Practical Issues in Applications of Multivariate Extreme Values

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with

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Two Applications

- **Sea-surge data**

Modelling of surge process over space for joint flood risk assessment for coastal sites and for offshore sites needed for insurance industry

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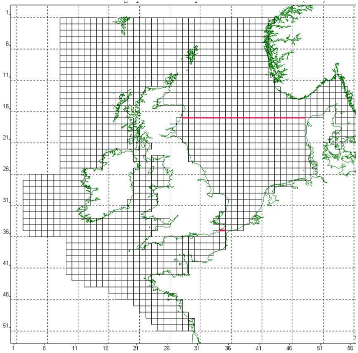
Modelling of surge process over space for joint flood risk assessment for coastal sites and for offshore sites needed for insurance industry
- **River flow data**
Modelling of river flow for network for joint flood risk assessment for planning purposes and insurance

Surge Data

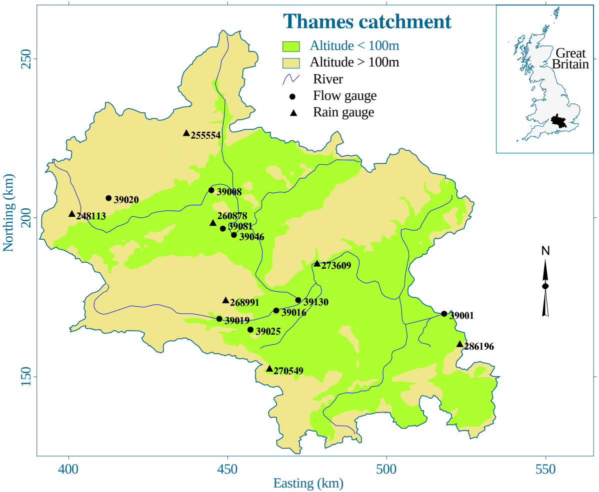
Hindcast output from the CSX model, a 2d numerical surge model for the European Continental Shelf forced by DNMI pressure data for the period 1955-2000



Data are: hourly maxima over 5-day blocks for 46 years at 259 sites

River Flow Data

Daily river flows for a network of sites in River Thames catchment in UK



Marginal Standardisation and Notation

X: univariate variable of most interest
$\mathbf{Y}$: d-dimensional variable

Transform marginals to Gumbel distributions

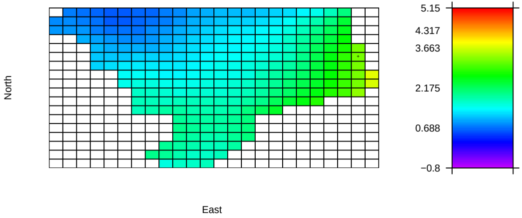
$$\Pr(X > x) = \Pr(Y_i > x) \sim \exp(-x) \text{ as } x \to \infty \text{ for } i = 1, \ldots, d$$

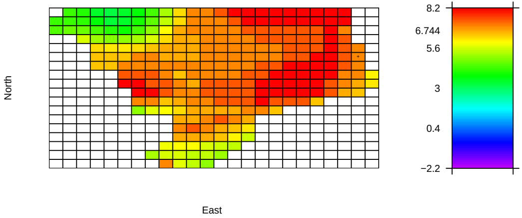
Lack of Memory Property

$$\Pr(X > t + x) \sim \exp(-t) \Pr(X > x) \text{ for large } x$$

Allows focus on dependence structure

A large surge event on the Danish coast in original and transformed margins





What is the Aim of Analysis?

- **Sea-surge data**

Simulation of surge events large at a given location
Estimation of spatial risk measure

$$E(\#\{\mathbf{Y} > x\} \mid X > x)$$

Dimension reduction for physical understanding

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- **River flow data**

Estimation of $\Pr(\mathbf{Y} > x \mid X > x)$

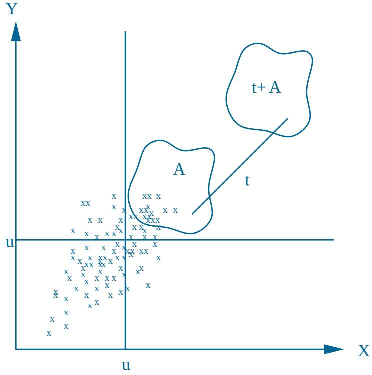
Schematic of Threshold Approach

Under assumption of asymptotic dependence

$$\lim_{x \to \infty} \Pr(Y > x \mid X > x) > 0$$

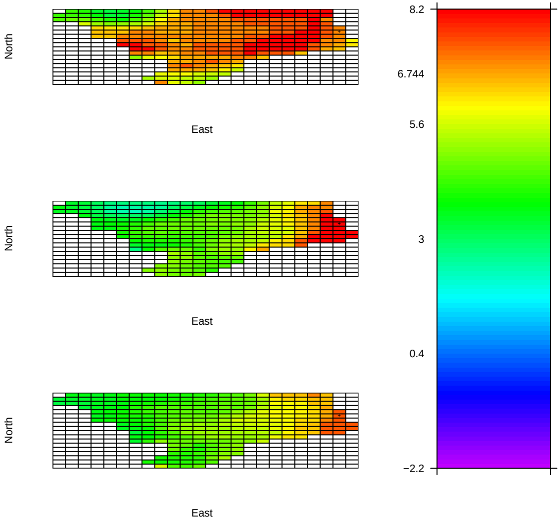
the following homogeneity property holds for all sets A extreme in at least one variable

$$\Pr((X, \mathbf{Y}) \in t + A) \approx \exp(-t) \Pr((X, \mathbf{Y}) \in A)$$



Is Surge Process Asymptotically Dependent?

X: Danish Site



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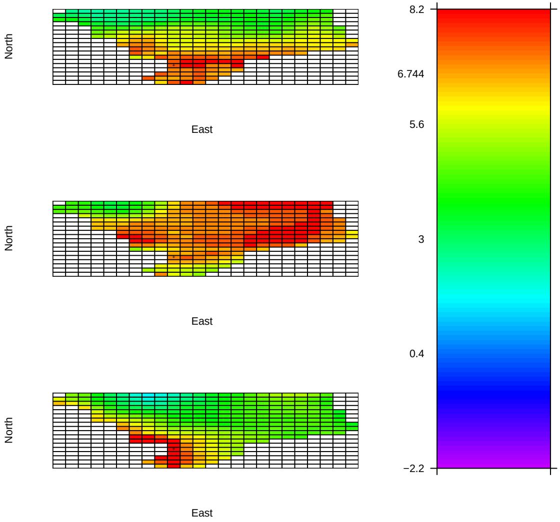

Sites Significant on Testing for Asymptotic Dependence

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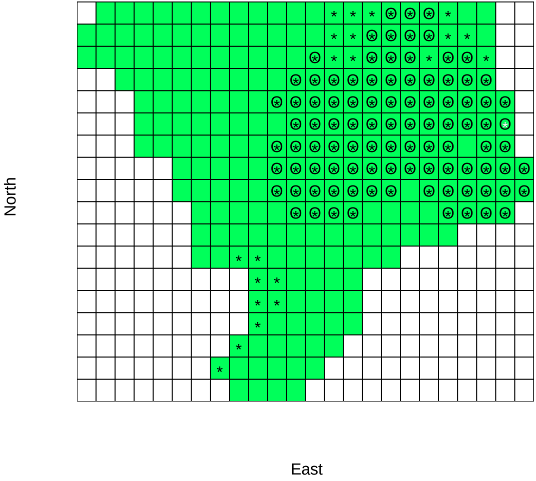

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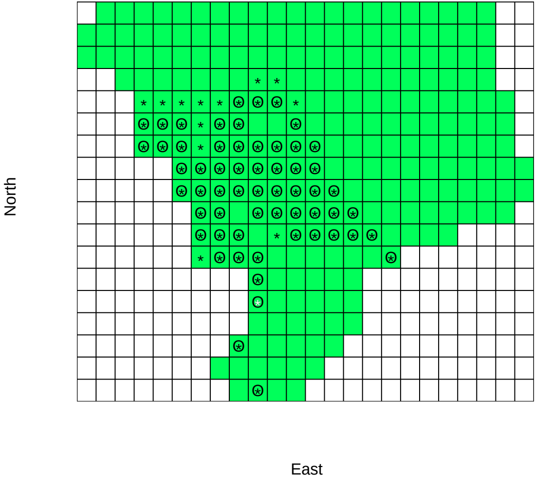
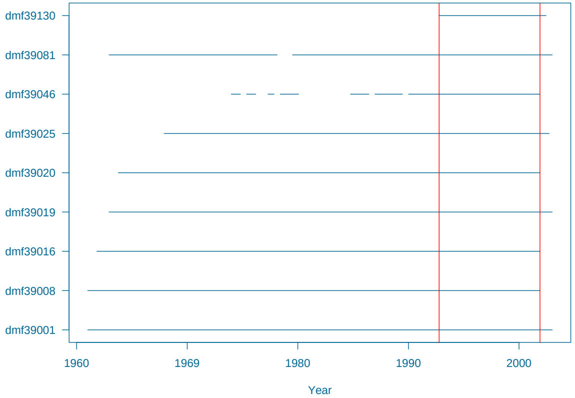


Problems for River Flow Application

Plot of data availability for Thames catchment sites



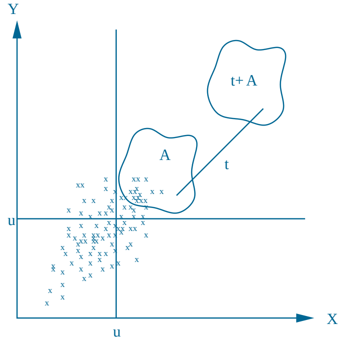
Regression Interpretation of Threshold Method

For $X > u$

$$\mathbf{Y} = X + \mathbf{Z}$$

where $\mathbf{Z}$ is independent of X

$$\hat{\Pr}((X, \mathbf{Y}) \in t + A) = \exp(-v) \int_v^\infty \frac{1}{m} \sum_{i=1}^m 1_{\{(x, x+\mathbf{z}_i) \in t+A\}} \exp(-x) dx$$



Extension of Regression/Conditional Method

Heffernan and Tawn (2004,JRSS B)

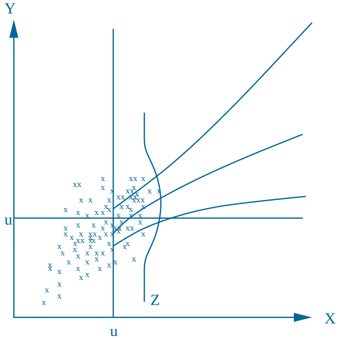
For $X > u$

$$\mathbf{Y} = \mathbf{a}X + X^{\mathbf{b}}\mathbf{Z}$$

where $\mathbf{Z}$ **is independent of** X

d**-dimensional parameters** $\mathbf{0} \leq \mathbf{a} \leq \mathbf{1}$ **and** $\mathbf{b}$

Nonparametric model for $\mathbf{Z}$



Theoretical Examples

$$\mathbf{Y} = \mathbf{a}X + X^{\mathbf{b}}\mathbf{Z}$$

Asymptotic Dependence

$$\mathbf{a} = \mathbf{1} \text{ and } \mathbf{b} = \mathbf{0}$$

Asymptotic Independence with Y_j

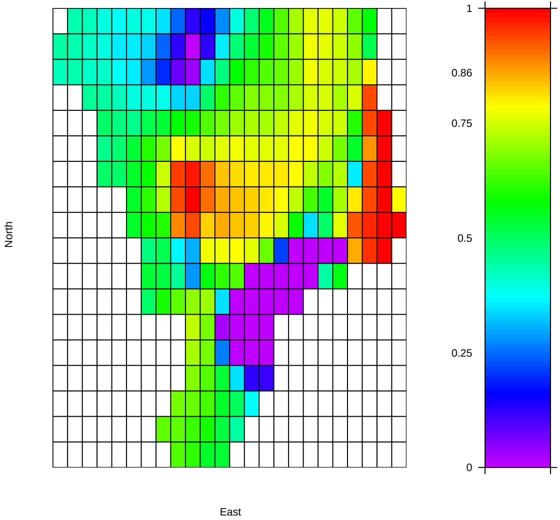
$$a_j < 1$$

Multivariate Normal Copula

$$a_j = \rho_j^2 \text{ and } b_j = \frac{1}{2} \text{ for } j = 1, \ldots, d$$

Estimates of a

X: Danish Site



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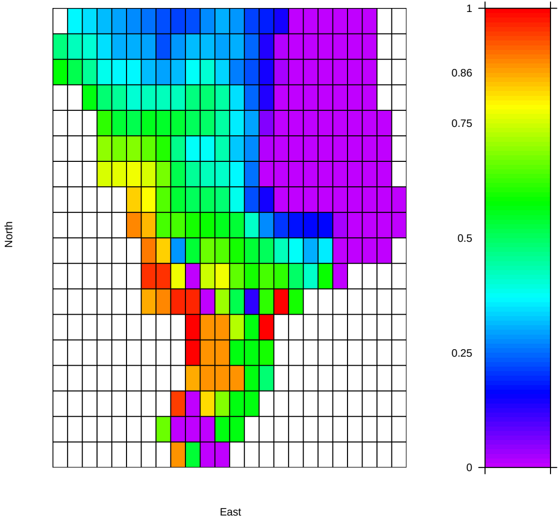

Which Sites are Asymptotically Dependent?

Test $a_j = 1, b_j = 0$

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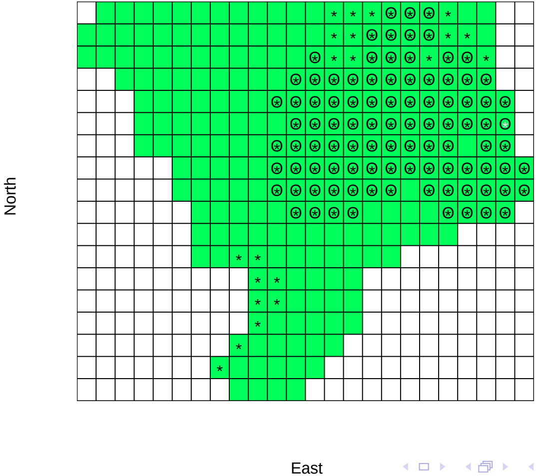

Search for Parsimonious Model

Dimension of model parameters currently $259 \times 258 \times 2$

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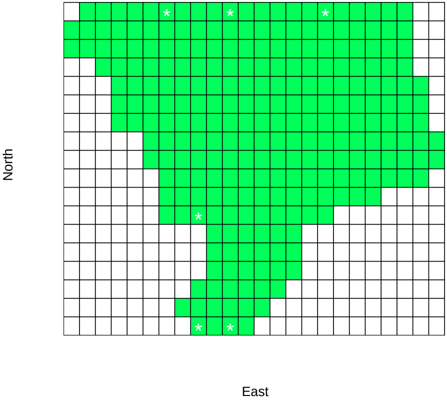


Parsimonious Spatial Model

Partition $(X, \mathbf{Y}) = (\mathbf{X}_C, \mathbf{Y}_C)$ where
$\mathbf{X}_C$ the six conditioning sites
$\mathbf{Y}_C$ the remaining sites

Then

$$[\mathbf{X}_C, \mathbf{Y}_C] = [\mathbf{X}_C][\mathbf{Y}_C \mid \mathbf{X}_C]$$

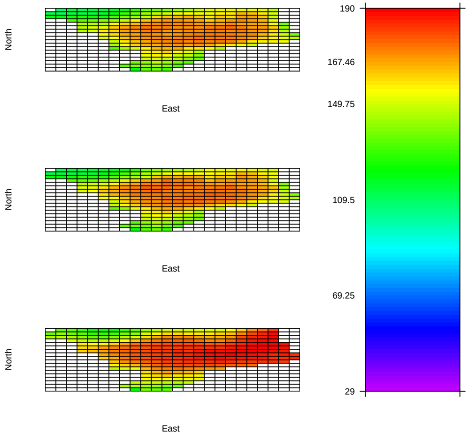
where $[\mathbf{X}_C]$ is low dimensional, and
$[\mathbf{Y}_C \mid \mathbf{X}_C]$ is simpler due to asymptotic dependence property

Extremes for $[\mathbf{Y}_C]$ only arise when $[\mathbf{X}_C]$ is extreme in at least only component

Spatial Risk Measure

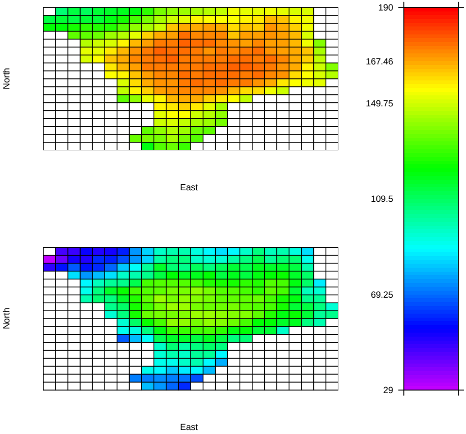
$E(\#\{\mathbf{Y} > x\} \mid X > x)$ **where x is the 97% quantile**

Comparison of empirical, global model, parsimonious model



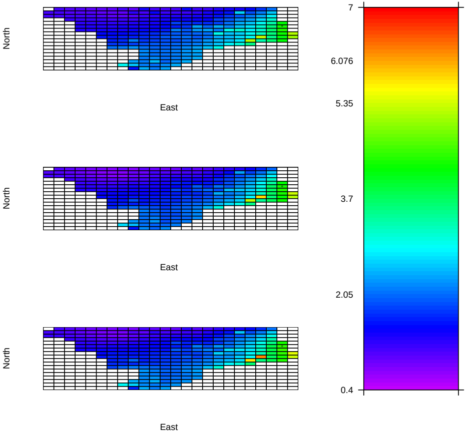
Extrapolation of Spatial Risk Measure

$E(\#\{\mathbf{Y} > x\} \mid X > x)$ **where x is the 97% and 99.9% quantiles for global model**



Simulated Fields on Original Scale

Exceeds 1000 year level on Danish coast site



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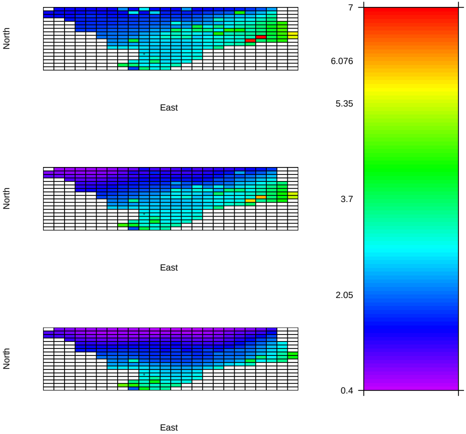

Handling Missing Data for River Flows

Partition $\mathbf{Y} = (\mathbf{Y}_M, \mathbf{Y}_O)$ where $\mathbf{Y}_M$ missing; $\mathbf{Y}_O$ observed
Also $\mathbf{Z} = (\mathbf{Z}_M, \mathbf{Z}_O)$

Then need to model $[\mathbf{Z}_M \mid \mathbf{Z}_O]$

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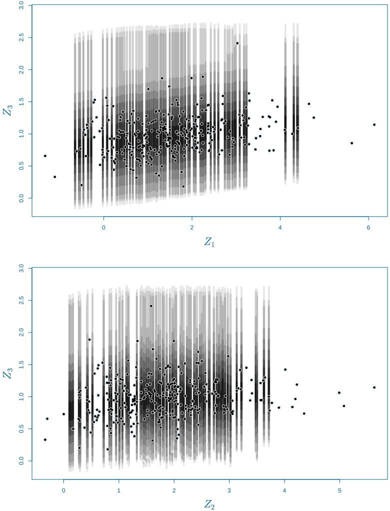
- Back transform sample and downweight values in sample $\hat{\mathbf{Z}}_M = T^{-1}(\hat{\mathbf{Z}}_M^N)$

Example of Handling Missing Data

Joint distribution model for $\mathbf{Z} = (Z_1, Z_2, Z_3)$ with infilled sample to replace missing Z_3 values

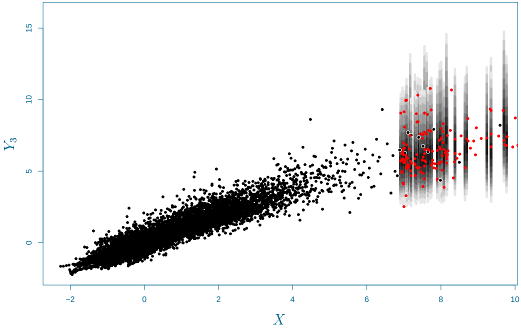


Extrapolation with Missing Data

Recall conditional model is for $X > u$

$$\mathbf{Y} = \mathbf{a}X + X^{\mathbf{b}}\mathbf{Z}$$

Extrapolation: simulate $X > v$ and independently simulate $\mathbf{Z}$ then join as above to give Y



Simulation Study to Assess Infill Method

Consider 3 different patterns of missingness with

$$X : \textbf{Full data};\ Y_1 : \textbf{50\%};\ Y_2 : \textbf{90\%};\ Y_3 : \textbf{80\%};$$

9 true distributions of $\mathbf{Z}$

Methods:
Use overlapping data only $\star$
Infill method $\circ$

Compare Estimators of:

$$P_i = \Pr(Y_i > x \mid X > x) \textbf{ for } i = 1, 2, 3$$

by RMSE efficiency relative to the Full Data case

Efficiency Results for Handling Missing Data

Results for P_1, P_2, P_3 respectively

The infill method does well!