Numerical bounds for the distribution of the maximum of a one- or two-parameter process.

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Goal and motivations

X stochastic process, $M_T = \sup_{t\in[0,T]} X_t$.

Goal:

Numerical estimation of $\mathbb{P}(M_T \geq u)$ for every u.

Motivations:

- LRT in non identifiable mixture:

  Definition of a test,
  Computation of its power.

- Estimation of the persistence exponent:

$$q(u) = \lim_{T\to+\infty} -\frac{1}{T} \ln \mathbb{P}\left(M_T < u\right).$$

- Computation of the significative height of wave.

Known Results

- Comparison Lemma, *Plackett (1954).*
- Some Processes based on the Brownian motion.
- Some stationary Gaussian processes (periodic covariance, OU).
- Rice's upper bound:

$$\mathbb{P}(M_T \geq u) \leq \mathbb{P}(X_0 \geq u) + \mathbb{E}(N_u),$$

  where N_u denotes the number of u-upcrossings.

- Extreme values:

$$\mathbb{P}\big(M_T \leq x\big) \sim_{T \to +\infty} \exp(-\exp(x/a_T + b_T)).$$

- Rice series, *Azaïs and Wschebor (2002):*

$$\mathbb{P}\big(M_T \geq u\big) = \mathbb{P}(X_0 \geq 0) + \sum_{m=1}^{+\infty} (-1)^{m+1} \frac{\mathbb{E}(U_u^{[m]} \mathbb{1}_{X_0 < u})}{m!}.$$

Statement of the integral formula

Framework:

X Gaussian with $\mathcal{C}^1$ sample paths and $\sigma_T^2 = \inf_{t\in[0,T]} \mathrm{Var}(X_t) > 0$.

Time of first passage:

$$\tau_u = \inf\{t \in (0,T), X_s < X_t, \forall s < t, X_t = u\}.$$

Formula:

$$\begin{aligned} \mathbb{P}(M_T \geq u) &= \mathbb{P}(X_0 \geq u) + \mathbb{P}(\tau_u \in (0,T)) \\ &= \mathbb{P}(X_0 \geq u) + \int_0^T \mathbb{E}(X_t'^{+} \mathbb{1}_{\{X_s<u,\forall s<t\}} / X_t = u) p_{X_t}(u) dt. \end{aligned}$$

Remark:

Rychlik (1987) gives density of τ_u.

Numerical procedure

MAGP tool-box gives bounds for $\mathbb{P}(M_T \geq u)$

rind.m: WAFO tool-box, *Brodtkorb et al. (2000).*

Arguments: (T, u, r).

Lower bound: For $\{t_1, \dots, t_m\}$ a subdivision of $[0, T]$,

$$1 - \mathbb{P}(X_{t_1} < u, \dots, X_{t_m} < u).$$

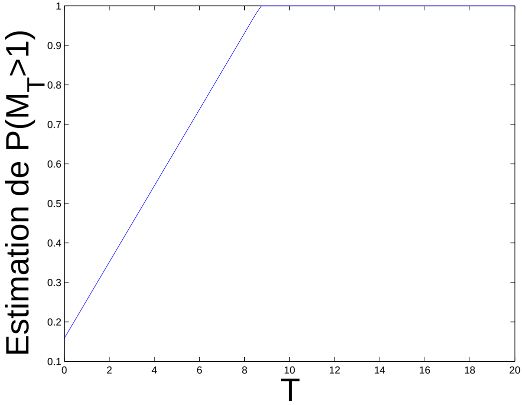
Upper bound: From integral formula

$$\mathbb{P}(X_0 \geq u) + \int_0^T \mathbb{E}(X_t'^{+} \mathbb{1}_{\{X_{s_k} < u, (s_k)_{k=1,\dots,m} \in (0,t)\}} / X_t = u) p_{X_t}(u) dt.$$

Numerical Results

Example: Gaussian process with the arguments $(T, 1, \exp(-t^2/2))$

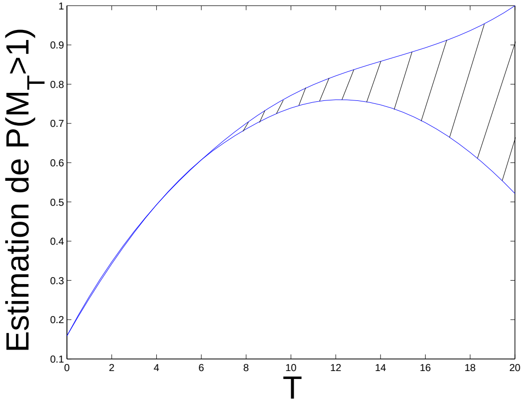
Application of the Rice's upper bound



Numerical Results

Example: Gaussian process with the arguments $(T, 1, \exp(-t^2/2))$

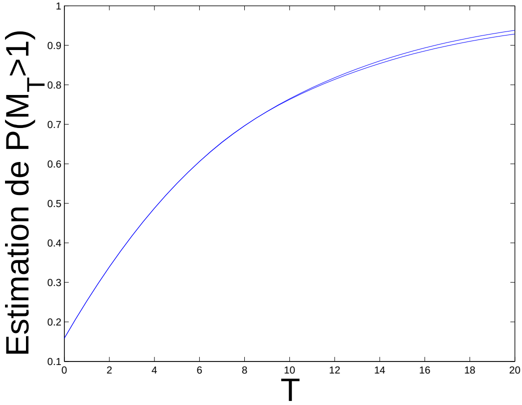
Application of the Rice series



Numerical Results

Example: Gaussian process with the arguments $(T, 1, \exp(-t^2/2))$

Application of the MAGP tool-box



Comments

Error of the procedure:
For $T \leq 25$ and $u \geq 1$ upper bounded by 10^{-3}.

Stationary and non stationary frameworks.

Non differentiable sample paths.

Estimation of the persistence exponent:

$$q(u) = \lim_{T \to +\infty} -\frac{1}{T} \ln \mathbb{P}\left(M_T < u\right).$$

Example, $r(t) = (\cosh(t))^{-1}$

- $q(0) \geq 0.25$ *Li-Shao (2002).*
- $q(0) \geq 0.365$ *Dembo, Poonen, Shao and Zeitouni (2002).*
- $q(0) \geq 0.371$

Equivalents when u is large

Framework: $M_S = \sup_{\mathbf{t}\in S} X(\mathbf{t})$,
X Gaussian process defined on $S \subset \mathbb{R}^2$ compact.

References: *Adler (1981); Adler and Taylor (2005).*

Notations: $\lambda(S)$ Lebesgue measure of S; $\Delta = \mathrm{Var}\big(X'(t)\big)$.

- One term:

$$\frac{\lambda(S)|\det(\Delta)|^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}} u e^{-\frac{u^2}{2}} + o\left(u e^{-\frac{u^2}{2}}\right)$$

- Two terms:

$$\forall\delta > 0: \quad 1-\Phi(u)+\left[\frac{T^2|\det(\Delta)|^{\frac{1}{2}}u}{(2\pi)^{\frac{3}{2}}}+\frac{T(\Delta_{11}^{\frac{1}{2}}+\Delta_{22}^{\frac{1}{2}})}{2\pi}\right]e^{-\frac{u^2}{2}}+o\left(e^{-(1+\delta)\frac{u^2}{2}}\right)$$

Extension of the integral formula

Integral formula:

↬ Study on the border of S,

↬ Study of the interior of S.

Extension of the integral formula

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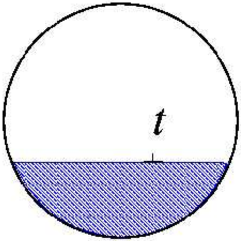
Past of $\mathbf{t}$*:*

$\Gamma_{\mathbf{t}} = \{\mathbf{s}, \mathbf{s_2} \leq \mathbf{t_2}\}$,

Record points:

$\{\mathbf{t}, X(\mathbf{s}) < X(\mathbf{t}), \forall \mathbf{s} \in \Gamma_{\mathbf{t}}\}$.



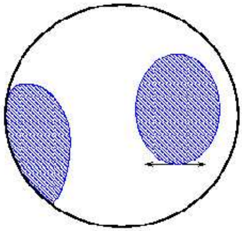
Extension of the integral formula

Integral formula:

↬ Study on the border of S,

↬ Study of the interior of S.

Crossings:
On the border: $X(\mathbf{t}) = u$,
In the interior: $\left(X(\mathbf{t}), \frac{\partial X}{\partial t_1}(\mathbf{t})\right) = (u, 0)$.



Extension of the integral formula

Integral formula:

↬ Study on the border of S,

↬ Study of the interior of S.

First explicit upper bound in the "unit-speed" case:

$$\mathbb{P}\left(M_{[0,T]^2} \geq u\right) \leq$$

$$1 - \Phi(u) + \frac{T}{\pi}\exp\left(-u^2/2\right) + \frac{T^2}{(2\pi)^{\frac{3}{2}}}[c\,\varphi(\frac{u}{c}) + u\,\Phi(\frac{u}{c})]\exp\left(-u^2/2\right)$$

with $c = (\mathrm{Var}(X_{20}) - 1)^{\frac{1}{2}}$

Numerical Procedure and Results

Upper bound: Integral formula and discretization

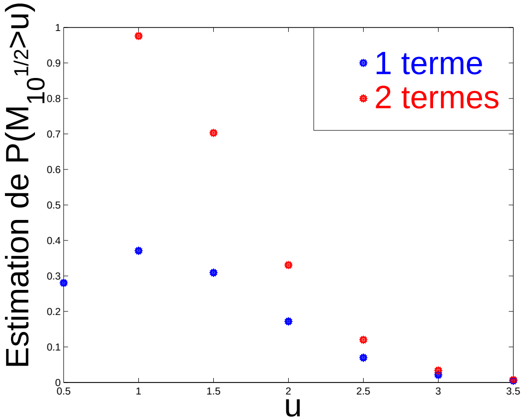
Lower bound: discretization

rind.m: WAFO tool-box, *Brodtkorb et al. (2000)*

Numerical Procedure and Results

Example: $S = [0, T]^2$ and arguments $(10, u, \exp(-\|s-t\|^2/2))$

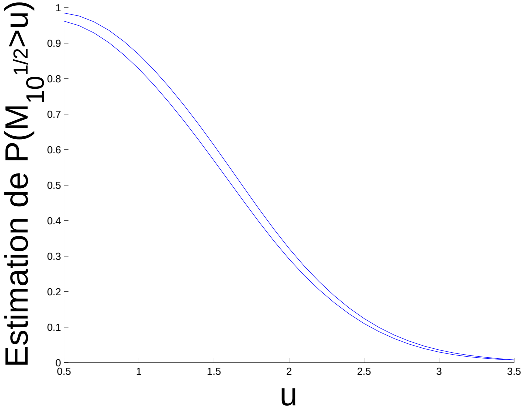
Application of equivalents



Numerical Procedure and Results

Example: $S = [0,T]^2$ and arguments $(10, u, \exp(-\|s-t\|^2/2))$

Application of the MAGP tool-box



Conclusion and perspectives

Conclusion

- $\rightsquigarrow$ Effective tool of computation.
- $\rightsquigarrow$ Geometry of the problem.

Perspectives

- $\rightsquigarrow$ rind.m and simultaneous statistics.
- $\rightsquigarrow$ Numerical extension $n = 3$.
- $\rightsquigarrow$ Explicit upper bound for all n.