Poisson Cluster process as a model for teletraffic arrivals and its extremes

Barbara González-Arévalo, University of Louisiana
Thomas Mikosch, University of Copenhagen
Gennady Samorodnitsky, Cornell University

In the last 5 years or so a large number of publications have studied various models for communication networks in order to explain the empirical findings of long memory and scaling in such networks.

W. Willinger, M.S. Taqqu, R. Sherman and D. Wilson (1995) "Self–similarity through high variability: statistical analysis of ethernet LAN traffic at the source level"

V. Pipiras and M. S. Taqqu (2002) "The limit of a renewal reward process with heavy-tailed rewards is not a linear fractional stable motion"

T. Mikosch, S. Resnick, H. Rootzén and A.W. Stegeman (2002) "Is network traffic approximated by stable Lévy motion or Fractional Brownian motion?"

Most of the models considered are based either on various renewal arrival processes, with various rules for the amount of work each arrival brings.

We are suggesting an intuitive cluster arrival model that has been used before, but, apparently, not in the communication network context.

In this model we assume that the first packet in each cluster (flow) arrives at the points Γ_j of a rate λ Poisson process on $\mathbb{R}$.

Each flow then consists of several packets, which arrive at the times $Y_{jk} = \Gamma_j + S_{jk}$, where for each j

$$S_{jk} = \sum_{i=1}^{k} X_{ji}\,, \quad 0 \le k \le K_j\,.$$

(X_{ji}) are iid non-negative random variables (K_j) are iid integer-valued random variables. We assume that (Γ_j), (K_j) and (X_{ji}) are mutually independent.

Let $N(a,b)$ denote the number of packets arriving in the interval $(a,b]$ for $a<b$ and $N(t)=N(0,t)$, $t>0$.

Some of the questions:

- Is $N(t)$ finite with probability 1?
- What are the tails of the number of arrivals in a finite interval?
- What are the correlations between the numbers of arrivals in intervals of the same length but far apart?
- What are the scaling limits for such a model?

Claim 1. A necessary and sufficient condition for $N(t) < \infty$ with probability 1 for some $t > 0$ is $EK < \infty$. Under that assumption $EN(t)^p < \infty$ for all $t > 0$ and $p > 0$.

That is, the number of arrivals in any interval of finite length has light tails.

To study the length of the memory in the packet arrival process we consider the stationary process

$$N((h, h+1)),\ \ h = 0, 1, 2, \ldots$$

of arrivals in consecutive intervals of unit length.

Let $\gamma_N(h) = \text{cov}(N(0,1], N(h, h+1])$ be the covariance function of this process.

Claim 2.

$$\int_0^\infty \gamma_N(h)\, dh < \infty \text{ if and only if } EK^2 < \infty.$$

The divergence of the integral is often taken is an indication of long range dependence.

In this sense, presence or absence of long memory in the arrivals of a cluster process depends only on the cluster size K, but not on the in-cluster interarrival time X.

To know more about the rate of decay of the covariance function one does need to have information about the in-cluster interarrival times.

If $EK^2 = \infty$ then the actual rate of decay of the covariance function depends on the tail of K, and on in-cluster interarrival time distribution.

Here is one possible situation.

Theorem 1. Assume that $P(K > k)$ is regularly varying with index $\alpha \in (1, 2)$ or $\alpha = 1$ and $EK < \infty$. Assume also that X has a non-arithmetic distribution and $EX < \infty$. Then

$$\begin{aligned}\gamma_N(h) &\sim \lambda(EX)^{\alpha-2} \int_h^\infty \overline{F}_K(y)\, dy \\ &\sim \lambda(EX)^{\alpha-2} \frac{1}{\alpha - 1}\, h\, \overline{F}_K(h)\,, \quad \text{if} \quad \alpha > 1\end{aligned}$$

as $h \to \infty$.

The ergodicity of the stationary point process N immediately implies that the number of arrivals $N(t)$ grows roughly linearly with t:

$$\frac{N(t)}{t} \stackrel{\text{a.s.}}{\to} \lambda\,(EK+1)\,, \quad t \to \infty$$

However, the deviations of the number of arrivals from the straight line may look differently depending, mostly, on the cluster size distribution.

Theorem 2. Assume $EK^2 < \infty$. Then N satisfies the functional central limit theorem

$$\left(\frac{N(rt) - \lambda rt(EK+1)}{\sqrt{\lambda r E[(K+1)^2]}}\,,\ 0 \le t \le 1 \right)$$

$$\Rightarrow (B(t)\,,\ 0 \le t \le 1)\,, \quad \text{as} \quad r \to \infty$$

in terms of convergence of the finite-dimensional distributions, where $(B(t)\,, 0 \le t \le 1,)$ is the standard Brownian motion.

On the other hand, if $EK^2 = \infty$, then the tail of K affects the limit, and, as usual, the regular variation of the tail is associated with a stable limit.

Theorem 2. Assume that $P(K > k)$ is regularly varying with index $\alpha \in (1, 2)$. Assume also that $EX < \infty$. Then N satisfies the functional central limit theorem:

$$\left(\frac{N(rt) - \lambda rt(EK + 1)}{\Theta(r)},\ 0 \le t \le 1 \right)$$

$$\Rightarrow (L_\alpha(t)\,,\ 0 \le t \le 1)\,, \quad \text{as} \quad r \to \infty$$

in terms of convergence of the finite-dimensional distributions, where $\Theta : (0, \infty) \to (0, \infty)$ is a nondecreasing function such that

$$\lim_{r \to \infty} r\, P(K > \Theta(r)) = 1$$

and $(L_\alpha(t)\,, 0 \le t \le 1,)$ is a spectrally positive α-stable Lévy motion with $L_\alpha(1) \sim S_\alpha(\sigma_\alpha, 1, 0)$, with $\sigma_\alpha > 0$.

How long is the length of each cluster?

The answer to this question is of importance for various structural properties of the cluster process, in particular for its dependence structure.

Obviously,

$$S_K = \sum_{i=1}^{K} X_i \,.$$

Under what conditions is S_K regularly varying?

We start with the case when X has heavier tail than K.

Proposition 1. Assume that $P(X > x)$ is regularly varying for some $\alpha > 0$, $EK < \infty$ and $P(K > x) = o(P(X > x))$. Then, as $x \to \infty$,

$$P(S_K > x) \sim EK\, P(X > x)\,.$$

Perhaps more natural in applications to communication networks is the situation where the tail K has heavier tail than X.

Proposition 2. Assume $P(K > k)$ is regularly varying with index $\beta \geq 0$. If $\beta = 1$, assume that $EK < \infty$. Moreover, assume that $EX < \infty$ and $P(X > x) = o(P(K > x))$. Then, as $x \to \infty$,

$$P(S_K > x) \sim P(K > (EX)^{-1}\, x)$$

$$\sim (EX)^{\beta}\, P(K > x)\,.$$

Interestingly, the statement of the proposition fails, in general, in the case $\beta = 1$ and $EK = \infty$, but will still hold under even stronger assumptions on X.

The reverse problem: what causes regularly varying tails of the cluster length?

First a a situation where K has a sufficiently light tail.

In that situation it turns out that the tail of X must be regularly varying of the same order as that of S_K.

Proposition 3. Assume $P(S_K > x)$ is regularly varying with index $\alpha > 0$ and $EK^{\max(1,\alpha+\delta)} < \infty$ for some positive δ. Then $P(X > x)$ is regularly varying with index α and

$$P(S_K > x) \sim EK\, P(X > x)$$

.

What happens if the tail of X is sufficiently light?

Than the tail of K must be regularly varying of the same order as that of S_K.

Proposition 4. Assume $P(S_K > x)$ is regularly varying with index $\alpha > 0$. Suppose that $EX < \infty$ and $P(X > x) = o(P(S_K > x))$ as $x \to \infty$. In the case $\alpha = 1$ and $ES_K = \infty$, assume that $xP(X > x) = o(P(S_K > x))$ as $x \to \infty$. Then K is regularly varying with index α and

$$P(S_K > x) \sim (EX)^\alpha P(K > x)\,.$$

Again, the case $\alpha = 1$ and infinite mean is special.

Statistical issues

To fit a cluster model to data it useful to know the stationary (Palm) distribution of the inter-arrival times. This can be computed explicitly:

$$\overline{F}_0(t) =: P_0(T_1 > t) =$$

$$\frac{1}{EK+1}\,(1 + EKP(X > t))$$

$$\exp\left\{-\lambda(t + EK\int_0^t P(X > x)\,dx)\right\}\,.$$