Some results on extremal and maximal processes associated with a Lévy process

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SUMMARY

We compare the largest jump in a Lévy process X_t up till time t, i.e,

$$Y_t = \sup\{|X_s - X_{s-}| : \ s \le t\},$$

to the two-sided maximal value of the process,

$$M_t = \sup\{|X_s| : \ s \le t\}.$$

Let

$$T(r) = \inf\{t > 0 : |X_t| > r\}, \ r > 0,$$

be the two-sided passage time out of the two-sided strip $[-r, r]$. Then we show that Y_t is negligible with respect to M_t for small times, i.e.,

$$\lim_{t \downarrow 0} \frac{Y_t}{M_t} = 0 \quad \text{a.s.},$$

iff the overshoot $X_{T(r)} - r$ is relatively stable in the sense that

$$\lim_{r \downarrow 0} \frac{|X_{T(r)}|}{r} = 1 \quad \text{a.s.},$$

These are further equivalent to
(i) the a.s. convergence of the (stochastic) integral

$$\int_0^1 \overline{\Pi}(M_t) dt,$$

where $\overline{\Pi}(\cdot)$ is the Lévy measure associated with X, and to
(ii) the bounded variation (with nonzero drift) of X.

Negligibility of Y_t with respect to M_t as $t \to \infty$ can similarly be characterised.

RESULTS

Let $X = \{X_t\}$ be a Lévy process on $\mathbb{R}$ starting at $X_0 = 0$. We adopt the usual probability set-up as described in Bertoin (1996), Sato (1999).

X has *Lévy exponent* $\psi(\theta) \equiv \log \mathbf{E} e^{i\theta X_t}/t =$

$$i\gamma\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{-\infty}^{\infty} \left(e^{i\theta x} - 1 - i\theta x 1_{\{|x|\le 1\}}\right) \Pi(dx),$$

for $\theta \in \mathbb{R}$, $t > 0$, with $\gamma \in \mathbb{R}$ and $\sigma \geq 0$.
Π is a measure on $\mathbb{R}$ which satisfies

$$\int_{-\infty}^{\infty} (x^2 \wedge 1)\Pi(dx) < \infty.$$

Assume throughout that

$$\Pi\{(-1, 1)\} = \infty.$$

This guarantees that X has infinitely many small jumps, a.s., in every non-degenerate time interval.

Define the maximal process by

$$M_t = \sup\{|X_s| : \, s \leq t\}, \; t \geq 0.$$

The large and small time behaviour of M_t and the jumps of of the Lévy process relate to certain stochastic integrals in natural ways. These integrals are of interest and application in many areas; e.g, Bertoin, Biane, & Yor (2004), Bertoin & Yor, M. (2002), Erickson & Maller (2004), Lindner & Maller (2005), $\cdots$

In analysing the dominance of the two-sided maximum process of X over its largest jump process, we are led to examine the a.s. convergence at zero and at infinity of integrals of certain functions of M_t.

Denote the tail of Π by

$$\overline{\Pi}(x) = \overline{\Pi}^-(x) + \overline{\Pi}^+(x), \text{ for } x > 0,$$

where

$$\overline{\Pi}^-(x) = \Pi\{(-\infty, -x)\} \quad \text{and} \quad \overline{\Pi}^+(x) = \Pi\{(x, \infty)\}.$$

Define, for $x > 0$,

$$A(x) = \gamma + (\overline{\Pi}^+(1) - \overline{\Pi}^-(1)) + \int_1^x (\overline{\Pi}^+(y) - \overline{\Pi}^-(y))dy,$$

and let

$$\Delta X_t = X_t - X_{t-}.$$

Theorem 1 *The following are equivalent:*

$$\int_0^1 \overline{\Pi}(M_t)dt < \infty \text{ a.s.}; \tag{0.1}$$

$$\int_0^1 \mathbf{E}\overline{\Pi}(M_t)dt < \infty; \tag{0.2}$$

$$\lim_{t\downarrow 0}\left(\frac{M_t}{\sup_{0<s\le t}|\Delta X_s|}\right) = \infty \text{ a.s.}; \tag{0.3}$$

$$\lim_{r\downarrow 0}\left(\frac{|X_{T(r)}|}{r}\right) = 1 \text{ a.s.}; \tag{0.4}$$

$$\sigma^2 > 0; \text{ or } \quad \sigma^2 = 0, \int_0^1 \overline{\Pi}(x)dx < \infty, \ \& \ \lim_{x\downarrow 0} A(x) = \delta \neq 0. \tag{0.5}$$

Remarks. (i) In Thm 1, X has inf. many jumps in $(0, t)$, so the denominator in (0.3) stays positive, a.s.

(ii) $|X_{T(r)}| - r$ is the (two-sided) *overshoot* of $|X|$ over level $r > 0$. (0.4) says that the overshoot is a.s. *relatively stable.*

The equivalence of (0.4) with (0.5) is in Doney and Maller (2002). (0.5) says that X has a Brownian component or else is of bounded variation with nonzero drift.

Thus jumps are "not too big". (0.3) quantifies this.

We have a version of Theorem 1 at infinity, as follows:

Theorem 2 *The following are equivalent:*

$$\int_1^\infty \overline{\Pi}(M_t)dt < \infty \text{ a.s.}; \tag{0.6}$$

$$\lim_{t\to\infty}\left(\frac{M_t}{\sup_{0<s\le t}|\Delta X_s|}\right) = \infty \text{ a.s.}; \tag{0.7}$$

$$\lim_{r\to\infty}\left(\frac{|X_{T(r)}|}{r}\right) = 1 \text{ a.s.}; \tag{0.8}$$

$$\mathbf{E}X_1^2 < \infty \text{ and } \mathbf{E}X_1 = 0; \quad \text{or} \quad \mathbf{E}|X_1| < \infty \text{ and } \mathbf{E}X_1 \neq 0. \tag{0.9}$$

The proofs of Theorems 1 and 2 can be deduced from the following result which gives NASC for convergence of the stochastic integrals in the theorem. More generally, introduce a nonstochastic, real-valued, right-continuous nonincreasing strictly positive function g on $(0,\infty)$. We allow $g(0+) = \infty$.

The stochastic integral $\int_{c_1}^{c_2} g(M_t)dt$ can be defined pathwise (e.g. Protter 2004), and is finite a.s. for $0 < c_1 \le c_2 < \infty$ since it is not larger than $g(M_{c_1})(c_2 - c_1)$. But $g(0+)$ may be infinite so the finiteness for $c_1 \to 0$ or $c_2 \to \infty$ is an issue.

We also need the functions, on $x > 0$,

$$W(x) = \sigma^2 + 2\int_0^x y\overline{\Pi}(y)dy,$$

and

$$k(x) = \frac{x|A(x)| + W(x)}{x^2}.$$

This is useful in measuring the magnitude of M_t via relations of Pruitt (1981): for some constants $c_i > 0$, $C_i > 0$, each $t > 0$ and $x > 0$,

$$\mathbf{P}(M_t > x) \le c_1 t k(x)$$

and

$$\mathbf{P}(M_t \le x) \le \frac{C_1}{tk(x)},$$

and also, for each $r > 0$,

$$\frac{c_2}{k(r)} \le \mathbf{E}T(r) \le \frac{C_2}{k(r)}.$$

Provided X_t is not identically equal to a constant process, we have $0 < k(x) < \infty$ for all $x > 0$,
$\lim_{x\downarrow 0} x^2 k(x) = \sigma^2$, $\lim_{x\to\infty} k(x) = 0$, and, further, $k(\lambda x) \asymp k(x)$, $\lambda > 0$, $x > 0$.
When $\sigma^2 > 0$ or $\Pi\{(-1,1)\} = \infty$, we have, in addition, $\lim_{x\downarrow 0} k(x) = \infty$.

Theorem 3 *(i) Assume $\sigma^2 > 0$ or $\Pi\{(-1,1)\} = \infty$. Let T be a positive finite random variable, on the same probability space as X. Then*

$$\int_0^T g(M_t)dt < \infty \quad \text{a.s.}$$

iff

$$\int_0^1 \left(\frac{1}{k(x)}\right) |dg|(x) < \infty.$$

(ii) Alternatively,

$$\int_T^\infty g(M_t)dt < \infty \quad \text{a.s.}$$

iff

$$\int_1^\infty \left(\frac{1}{k(x)}\right) |dg|(x) < \infty.$$

Remarks. The convergences in (i) are also equivalent to

$$\mathbf{E}\int_0^1 g(M_t)dt < \infty,$$

and similarly for (ii) (under a subsidiary condition).

We can replace the rv T in Theorem 3 by any finite nonzero constant. We can also replace the ranges $(0,1)$ and $(1,\infty)$ by $(0,c)$ and (c,∞), for any $c > 0$.

We can replace $g(x)$ by $g(\lambda x)$, $\lambda > 0$, in Theorem 3.

The compound Poisson case (i.e, when $\gamma = \sigma^2 = 0$ and $\Pi\{(-\infty, \infty)\}$ is finite) is excluded from Part (i) of Theorem 3 by our assumptions. In this case, $X_t = 0 = M_t$ for all $0 \le t \le$ some random τ.

We have $\int_0^\infty g(M_t)dt < \infty$ a.s. iff both $\int_0^T g(M_t)dt < \infty$ a.s. and $\int_T^\infty g(M_t)dt < \infty$ a.s., so;

Corollary to Theorem 3:

$$\int_0^\infty g(M_t)dt < \infty \text{ a.s. iff } \int_0^\infty \left(\frac{1}{k(x)}\right) |dg|(x) < \infty.$$

When $\sigma^2 > 0$, we have $k(x) \sim \sigma^2 x^{-2}$ as $x \downarrow 0$, while when $v := \mathbf{E}X_1^2 < \infty$ and $\mathbf{E}X_1 = 0$, we have $k(x) \sim vx^{-2}$ as $x \to \infty$. In these cases, for $a = 0$ and/or $b = \infty$,

$$\int_a^b g(M_t)dt < \infty \text{ a.s. iff } \int_a^b xg(x)dx < \infty.$$

Thus, e.g., $g(x) = \frac{1}{x^2}$ gives divergence at 0 and ∞; $g(x) = \frac{1}{x}$ gives convergence at 0 but divergence at ∞.

When $X \in bv$ with nonzero drift, we have $\lim_{t \to 0} X_t/t = \delta$, so $k(x) \sim \delta x^{-1}$ as $x \downarrow 0$. When $\mathbf{E}|X_1| < \infty$ and $\mu := \mathbf{E}X_1 \neq 0$, we have $k(x) \sim |\mu| x^{-1}$ as $x \to \infty$. In these cases, for $a = 0$ and/or $b = \infty$,

$$\int_a^b g(M_t)dt < \infty \text{ a.s. iff } \quad \int_a^b g(x)dx < \infty.$$

There are situations in which the function $A(x)$ dominates $W(x)$, and vice-versa. In the first case we can replace $k(x)$ by $|A(x)|/x$ and in the second case by $W(x)/x^2$.

Example 1. Assume $\Pi\{(-1,1)\} = \infty$, $\sigma^2 = 0$ and $\lim_{t\downarrow 0} \mathbf{P}(X_t \geq 0) = 1$. Then by a result of Doney (2005), $A(x) > 0$ for all small $x > 0$, $x \leq x_0$, say, and $W(x) = O(xA(x))$ as $x \downarrow 0$. Thus $A(x)$ dominates $W(x)$ and $k(x) \asymp A(x)/x$ as $x \downarrow 0$. So the test integral can be taken to be $\int_0^{x_0} (x/A(x)|dg|(x)$.

Similarly, given $\Pi\{(-1,1)\} = \infty$, $\sigma^2 = 0$ and $\lim_{t\downarrow 0} \mathbf{P}(X_t \leq 0) = 1$, we get $A(x) < 0$ for $x \leq x_0$, say, and the test integral is $\int_0^{x_0} (x/|A(x)||dg|(x)$. These hold in particular if X is a subordinator or the negative of a subordinator. Similar results hold for $t \to \infty$ using results of Doney & Maller (2002).

Example 2. At the other extreme, $W(x)$ may dominate $|A(x)|$. Let X be a symmetric stable process with index α, i.e., having $0 < \alpha < 2$, $\sigma^2 = 0$, $A(x) = 0$, and $\overline{\Pi}(x) = cx^{-\alpha}$

for some $c > 0$, or else $\alpha = 2$, $\sigma^2 > 0$, $A(x) = 0$, and $\overline{\Pi}(x) \equiv 0$. Then $k(x) \asymp x^{-\alpha}$ as $x \downarrow 0$ or as $x \to \infty$. For this process it follows from Theorem 3 that, for any finite rv T,

$$\int_0^T g(M_t)dt < \infty \text{ a.s. } \iff \int_0^1 x^\alpha \left|dg\right|(x) < \infty.$$

There is an analogous equivalence for $\int_T^\infty g(M_t)dt < \infty$ a.s.

Extensions. Similar methods can be used to get information on, eg.:

$$\lim_{t \downarrow 0} \left(\frac{M_t}{f(\sup_{0<s\leq t} |\Delta X_s|)} \right) = \infty \text{ a.s.}$$

for certain (monotone, measurable) functions $f(\cdot)$. Again these relate to the convergence of a stochastic integral.

Also interesting would be to give conditions for

$$\lim_{t \downarrow 0} \left(\frac{S_t}{f(\sup_{0<s\leq t} |\Delta X_s|)} \right) = \infty \text{ a.s.}$$

where

$$S_t = \sup_{0\leq y\leq t} X_y.$$

Does this relate to the convergence of a stochastic integral?

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