ON MAXIMA OF COMPLETE AND INCOMPLETE SAMPLES FROM STATIONARY SEQUENCES

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1. INTRODUCTION

Let (X_n) be a strictly stationary random sequence with the marginal distribution function $F(x) = P\{X_1 \leq x\}$. Suppose that some of random variables $X_1, X_2, X_3, \dots$ can be observed, and let ε_k be the indicator of the event that random variable X_k is observed and $S_n = \varepsilon_1 + \cdots + \varepsilon_n$. We suppose that ε_k is independent of the sequence (X_n). Denote $M_n = \max\{X_1, \dots, X_n\}$ and

$$\widetilde{M}_n = \begin{cases} \max\{X_j : 1 \leqslant j \leqslant n, \varepsilon_j = 1\}, & \text{if } S_n \geqslant 1, \\ inf\{t | \, F(t) > 0\}, & \text{if } S_n = 0. \end{cases}$$

Problem. Determine the limiting distribution of the random vector $(\widetilde{M}_n, M_n)$ as $n \to \infty$.

The associated independent sequence. Following Loynes (1965), for a given stationary sequence (X_n) let us define the associated independent sequence (X_n^*) to be i.i.d. with the same distribution function $F(x) = P\{X_1^* \leqslant x\} = P\{X_1 \leqslant x\}$. Let us denote

$$\widetilde{M}_n^* = \max\{X_j^* \, : \, 1 \leqslant j \leqslant n, \, \varepsilon_j = 1\}, \; M_n^* = \max\{X_1^*, \dots, X_n^*\}.$$

Under some conditions of weak dependence of random variables in the sequence (X_n) Leadbetter (1974) proved that random variables M_n and M_n^* have the same limiting distributions with the same normalizing constants. We shall first determine the limiting distribution of the random vector $(\widetilde{M}_n^*, M_n^*)$.

2. SOME PRELIMINARIES AND EXAMPLES

A distribution function F belongs to the domain of attraction of a non-degenerate distribution function G (notation $F \in D(G)$) if there exist sequences $a_n > 0$ and $b_n \in R$, $n \in \mathbf{N}$, such that the equality

$$\lim_{n\to\infty} F^n(a_n x + b_n) = G(x) \tag{2.1}$$

holds for every continuity point of G. Every distribution function with non-empty domain of attraction is of one of the following three types:

$$\Lambda(x) = \exp(-e^{-x}), \qquad -\infty < x < +\infty; \tag{2.2}$$

$$\Phi_\alpha(x) = \begin{cases} 0, & \text{if } x < 0, \\ \exp(-x^{-\alpha}), & \text{if } x \geqslant 0, \end{cases} \quad (\alpha > 0), \tag{2.3}$$

$$\Psi_\alpha(x) = \begin{cases} \exp(-(-x)^{\alpha}), & \text{if } x < 0, \\ 1, & \text{if } x \geqslant 0, \end{cases} \quad (\alpha > 0). \tag{2.4}$$

We shall refer to $\Lambda(x)$, $\Phi_\alpha(x)$ and $\Psi_\alpha(x)$ as extreme value distribution functions. The characterization of domains of attraction can be given in terms of the regular varying of tails of corresponding distribution functions. For example, $F \in D(\Phi_\alpha)$, for some $\alpha > 0$, if and only if

$$\lim_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \quad \text{for every } x > 0. \tag{2.5}$$

For more details about domains of attraction of extreme value distribution functions see Gnedenko (1943), Mejzler (1949), de Haan (1970) and Resnick (1987).

The result concerning limiting distribution of the random vector $(\widetilde{M}_n^*, M_n^*)$ will be formulated under some conditions on the sequence (ε_n).

The Probability $P\{\widetilde{M}_n^* \leqslant u_n, M_n^* \leqslant v_n\}$

Let (ε_n) be an i.i.d. sequence, independent of (X_n^*), and

$$P\{\varepsilon_k = 1\} = p, \quad P\{\varepsilon_k = 0\} = 1 - p,$$

where $0 < p < 1$. Then $S_n \in \mathcal{B}(n,p)$, i.e. S_n is a binomial random variable with parameters n and p. If $u_n = a_n x + b_n$, $v_n = a_n y + b_n$, where $a_n > 0$, $b_n \in \mathbf{R}$, and $x < y$, then the following equalities hold:

$$\begin{aligned} P\{\widetilde{M}_n^* \leqslant u_n, M_n^* \leqslant v_n\} \\ &= \sum_{k=0}^{n} P\{S_n = k\} P\{\widetilde{M}_n^* \leqslant u_n, M_n^* \leqslant v_n \mid S_n = k\} \\ &= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} (F(u_n))^k (F(v_n))^{n-k} \\ &= (pF(u_n) + (1-p)F(v_n))^n. \end{aligned} \tag{2.6}$$

Example 1.

Let (X_n^*) be an i.i.d. sequence with the common distribution function $F(t) = 1 - e^{-t}$. Then, for every real x,

$$P\{M_n^* \leqslant x + \ln n\} \to \exp(-e^{-x}), \quad \text{as} \quad n \to \infty,$$

and $F \in D(\Lambda)$. For $x < y$ let us denote $u_n = x + \ln n$ and $v_n = y + \ln n$. If (ε_n) is an i.i.d. sequence, independent of (X_n^*), and $S_n \in \mathcal{B}(n,p)$, where $0 < p < 1$, then

$$\begin{aligned}
P\{\widetilde{M}_n^* \leqslant u_n, M_n^* \leqslant v_n\} \\
&= (pF(u_n) + (1-p)F(v_n))^n \\
&= \left\{p\left(1 - e^{-(x+\ln n)}\right) + (1-p)\left(1 - e^{-(y+\ln n)}\right)\right\} \\
&= \left\{1 - \frac{pe^{-x} + (1-p)e^{-y}}{n}\right\}^n \\
&\to e^{-pe^{-x}} e^{-(1-p)e^{-y}}, \quad \text{as } n \to \infty. \qquad (2.7)
\end{aligned}$$

Example 2.

Let $F \in D(\Phi_\alpha)$, where $\alpha > 0$, and $0 < x \leqslant y < +\infty$,

$$a_n = \left(\frac{1}{1-F}\right)^{-1}(n) = \inf\left\{t:\ F(t) \geqslant 1 - \frac{1}{n}\right\}. \tag{2.8}$$

If (ε_n) is an i.i.d. sequence, independent of (X_n^*), and $S_n \in \mathcal{B}(n,p)$, where $0 < p < 1$, then we shall prove that the following equality holds:

$$\lim_{n\to\infty} P\{\widetilde{M}_n^* \leqslant a_n x, M_n^* \leqslant a_n y\} = e^{-px^{-\alpha}} e^{-(1-p)y^{-\alpha}}. \tag{2.9}$$

If the constant a_n is given by (2.8), then $a_n \to \infty$ and $1 - F(a_n) \sim \frac{1}{n}$ as $n \to \infty$.
Consequently, using (2.5) and (2.6), we obtain that

$$\begin{aligned}
&P\{\widetilde{M}_n^* \leqslant a_n x, M_n^* \leqslant a_n y\} = (pF(a_n x) + (1-p)F(a_n y))^n \\
&= \left\{1 - p\frac{1-F(a_n x)}{1-F(a_n)}(1-F(a_n)) \right. \\
&\qquad\qquad \left. - (1-p)\frac{1-F(a_n y)}{1-F(a_n)}(1-F(a_n))\right\}^n \\
&= \left\{1 - \frac{px^{-\alpha}}{n}(1+o(1)) - \frac{(1-p)y^{-\alpha}}{n}(1+o(1))\right\}^n \\
&= \left\{1 - \frac{px^{-\alpha} + (1-p)y^{-\alpha}}{n} + o\left(\frac{1}{n}\right)\right\}^n \\
&\to e^{-px^{-\alpha}} e^{-(1-p)y^{-\alpha}}, \quad \text{as } n \to \infty.
\end{aligned} \tag{2.10}$$

3a. THE LIMITING DISTRIBUTION OF THE RANDOM VECTOR $(\widetilde{M}_n^*, M_n)$

Theorem 1.

Let us suppose that the following conditions are satisfied:

(a) $F \in D(G)$, i.e. for some constants $a_n > 0$ and $b_n \in \mathbf{R}$, $n \in \mathbf{N}$ and every real x the equality (2.1) holds.

(b) (ε_n) is a sequence of indicators that is independent of (X_n^*) and such that

$$\frac{S_n}{n} \xrightarrow{P} p \in [0,1] \quad \text{as } n \to \infty. \tag{3.1}$$

Then, the following equality holds for all real $x < y$:

$$\lim_{n\to\infty} P\{\widetilde{M}_n^* \leqslant a_n x + b_n, M_n^* \leqslant a_n y + b_n\} = G^p(x) G^{1-p}(y). \tag{3.2}$$

Remark. Maxima $\widetilde{M}_n^*$ and M_n^* are asymptotically independent.

CONDITIONS OF WEAK DEPENDENCY

The condition $D(u_n)$

Definition 1. [Leadbetter (1974)] Let (X_n) be a strictly stationary random sequence and (u_n) a sequence of real numbers. The condition $D(u_n)$ is satisfied if for all positive integers

$$1 \leqslant j_1 < \cdots < j_k < j_{k+1} < \cdots < j_{k+m} \leqslant n,$$

where $j_{k+1} - j_k \geqslant l$, the inequality

$$\left| P\Big(\bigcap_{s=1}^{k+m} \{X_{j_s} \leqslant u_n\} \Big) - \right.$$
$$\left. - P\Big(\bigcap_{s=1}^{k} \{X_{j_s} \leqslant u_n\} \Big) P\Big(\bigcap_{s=k+1}^{k+m} \{X_{j_s} \leqslant u_n\} \Big) \right| \leqslant \alpha_{n,l},$$

holds and $\alpha_{n,l_n} \to 0$ as $n \to \infty$ for some $l_n = o(n)$.

The condition $D'(u_n)$

Definition 2. [Loynes (1965)] Let (X_n) be a strictly stationary random sequence and (u_n) a sequence of real numbers. The condition $D'(u_n)$ is satisfied if

$$\lim_{k\to\infty} \limsup_{n\to\infty} \; n \cdot \sum_{j=2}^{[n/k]} P\{X_1 > u_n, X_j > u_n\} = 0.$$

The condition $D(u_n, v_n)$

Definition 3. Let (X_n) be a strictly stationary random sequence, (u_n) and (v_n) two sequences of real numbers, and $\mathbf{N}_n = \{1, 2, \ldots, n\}$. The condition $D(u_n, v_n)$ is satisfied, if for all A_1, A_2, B_1, $B_2 \subset \mathbf{N}_n$, such that

$$b - a \geqslant l, \quad \text{for all } \; a \in A_1 \cup A_2, \;\; b \in B_1 \cup B_2, \\ A_1 \cap A_2 = \varnothing, \quad B_1 \cap B_2 = \varnothing,$$

the following inequality holds

$$\Bigg| P\Big(\bigcap_{j \in A_1 \cup B_1} \{X_j \leqslant u_n\} \cap \bigcap_{j \in A_2 \cup B_2} \{X_j \leqslant v_n\} \Big) - \\ - P\Big(\bigcap_{j \in A_1} \{X_j \leqslant u_n\} \cap \bigcap_{j \in A_2} \{X_j \leqslant v_n\} \Big) \times \\ \times P\Big(\bigcap_{j \in B_1} \{X_j \leqslant u_n\} \cap \bigcap_{j \in B_2} \{X_j \leqslant v_n\} \Big) \Bigg| \leqslant \alpha_{n,l},$$

and $\alpha_{n,l_n} \to 0$ as $n \to \infty$ for some $l_n = o(n)$.

Remark. If the condition $D(u_n, v_n)$ is satisfied, then it is obvious that the condition $D(u_n)$ is also satisfied.

3b. THE MAIN RESULT

THE LIMITING DISTRIBUTION OF THE RANDOM VECTOR $(\widetilde{M}_n, M_n)$

Theorem 2.

Let us suppose that the following conditions are satisfied:

(a) $F \in D(G)$, i.e. for some constants $a_n > 0$ and $b_n \in \mathbf{R}$, $n \in \mathbf{N}$ and every real x the equality (2.1) holds.

(b) (X_n) is a strictly stationary random sequence, such that conditions $D(u_n, v_n)$ and $D'(u_n)$ are satisfied for $u_n = a_n x + b_n$ and $v_n = a_n y + b_n$, where $x < y$.

(c) (ε_n) is a strictly stationary sequence of indicators that is independent of (X_n) and such that

$$\frac{S_n}{n} \xrightarrow{P} p \in [0,1] \quad \text{as} \quad n \to \infty, \tag{3.3}$$

Then, the following equality holds for all real $x < y$:

$$\lim_{n\to\infty} P\{\widetilde{M}_n \leqslant a_n x + b_n, M_n \leqslant a_n y + b_n\} = G^p(x) G^{1-p}(y). \tag{3.4}$$

Remark 1. The random variable S_n in Theorems 1 and 2 is not necessarily binomial one.

Remark 2. Limit theorem for joint distribution of maxima of a Gaussian process in continuous and discrete time was proved by Piterbarg (2002).

4. Some auxiliary results.

In this section we shall formulate some lemmas needed for proving Theorems 1 and 2.

Lemma 1.

Let the condition (a) of Theorem 2 is satisfied, $u_n = a_n x + b_n$, $v_n = a_n y + b_n$, where $x < y$ and $0 < G(x) \leqslant G(y) \leqslant 1$.

(a) The following equality holds:

$$\lim_{n\to\infty} n(1 - pF(u_n) - (1-p)F(v_n)) \\ = -p\ln G(x) - (1-p)\ln G(y). \tag{4.1}$$

(b) If k is a fixed positive integer and $m = \left[\frac{n}{k}\right]$, then the following equality holds:

$$\lim_{n\to\infty} m(F(u_n) - F(v_n)) = \frac{\ln G(x) - \ln G(y)}{k}. \tag{4.2}$$

Lemma 2.

Let (X_n) be a strictly stationary random sequence such that the condition $D(u_n, v_n)$ is satisfied for $u_n = a_n x + b_n$ and $v_n = a_n y + b_n$, where $x < y$. Let $I_1, I_2, \dots, I_k$ be subsets of $\mathbf{N}_n = \{1, 2, \dots, n\}$, such that $|b - a| \geqslant l$ for all $a \in I_s$, $b \in I_t$, where $s \neq t$, and suppose that (ε_n) is a strictly stationary sequence of indicators that is independent of (X_n). If we denote

$$M(I_s) = \max\{X_j : j \in I_s\},$$
$$\widetilde{M}(I_s) = \max\{X_j : j \in I_s, \varepsilon_j = 1\},$$

then the following inequality holds:

$$\left| P\left(\bigcap_{s=1}^{k} \{\widetilde{M}(I_s) \leqslant u_n, M(I_s) \leqslant v_n\} \right) - \right.$$
$$\left. - \prod_{s=1}^{k} P\{\widetilde{M}(I_s) \leqslant u_n, M(I_s) \leqslant v_n\} \right| \leqslant (k-1)\alpha_{n,l}. \qquad (4.3)$$

Lemma 3.

Let conditions (a) and (b) of theorem 2 be satisfied, and let (ε_n) be a strictly stationary sequence of indicators that is independent of (X_n). If k is a fixed positive integers, then the following equality holds:

$$\lim_{n \to \infty} \left(P\{\widetilde{M}_n \leqslant u_n, M_n \leqslant v_n\} \right.$$
$$\left. - P^k\{\widetilde{M}_{[n/k]} \leqslant u_n, M_{[n/k]} \leqslant v_n\} \right) = 0. \qquad (4.4)$$

5. A STORAGE PROCESS IN DISCRETE TIME

Let $X(t)$, $t \geqslant 0$, $X(0) = 0$ a.s., be a fractional Brownian motion, that is Gaussian zero mean process with stationary increments such that $E(X(t) - X(s))^2 = |t-s|^{2H}$, $0 < H \leqslant 1$. Consider a random sequence

$$Y_k = \sup_{l \geqslant k}(X(j) - X(k) - c(l-k)),$$

where $c > 0$ and $k, l \in Z$. Since X has stationary increments, Y_k is a stationary sequence, we call it a discrete time storage process with fractional Brownian motion as input. The storage process with continuous time

$$Y(t) = \sup_{\sigma \geqslant t}(X(\sigma) - X(t) - c(\sigma - t))$$

was studied by Piterbarg (2001). We have,

$$P\left\{\max_{k=1,\ldots,K} Y_k \leqslant u\right\} = P\{X(u(s+\tau)) - X(su) \leqslant u + cu\tau,$$

$$\text{for some } \tau \geqslant 0, s \in [0, u^{-1}K], su, \tau u \in Z\}$$

$$= P\left\{\sup_{\tau>0, s\in[0,u^{-1}K], su, \tau u \in Z} \frac{X(u(s+\tau)) - X(su)}{\tau^H u^H v(\tau)} \leqslant u^{1-H}\right\},$$

where $v(\tau) = \tau^{-H} + c\tau^{1-H}$. The Gaussian random field

$$Z(s,\tau) := \frac{X(u(s+\tau)) - X(su)}{\tau^H u^H v(\tau)}, \quad \tau > 0, \;\; s \in R,$$

was studied by Piterbarg (2001).

Its distribution does not depend of u, $Z(s,\tau)$ is stationary in s but not in τ, its variance $\sigma_Z^2(\tau)$ depends only of τ and has a single maximum point at $\tau_0 = H/(c(1-H))$, moreover,

$$\sigma_Z(\tau) = v^{-1}(\tau) = \frac{1}{A} - \frac{B}{2A^2}(\tau-\tau_0)^2 + O((\tau-\tau_0)^3)$$

as $\tau \to \tau_0$, where

$$A := \frac{1}{1-H}\left(\frac{H}{c(1-H)}\right)^{-H} = v(\tau_0),$$
$$B := H\left(\frac{H}{c(1-H)}\right)^{-H-2} = v''(\tau_0).$$

For the correlation function $r(s,\tau;s',\tau')$ of Z one has

$$r(s,\tau;s',\tau') = 1 - \frac{1+o(1)}{2\tau_0^{2H}}(|s-s'+\tau-\tau'|^{2H} + |s-s'|^{2H}),$$

as $s-s' \to 0$, $\tau-\tau' \to 0$, and, for sufficiently large $|s-s'|$ and an absolute constant C,

$$|r(s,\tau;s',\tau')| \leqslant C|s-s'|^{2H-2},$$

if $2H \neq 1$. For $2H = 1$, we have $r(s,\tau;s',\tau') = 0$ for large $|s-s'|$ since the increments of the Brownian motion on disjoint intervals are independent. Details can be found in Piterbarg (2001), and Hüsler and Piterbarg (2004).

Here we consider the Gaussian field $Z(s,\tau)$ on the grid $\mathcal{R}_u = u^{-1}(Z_+ \times Z)$.

Lemma 4. Assume that $H > 1/2$. There exists $\delta > 0$ such that

$$P\left\{\max_{k=1,\dots,K} Y_k > u\right\} \sim P\left\{\max_{t\in[0,K]} Y(t) > u\right\}$$
$$\sim P\{Y(0) > u\} \sim P\{Y_1 > u\}$$

as $u \to \infty$, where K may tend to infinity such that $K = O(e^{\delta u^2})$.

Lemma 5. Let $u_n, v_n \to +\infty$ as $n \to \infty$, such that u_n/v_n is bounded from zero and infinity. Then the condition $D(u_n, v_n)$ is fulfilled for the sequence (Y_k).

Corollary. Let (ε_k) be a sequence of indicators that is independent of the sequence (Y_k), and let us denote $S_n = \varepsilon_1 + \cdots + \varepsilon_n$ and

$$\widetilde{M}_n = \max\{Y_j : 1 \leqslant j \leqslant n, \varepsilon_j = 1\}, \quad M_n = \max\{Y_1, \dots, Y_n\}.$$

If $P\{S_n = 0\} \to 0$ as $n \to \infty$, then a consequence of Lemma 4 is the complete dependence of $\widetilde{M}_n$ and M_n.

A similar conclusion can be derived for a discrete time storage process with a self-similar and infinitely divisible input. A continuous time storage process with a self-similar and infinitely divisible input was studied by Albin and Samorodnitsky (2004).

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ON ESTIMATION OF THE EXPONENT OF REGULAR VARIATION USING A SAMPLE WITH MISSING OBSERVATIONS

The Class $\mathcal{F}_\alpha^0$. For every $\alpha > 0$ let $\mathcal{F}_\alpha^0$ be the set of all distribution functions which satisfy the following conditions:

$$\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = x^{-\alpha} \quad \text{for all } x > 0;$$
$$F(0) = 0.$$

A sample with missing observations. Let (X_n) be a sequence of possibly dependent random variables with the same marginal distribution function $F \in \mathcal{F}_\alpha^0$ for some $\alpha > 0$. We are interested in the estimation of α using a sample obtained by observing some of random variables $X_1, \ldots, X_n$. Let S_n be the number of observed random variables from the set $\{X_1, \ldots, X_n\}$. Denete observed random variables by $\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_{S_n}$.

Assumption A. The random variable S_n does not depend of $X_1, X_2, \ldots$ and there exists a sequence of real numbers (γ_n) such that the following conditions are satisfied:

$$\lim_{n\to\infty} \gamma_n = +\infty,$$
$$\frac{S_n}{\gamma_n} \xrightarrow{P} c_0 > 0 \quad \text{as } n \to \infty.$$

Let (β_n) be a sequence of real numbers such that

$$\lim_{n\to\infty} \beta_n = +\infty, \qquad \lim_{n\to\infty} \beta_n/\gamma_n = 0.$$

The Hill estimator. Suppose that S_n satisfies Assumption A. Let $X_{1,S_n} \geqslant X_{2,S_n} \geqslant \ldots \geqslant X_{S_n,S_n}$ be the sequence of order statistics defined by S_n observed random variables from the set $\{X_1, \ldots, X_n\}$. We shall consider the Hill estimator of the parameter $\frac{1}{\alpha}$ which is given by

$$H_{S_n} = \frac{1}{M_n} \sum_{j=1}^{M_n} \ln X_{j,S_n} - \ln X_{M_n+1,S_n} \quad \text{if } S_n \geqslant \beta_n,$$

and $H_{S_n} = 0$ if $S_n < \beta_n$, where $M_n = [S_n/\beta_n]$, $0 < \beta_n < \gamma_n$. Let $B_n = M_n/S_n$ if $S_n \geqslant 1$ and $B_n = 0$ if $S_n = 0$.

Theorem. Let us denote

$$T_{ni} = (\ln \widetilde{X}_i - \ln F^{-1}(1 - B_n))_+,$$
$$I_{ni} = I\{\ln \widetilde{X}_i - \ln F^{-1}(1 - \rho B_n) > \varepsilon\},$$

where $I\{\}$ is notation for an indicator function, and $\widetilde{X}_i$ is the i-th observed random variable. Suppose that Assumption A holds and the following conditions are satisfied:

- $I\{S_n \geqslant \beta_n\} \cdot \dfrac{1}{M_n} \sum_{i=1}^{S_n} (T_{ni} - ET_{ni}) \xrightarrow{P} 0$ as $n \to \infty$;
- $I\{S_n \geqslant \beta_n\} \cdot \dfrac{1}{M_n} \sum_{i=1}^{S_n} (I_{ni} - EI_{ni}) \xrightarrow{P} 0$ as $n \to \infty$, for all $\varepsilon \in R$ and $\rho \in J$, where J is some neighborhood of 1.

Then, $H_{S_n} \to 1/\alpha$ in probability as $n \to \infty$.

This is a generalization of the result of Hsing (1991).