conditions" such that

$$\text{"some conditions"}$$
$$\Downarrow \quad ?$$
$$\sup_{x \in D_{t,\rho}} w_f(t,x) \left| \frac{t a_0(t) f(a_0(t)x + b_0(t)) - (1+\gamma x)^{-\frac{1}{\gamma}-1}}{A_0(t)} + d_f(x) \right| = o(1)$$
$$\Downarrow \quad ?$$
$$\frac{t a_0(t) f(a_0(t)x + b_0(t))}{(1+\gamma x)^{-1/\gamma - 1}} \to 1$$

uniform for $-\frac{1}{\gamma \vee 0} \overset{\text{slower}}{\longleftarrow} x_t \leq x < \frac{1}{(-\gamma)\vee 0}$, and some result like

$$\sup_{x \in \dots} w(x) \left| \frac{\frac{d}{dx} F^n(a_0(n)x + b_0(n)) - \frac{d}{dx} G_\gamma(x)}{A_0(x)} - \text{bias function} \right| = o(1)\,?$$

Note:

1. These results are parallelized with those of Drees et. al. (2003, 2005), but do not follow directly from those results.
2. "some condition" should be stronger than (1).
3. $$\frac{d}{dx} F^n(a_0(n)x + b_0(n)) - \frac{d}{dx} G_\gamma(x) \to 0$$
   means that F is in the *differentiable* domain of attraction of G_γ (see Pickands (1986)).

2 Motivation/Background

Game theory: n goods; $i = 1, 2, ..., n$;

p_i: the price of i-th good to buy;

$\sigma_i X_i$: the profit from i-th good;

X_i is a random variable, $\sigma_i > 0$ is the scale;

$U_i = -p_i + \sigma_i X_i$:

utility sign for one consumer to choose i-th good;

$D_i = P(U_i > \max_{j \neq i} U_j)$:

demand for for one consumer to choose i-th good;

In order to calculate the *Nash equilibrium*, we need to approximate D_i. See Gabaix and Laibson (2003).

In general, assume $X_1, ..., X_n$ i.i.d. F (density f) and define

$$D_n = P\Big(- p_{n1} + \sigma_{n1} X_1 \geq \max_{2 \leq i \leq n} \{-p_{ni} + \sigma_{ni} X_i\}\Big).$$

$D_n = 1/n$: if $p_{ni} = p_{nj}$, $\sigma_{ni} = \sigma_{nj}$ for $i, j = 1, ..., n$.

How to approximate D_n for other cases?

Assume $F \in D(G_\gamma)$ with $\gamma \geq 0$ and $0 < F(x) < 1$ for $x \in \mathbb{R}$.

Then

$$\begin{aligned} D_n &= P(-p_{n1} + \sigma_{n1} X_1 \geq \max_{2 \leq i \leq n} \{-p_{ni} + \sigma_{ni} X_i\}) \\ &= \int_{-\infty}^{\infty} f(x) \prod_{i=2}^{n} F(\frac{\sigma_{n1} x - p_{n1} + p_{ni}}{\sigma_{ni}})\, dx \\ &= \sigma_{n1}^{-1} \int_{-\infty}^{\infty} f\Big(\frac{x + p_{n1}}{\sigma_{n1}}\Big) \prod_{i=2}^{n} F\Big(\frac{x + p_{ni}}{\sigma_{ni}}\Big)\, dx. \end{aligned}$$

First consider $\sigma_{ni} \equiv 1$. Let $a_n = a_0(n)$, $b_n = b_0(n)$.

Then replace x by $a_n x + b_n$,

$$\begin{aligned} D_n &= n^{-1} \int_{-\infty}^{\infty} n a_n f\Big(a_n(x + \frac{p_{n1}}{a_n}) + b_n\Big) \prod_{i=2}^{n} F\Big(a_n(x + \frac{p_{ni}}{a_n}) + b_n\Big)\, dx \\ &= n^{-1}\Big(\int_{l_n}^{\infty} \dots \; + \; \int_{-\infty}^{l_n} \Big) =: n^{-1}(I_{1n} + I_{2n}). \end{aligned}$$

If

$$\frac{n\bar{F}(a_n x + b_n)}{(1 + \gamma x)^{-1/\gamma}} \to 1, \qquad \frac{n a_n f(a_n x + b_n)}{(1 + \gamma x)^{-1/\gamma - 1}} \to 1$$

uniform for $l_n \leq x < \infty$, it may follow that

$$I_{1n} \sim \int_{l_n}^{\infty} (1 + \gamma x_1)^{-\frac{1}{\gamma} - 1} \prod_{i=2}^{n} \Big(1 - \frac{1}{n}(1 + \gamma x_i)^{-\frac{1}{\gamma}}\Big)\, dx$$

with $x_i = x + p_{ni}/a_n$, $i = 1, ..., n$.

This may help us to approximate D_n.

3 Main Results

Assume the second order condition in de Haan and Resnick (1996):

$$\begin{cases} U \textit{ is twice differentiable, } U' \textit{ is eventually positive}; \\ A(t) := \frac{tU''(t)}{U'(t)} - \gamma + 1 \textit{ has constant sign near infinity}; \\ A(t) \to 0 \textit{ as } t \to \infty \textit{ and } |A| \in \mathrm{RV}(\rho) \textit{ with } \rho \le 0. \end{cases} \tag{2}$$

(2)$\Longrightarrow$(1), thus the results on F (Drees et. al. (2003, 2005)) hold under the condition (2).

Let

$$a_0(t) = \begin{cases} tU'(t), & \gamma = \rho = 0, \\ \gamma U(t), & \gamma > 0 = \rho, \\ -\gamma(U(\infty) - U(t)), & \gamma < 0 = \rho, \\ ct^\gamma, & \rho < 0, \end{cases}$$

$$b_0(t) = \begin{cases} U(t) - a_0(t)A_0(t)\frac{1}{\gamma+\rho}, & \text{if } \rho < 0,\ \gamma + \rho \neq 0, \\ U(t), & \text{else} \end{cases}$$

for some suitable A_0, and for each $\beta, \delta > 0$ define

$$D_{t,\rho} := D_{t,\rho,\delta,\beta} := \begin{cases} \{x :\ (1+\gamma x)^{-1/\gamma} \le \beta t^{-\delta+1}\}, & \rho < 0, \\ \{x :\ (1+\gamma x)^{-1/\gamma} \le |A_0(t)|^{-\beta}\}, & \rho = 0. \end{cases}$$

Theorem 1. Suppose (2) holds. Then

$$\sup_{x\in D_{t,\rho}} w_f(t,x)\left|\frac{ta_0(t)f(a_0(t)x+b_0(t))-(1+\gamma x)^{-\frac{1}{\gamma}-1}}{A_0(t)}+d_f(x)\right| = o(1).$$

Note:

$$d_f(x) = {d_F}'(x)$$

$$\frac{w_f(t,x)}{w_F(t,x)} = 1+\gamma x, \quad \text{if not } \gamma=\rho=0.$$

Theorem 2. Suppose (2) holds with $\rho > -1$ but not $\gamma = \rho = 0$. Then

$$\sup_{\{x:\,(1+\gamma x)^{-1/\gamma}\le \log^2|A_0(t)|\}} w(x)$$

$$\times\left|\frac{\frac{d}{dx}F^n(a_0(n)x+b_0(n))-\frac{d}{dx}G_\gamma(x)}{A_0(x)}-\text{bias func.}\right| = o(1).$$

Note: $w(x)$ and bias function are complicated.

Based on Theorem 1 and Theorem 2, we can easily have some large deviation results on the density function.

4 Sketch of Proofs

By de Haan and Resnick (1996), (2) implies

$$U'(t) = kt^{\gamma-1} \exp\Big(\int_1^t \frac{A(u)}{u}\,du\Big),$$

where $k \neq 0$, and that

$$\frac{\frac{U'(tx)}{U'(t)} - x^{\gamma-1}}{A(t)} \to x^{\gamma-1}\frac{x^\rho - 1}{\rho}, \quad x > 0.$$

Then we prove that

<u>Lemma 1</u>. Suppose (2) holds. Then $\forall \varepsilon > 0$, $\exists\, t_\varepsilon > 0$ s.t. for all $\min\{t,\, tx\} > t_\varepsilon$

$$x^{-(\gamma+\rho-1)} e^{-\varepsilon|\log x|} \left| \frac{\frac{U'(tx)}{t^{-1}a_0(t)} - x^{\gamma-1}}{\tilde{A}(t)} - \frac{d}{dx}\Big(K_{\gamma,\rho}(x)\Big) \right| < \varepsilon.$$

Comparing it with

<u>Lemma 1*</u>. (Drees, 1998; Cheng and Jiang, 2001) Suppose (1) holds. Then $\forall \varepsilon > 0$, $\exists\, t_\varepsilon > 0$ s.t. for all $\min\{t,\, tx\} > t_\varepsilon$

$$x^{-(\gamma+\rho)} e^{-\varepsilon|\log x|} \left| \frac{\frac{U(tx)-U(t)}{a_0(t)} - \frac{x^\gamma - 1}{\gamma}}{A_0(t)} - K_{\gamma,\rho}(x) \right| < \varepsilon.$$

Lemma 1* is elementary to derive the results on F;

Lemma 1 is elementary to obtain the results on f.

Proof of Theorem 1:

Recall $U(t) = F^{\leftarrow}(1 - 1/t)$. Hence $F(U(t)) = 1 - 1/t$ and

$$f(U(t)) \cdot U'(t) = t^{-2}.$$

Replace t by

$$U^{\leftarrow}(a_0(t)x + b_0(t)) = \frac{1}{\bar{F}(a_0(t)x + b_0(t))},$$

then

$$f(a_0(t)x + b_0(t)) = \frac{\bar{F}^2(a_0(t)x + b_0(t))}{U'(1/\bar{F}(a_0(t)x + b_0(t)))}.$$

Using the results on F to approximate $\bar{F}(a_0(t)x + b_0(t))$, using Lemma 1 to approximate $U'(1/\bar{F}(a_0(t)x + b_0(t)))$ and by very complicated calculation (for different cases), we obtain Theorem 1.

Proof of Theorem 2:

Note that

$$\frac{d}{dx}F^n(a_0(n)x + b_0(n))$$
$$= na_0(n)f(a_0(n)x + b_0(n)) \cdot F^{n-1}(a_0(n)x + b_0(n))$$

and that

$$F^{n-1}(a_0(n)x + b_0(n))$$
$$= \exp\left\{(n-1)\log\left[1 - \bar{F}(a_0(n)x + b_0(n))\right]\right\}$$

Hence by Theorem 1 and the results on F (Drees et. al., 2003, 2005), Theorem 2 follows by complicated calculation.

5 Application

Theorem 2.1 in Li and de Vries (2005):

Suppose $X_1, \ldots, X_n$ i.i.d. F and F satisfies (2) with $\gamma > 0$. Then

$$D_n = P\Big(-p_{n1} + \sigma_{n1} X_1 \geq \max_{2 \leq i \leq n} \{-p_{ni} + \sigma_{ni} X_i\}\Big) \sim \frac{\sigma_{n1}^{1/\gamma}}{\sum_{i=2}^{n} \sigma_{ni}^{1/\gamma}}$$

as $n \to \infty$, provided by

$$\max_{1 \leq i \leq n} \sigma_{ni} = O(1),$$

$$\max_{1 \leq i \leq n} |p_{ni}| = o(n^{(1-\varepsilon)\gamma})$$

for some $\varepsilon > 0$.

More results in Li and de Vries (2005):

1. $\gamma = 0 > \rho$;
2. $\gamma = 0 = \rho$;
3. $\bar{F} \sim c x^{\beta} e^{-d x^{\alpha}}$ with $c, d, \alpha, \beta > 0$.