Kesten's counterexample to the Cramér-Wold device for regular variation

FILIP LINDSKOG

Royal Institute of Technology, Stockholm

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based on joint work with Henrik Hult

www.math.kth.se/∼lindskog

The Cramér-Wold device

Let $\mathbf{X}, \mathbf{X}_1, \mathbf{X}_2, \ldots$ be $\mathbb{R}^d$-valued random (column) vectors. It holds that

$$\mathbf{X}_n \xrightarrow{d} \mathbf{X} \quad \text{as } n \to \infty$$

if and only if

$$\mathbf{x}^{\mathrm{T}}\mathbf{X}_n \xrightarrow{d} \mathbf{x}^{\mathrm{T}}\mathbf{X} \quad \text{as } n \to \infty \text{ for all } \mathbf{x} \in \mathbb{R}^d,$$

where $\mathbf{x}^{\mathrm{T}}\mathbf{X}_n = x^{(1)}X_n^{(1)} + \cdots + x^{(d)}X_n^{(d)}$.

Let $h_{\mathbf{x}} : \mathbb{R}^d \to \mathbb{R}$ be given by $h_{\mathbf{x}}(\mathbf{z}) = \mathbf{x}^{\mathrm{T}}\mathbf{z}$. If μ denotes the distribution of $\mathbf{X}$, then $\mu h_{\mathbf{x}}^{-1}$ is the distribution of $\mathbf{x}^{\mathrm{T}}\mathbf{X}$:

$$\mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X} \le y) = \mathrm{P}(h_{\mathbf{x}}(\mathbf{X}) \in (-\infty, y]) = \mathrm{P}(\mathbf{X} \in h_{\mathbf{x}}^{-1}(-\infty, y]).$$

Notice that $h_{\mathbf{x}}^{-1}(-\infty, y] = \{\mathbf{z} \in \mathbb{R}^d : \mathbf{x}^{\mathrm{T}}\mathbf{z} \leq y\}$ is a half space.

The characteristic function of $\mathbf{x}^{\mathrm{T}}\mathbf{X}$ is

$$\int_{\mathbb{R}} e^{isy} \mu h_{\mathbf{x}}^{-1}(\mathrm{d}y) = \int_{\mathbb{R}^d} e^{ish_{\mathbf{x}}(\mathbf{z})} \mu(\mathrm{d}\mathbf{z}) = \int_{\mathbb{R}^d} e^{is\mathbf{x}^{\mathrm{T}}\mathbf{z}} \mu(\mathrm{d}\mathbf{z}) = \widehat{\mu}(s\mathbf{x}).$$

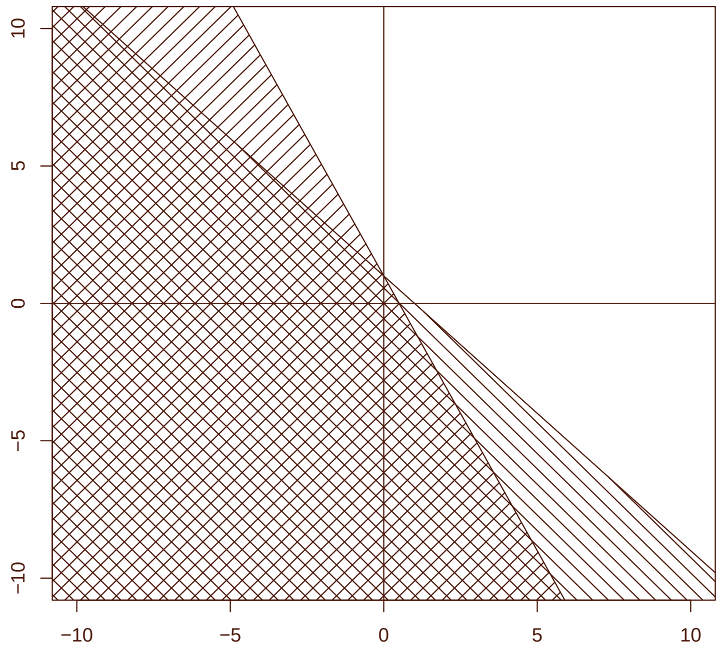
$\Rightarrow$ If we know the distribution $\mu h_{\mathbf{x}}^{-1}$ of $\mathbf{x}^{\mathrm{T}}\mathbf{X}$ for all $\mathbf{x}$, then we know the characteristic function $\widehat{\mu}$ of $\mathbf{X}$ in every point $\mathbf{x}$.

Since $\widehat{\mu}$ uniquely determines μ we find that:

A probability measure μ is uniquely determined by the values it gives to half spaces.

Notice that the set of half spaces is not a π-system, the intersection of two half spaces is not a half space.

Two half spaces

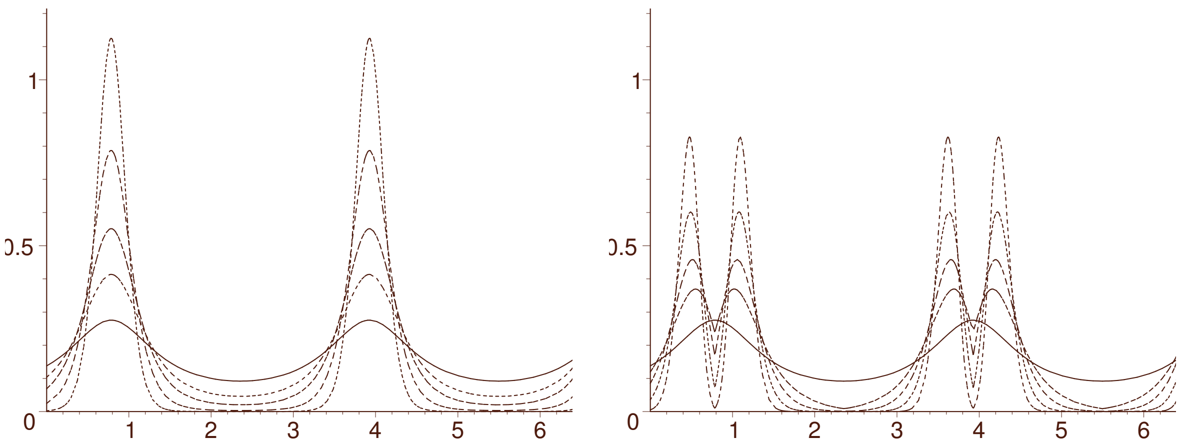


$$h^{-1}_{(1,1)}(-\infty, 1] = \{(x, y) : x + y \leq 1\}$$ and $$h^{-1}_{(2,1)}(-\infty, 1] = \{(x, y) : 2x + y \leq 1\}$$

Regular variation

An $\mathbb{R}^d$-valued random vector $\mathbf{X}$ is **regularly varying** with index $\alpha \in (0,\infty)$ and spectral measure σ on $\mathcal{B}(\mathbb{S}^{d-1})$, $\mathbb{S}^{d-1} = \{\mathbf{z} \in \mathbb{R}^d : |\mathbf{z}| = 1\}$, if

$$\frac{\mathrm{P}(|\mathbf{X}| > xt, \mathbf{X}/|\mathbf{X}| \in \cdot\,)}{\mathrm{P}(|\mathbf{X}| > t)} \xrightarrow{w} x^{-\alpha}\sigma(\cdot) \quad \text{på } \mathcal{B}(\mathbb{S}^{d-1}) \text{ as } t \to \infty \text{ for } x > 0.$$



Spectral measures with respect to the 2-norm and max-norm for bivariate $t(\alpha,\Sigma)$-distributions, $\alpha = 0, 2, 4, 8, 16$, $\Sigma_{11} = \Sigma_{22} = 1$ and $\Sigma_{12} = \Sigma_{21} = 1/2$.

Regular variation

A function L is **slowly varying** if $\lim_{t\to\infty} L(ut)/L(t) = 1$ for alla $u > 0$.

An $\mathbb{R}^d$-valued random vector is **regularly varying** with index $\alpha \in (0,\infty)$ if and only if there exist a slowly varying function L and a measure $\mu \neq 0$ such that

$$\lim_{t\to\infty} t^\alpha L(t)\,\mathrm{P}(\mathbf{X} \in tA) = \mu(A)$$

for all $A \in \mathcal{B}(\mathbb{R}^d)$ bounded away from $\mathbf{0}$ with $\mu(\partial A) = 0$.

It follows that $\mu(uA) = u^{-\alpha}\mu(A)$ for all $u > 0$ and $A \in \mathcal{B}(\mathbb{R}^d)$.

We write $\mathbf{X} \in \mathrm{RV}(\alpha, \mu)$.

Regular variation and linear combinations

Suppose that $\mathbf{X} \in \mathrm{RV}(\alpha, \mu)$, i.e. $\lim_{t\to\infty} t^\alpha L(t)\, \mathrm{P}(\mathbf{X} \in tA) = \mu(A)$.

Let $W_{\mathbf{x}} = \{\mathbf{z} \in \mathbb{R}^d : \mathbf{x}^{\mathrm{T}}\mathbf{z} > 1\}$ - a half space which does not contain $\mathbf{0}$.

Then it holds that for all $\mathbf{x} \neq \mathbf{0}$

$$\lim_{t\to\infty} t^\alpha L(t)\, \mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X} > t) = \lim_{t\to\infty} t^\alpha L(t)\, \mathrm{P}(\mathbf{X} \in tW_{\mathbf{x}}) = \mu(W_{\mathbf{x}}).$$

We have shown that (1) $\mathbf{X} \in \mathrm{RV}(\alpha, \mu)$ implies

$$(2) \begin{cases} \text{for all } \mathbf{x} \neq \mathbf{0}, \quad \lim_{t\to\infty} t^\alpha L(t)\, \mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X} > t) = h(\mathbf{x}) \text{ exists,} \\ h(\mathbf{x}) > 0 \text{ for some } \mathbf{x} \neq \mathbf{0}, \end{cases}$$

with $h(\mathbf{x}) = \mu(W_{\mathbf{x}})$. But does it hold that $(2) \Rightarrow (1)$? i.e.

Does the Cramér-Wold device for regular variation hold?

Is the measure μ determined by its values on half spaces?

A sufficient condition for $(1) \Leftrightarrow (2)$ is that

$$\mu(uW_{\mathbf{x}}) = u^{-\alpha}\mu(W_{\mathbf{x}}) \;\; u > 0, \mathbf{x} \neq \mathbf{0} \quad \text{and} \quad \mu(W_{\mathbf{x}}) = \widetilde{\mu}(W_{\mathbf{x}}) \;\; \mathbf{x} \neq \mathbf{0}$$

implies $\mu = \widetilde{\mu}$, i.e. that μ is determined by the values it gives to half spaces.

Problem: Since μ is not a finite measure we cannot use characteristic functions.

(Basrak, Davis, Mikosch 2002)
If α is not an integer, then μ is determined by the values it gives to half spaces.
Hence,
the Cramér-Wold device for regular variation holds if α is not an integer.

Problem if α is an integer

Consider $\mathbb{R}^2$-valued stochastic vectors $\mathbf{X}_1$ och $\mathbf{X}_2$ with

$$\mathbf{X}_1 \stackrel{\mathrm{d}}{=} R(\cos\Theta_1, \sin\Theta_1)' \quad \text{and} \quad \mathbf{X}_2 \stackrel{\mathrm{d}}{=} R(\cos\Theta_2, \sin\Theta_2)',$$

where R is Pareto-distributed, $\mathrm{P}(R > r) = r^{-\alpha}$ for $r > 1$, and independent of Θ_1, Θ_2.

We see that $\mathbf{X}_k$, $k = 1, 2$, are regularly varying with index α and spectral measures $\mathrm{P}((\cos\Theta_k, \sin\Theta_k) \in \cdot\,)$.

Let α be an integer and Θ_1 uniformly distributed on $[0, 2\pi)$. Let Θ_2 have density

$$f_2(\theta) = \frac{1}{2\pi} + c\sin((\alpha+2)\theta), \quad c \in (0, 1/2\pi).$$

Problem if α is an integer

Since Θ_1 and Θ_2 have different distributions it holds that $\mathbf{X}_1 \in \mathrm{RV}(\alpha, \mu_1)$ and $\mathbf{X}_2 \in \mathrm{RV}(\alpha, \mu_2)$ with $\mu_1 \neq \mu_2$.

It can be shown that $\mu_1(W_{\mathbf{x}}) = \mu_2(W_{\mathbf{x}})$ for all $\mathbf{x} \neq \mathbf{0}$, i.e. μ_1 and μ_2 agree on half spaces.

Conclusion:
μ is not determined by the values it gives to half spaces if α is an integer!

Notice that this does not answer the question:

Does regular variation with index α of $\mathbf{x}^{\mathrm{T}}\mathbf{X}$ for all $\mathbf{x} \neq \mathbf{0}$ imply that $\mathbf{X}$ is regularly varying with index α?

Harry Kesten's remark

In (Kesten 1973) a remark says that for $\alpha = 1$ regular variation of $\mathbf{x}^{\mathrm{T}}\mathbf{X}$ for all $\mathbf{x} \neq \mathbf{0}$ is not a sufficient condition for regular variation of $\mathbf{X}$.

It turns out that unpublished notes by Kesten contains the idea for showing that:

(Hult, Lindskog 2005)
If α is an integer, then one can find a random vector $\mathbf{X}$ which is not regularly varying for which $\mathbf{x}^{\mathrm{T}}\mathbf{X}$ is regularly varying with index α for all $\mathbf{x} \neq \mathbf{0}$.

Hence,

The Cramér-Wold device for regular variation does not hold!

Stochastic recurrence equations

If $(\mathbf{A}_1, \mathbf{B}_1), (\mathbf{A}_2, \mathbf{B}_2), \ldots$ are independent and identically distributed, $\mathbf{A}_k \in \mathbb{R}^{d \times d}$ and $\mathbf{B}_k \in \mathbb{R}^d$, then the stationary solution $\mathbf{X}_\infty$ to

$$\mathbf{X}_{n+1} = \mathbf{A}_n \mathbf{X}_n + \mathbf{B}_n,$$

under mild conditions on $(\mathbf{A}_1, \mathbf{B}_1)$, satisfies

$$\begin{cases} \text{for all } \mathbf{x} \neq \mathbf{0}, \quad \lim_{t \to \infty} t^\alpha \, \mathrm{P}(\mathbf{x}^{\mathrm{T}} \mathbf{X}_\infty > t) = h(\mathbf{x}) \text{ exists,} \\ h(\mathbf{x}) > 0 \text{ for some } \mathbf{x} \neq \mathbf{0}, \end{cases}$$

Hence, from the above condition it does not follow that $\mathbf{X}_\infty$ is regularly varying if α is an integer.

References

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