On Univariate Extreme Value Statistics and the Estimation of Reinsurance Premiums

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Göteborg 2005

Or ...

... on univariate extreme value statistics and the estimation of insurance premiums for excess-of-loss reinsurance policies in excess of a high retention level ... with special attention to heavy-tailed distributions and Wang's premium principle as a generalization to the net premium principle.

Overview

- (Re)insurance premium calculation
  - Net premium principle
  - Wang's premium principle
    $\rightsquigarrow$ applied to excess-of-loss reinsurance setting
- Extreme value statistics
  - Motivation
  - Extreme value theory (first order framework)
  - Estimating reinsurance premiums
- Finite sample behavior
  - Simulated data (Fréchet, Burr)
  - Reinsurance premiums (net premium, dual-power transform)

Overview

- Asymptotic results
  - Motivation
  - Extreme value theory (second order framework)
  - Premium approximation error
  - Asymptotic normality and bias
- Finite sample behavior
  - Secura Belgian Re data
  - Simulated data (Fréchet, Burr)
  - Reinsurance premiums (net premium, dual-power transform)

Net premium principle

Insurance premium calculation ... at the heart of actuarial science ...

$X \rightsquigarrow$ non-negative random variable denoting the total claim amount resulting from a single insurance policy
$\big($decumulative distribution function $\bar{F}(x) = P(X > x)\big)$

Net premium principle

Under the assumption that risk is essentially non-existing if the insurer sells enough identically distributed and independent policies, on average the insurer will not lose any money when using premium

$$\Pi = E(X) = \int_0^\infty \bar{F}(x)dx$$

Wang’s premium principle

However ... experienced losses hardly ever equal expected losses ...

$\rightsquigarrow$ extra loading for risk is desirable if the insurer on average does not want to lose any money $\left(\Pi \geq E(X)\right)$

The search for sound premium calculation principles has been the subject of numerous actuarial papers and remains debatable with respect to choice ...

Wang’s premium principle $\left(\text{Wang, } 1996\right)$

With g an increasing, concave function, called the distortion, that maps $[0,1]$ onto $[0,1]$

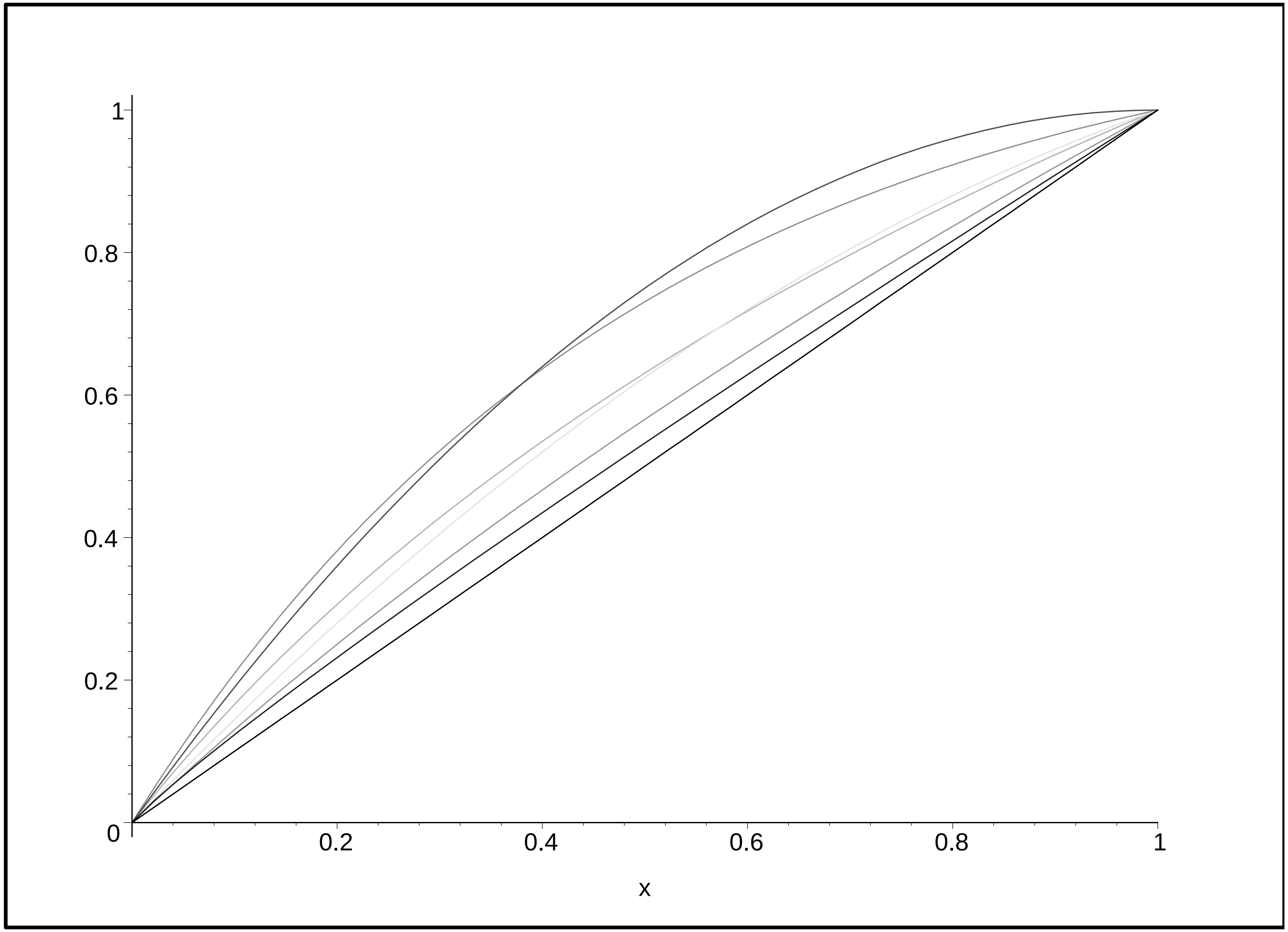
$$\Pi = \int_0^{\infty} g\left(\bar{F}(x)\right) dx$$

Wang's premium principle

Examples:

- $g(x) = x \rightsquigarrow$ Net premium principle $(\Pi = EX)$
- $g(x) = x^{1/\alpha} \quad (\alpha \geq 1) \rightsquigarrow$ Proportional hazard transform principle
- $g(x) = 1 - (1-x)^{\alpha} \quad (\alpha \geq 1) \rightsquigarrow$ Dual-power transform principle
- $g(x) = (1+\alpha)x - \alpha x^2 \quad (0 \leq \alpha \leq 1) \rightsquigarrow$ Gini principle
- $g(x) = \frac{\sqrt{1+\alpha x}-1}{\sqrt{1+\alpha}-1} \quad (\alpha > 0) \rightsquigarrow$ Square root function principle
- $g(x) = \frac{1-e^{-\alpha x}}{1-e^{-\alpha}} \quad (\alpha > 0) \rightsquigarrow$ Exponential function principle
- $g(x) = \frac{\log(1+\alpha x)}{\log(1+\alpha)} \quad (\alpha > 0) \rightsquigarrow$ Logarithmic function principle

Wang's premium principle



Excess-of-loss reinsurance setting

$$X_R = (X - R)_+ = \max(0, X - R)$$

$\rightsquigarrow$ total claim amount resulting from a single excess-of-loss reinsurance policy in excess of a high retention level R

(decumulative distribution function $\bar{F}_R(x) = \bar{F}(x + R)$)

$$\Downarrow$$

$$\Pi(R) = \int_R^\infty g\left(\bar{F}(x)\right) dx$$

$\rightsquigarrow$ premium given as a function of the decumulative distribution function $\bar{F}$ of the original claim amount for the layer from R to infinity

Motivation

In reinsurance applications, emphasis often lies on the modelling of extreme events, i.e. mostly events with

- low frequency
- high and often disastrous impact

A very practical tool in the analysis of such extreme events can be found in extreme value statistics, where the tail behavior of a distribution is characterized mainly by the extreme value index γ

- theoretical framework
- some estimators (extreme value index, small exceedance probabilities ... reinsurance premiums)

Extreme value theory (first order framework)

Consider $X_{1,n} \leq \ldots \leq X_{n,n}$ independent and identically distributed random variables with common distribution function F

- Maximum domain of attraction condition

$$\lim_{n\to\infty} P\left(\frac{X_{n,n}-b_n}{a_n} \leq x\right) = H(x)$$

- Limit necessarily of generalized extreme value type

$$H_\gamma(x) = \begin{cases} \exp\left(-(1+\gamma x)^{-\frac{1}{\gamma}}\right), & 1+\gamma x > 0, \gamma \neq 0 \\ \exp\left(-\exp(-x)\right), & x \in \mathbb{R}, \gamma = 0 \end{cases}$$

Then F is said to belong to the maximum domain of attraction of the extreme value distribution H_γ, denoted as $F \in \mathcal{D}(H_\gamma)$

Extreme value theory (first order framework)

- $\gamma < 0 \rightsquigarrow F$ belongs to the Weibull class
  (e.g. uniform, beta and reversed Burr distribution)
- $\gamma = 0 \rightsquigarrow F$ belongs to the Gumbel class
  (e.g. exponential, normal and gamma distribution)
- $\gamma > 0 \rightsquigarrow F$ belongs to the Fréchet class
  (e.g. Fréchet, Pareto and Burr distribution)

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$$\Updownarrow$$

$$\bar{F}(x) = x^{-1/\gamma} l_F(x) \rightsquigarrow \text{Pareto-type tail}$$

(l_F slowly varying at infinity, i.e. $\frac{l_F(tx)}{l_F(t)} \stackrel{t\to\infty}{\longrightarrow} 1$ for all $x > 0$)

$$U(x) = Q(1 - 1/x) = x^{\gamma} l_U(x)$$

(l_U slowly varying at infinity)

Estimating reinsurance premiums (Karamata theorem)

Assume $G(t) = g(1/t)$ to be regularly varying at infinity with index of regular variation β, i.e. $G(t) = t^{\beta} l_G(t)$

(l_G slowly varying at infinity)

$$\Downarrow \bar{F}(x) = x^{-1/\gamma} l_F(x)$$

$$\Pi(R) = \int_R^{\infty} x^{\beta/\gamma} l_{GF}(x) dx$$

(l_{GF} slowly varying at infinity)

Karamata $\Downarrow \gamma < -\beta$

$$\Pi(R) \sim \dot{\Pi}(R) = \frac{1}{-\beta/\gamma - 1} R g\left(\bar{F}(R)\right)$$

when the retention level R tends to infinity

Estimating reinsurance premiums

$$\Pi(R) \sim \dot{\Pi}(R) = \frac{1}{-\beta/\gamma - 1} R g(\bar{F}(R))$$

$$\Downarrow$$

Premium estimator: $\hat{\Pi}(R) = \frac{1}{-\beta/\hat{\gamma}-1} R g(\hat{p}_R)$

$\rightsquigarrow$ large retention level

$\rightsquigarrow$ regularly varying distortion

Estimating reinsurance premiums

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Net premium principle

$$g(x) = x \quad \rightsquigarrow \begin{cases} G(x) = x^{-1}, \\ \beta = -1, \\ l_G(x) = 1. \end{cases}$$

Estimating reinsurance premiums

$$\mathit{\Pi}(R) \sim \dot{\mathit{\Pi}}(R) = \frac{1}{-\beta/\gamma - 1} R g(\bar{F}(R))$$

$$\Downarrow$$

Premium estimator: $\hat{\mathit{\Pi}}(R) = \frac{1}{-\beta/\hat{\gamma}-1} R g(\hat{p}_R)$

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$\rightsquigarrow$ regularly varying distortion

Dual-power transform principle ($\alpha > 1$)

$$g(x) = 1 - (1-x)^{\alpha} \quad \rightsquigarrow \quad \begin{cases} G(x) = x^{-1}\{\alpha - \frac{\alpha(\alpha-1)}{2} x^{-1} + o\left(x^{-1}\right)\} \\ \beta = -1 \\ l_G(x) = \alpha - \frac{\alpha(\alpha-1)}{2} x^{-1} + o\left(x^{-1}\right) \end{cases}$$

Estimating reinsurance premiums

$$\mathit{\Pi}(R) \sim \dot{\mathit{\Pi}}(R) = \frac{1}{-\beta/\gamma - 1} Rg(\bar{F}(R))$$

$$\Downarrow$$

Premium estimator: $\hat{\mathit{\Pi}}(R) = \frac{1}{-\beta/\hat{\gamma}-1} Rg(\hat{p}_R)$

$\rightsquigarrow$ large retention level

$\rightsquigarrow$ regularly varying distortion

Needed:

- estimator $\hat{p}_R$ of small exceedance probability $p_R = \bar{F}(R)$
- estimator $\hat{\gamma}$ of the tail index γ

Estimating reinsurance premiums ($p_{R_n} \to 0$, as $n \to \infty$)

$\bar{F}(x) = x^{-1/\gamma} l_F(x)$ with l_F slowly varying at infinity

$$\lim_{t\to\infty} \frac{\bar{F}(\lambda t)}{\bar{F}(t)} = \lambda^{-1/\gamma} \lim_{t\to\infty} \frac{l_F(\lambda t)}{l_F(t)} = \lambda^{-1/\gamma}$$

for all $\lambda > 0$

- for large values $y = \lambda t$ we can approximate $\bar{F}(y)$ by

$$\bar{F}(y) \approx \bar{F}(t) \left(\frac{y}{t}\right)^{-1/\gamma}$$

- with a large threshold $t = X_{n-k,n}$ and estimator $\hat{\gamma}_k$ this leads to

$$\hat{p}_{k,R_n} = \left(\frac{k+1}{n+1}\right) \left(\frac{R_n}{X_{n-k,n}}\right)^{-1/\hat{\gamma}_k}$$

Estimating reinsurance premiums ($\gamma > 0$)

Consider scaled log-spacings $Z_{jk} = j\left(\log X_{n-j+1,n} - \log X_{n-j,n}\right)$ for $j = 1, \ldots, k$ above a large threshold $t = X_{n-k,n}$

- $\gamma > 0$: Feuerverger and Hall (1999), Beirlant *et al.* (1999)

$$Z_{jk} \approx_d \gamma g_{jk}$$

  with g_{jk} i.i.d. standard exponential random variables

- Maximum likelihood estimator on the scaled log-spacings Z_{jk} for large threshold $X_{n-k,n}$ ($1 \leq j \leq k$)

$$\hat{\gamma}_{k,H} = \frac{1}{k}\sum_{j=1}^{k} Z_{jk}$$

  Hill (1975)

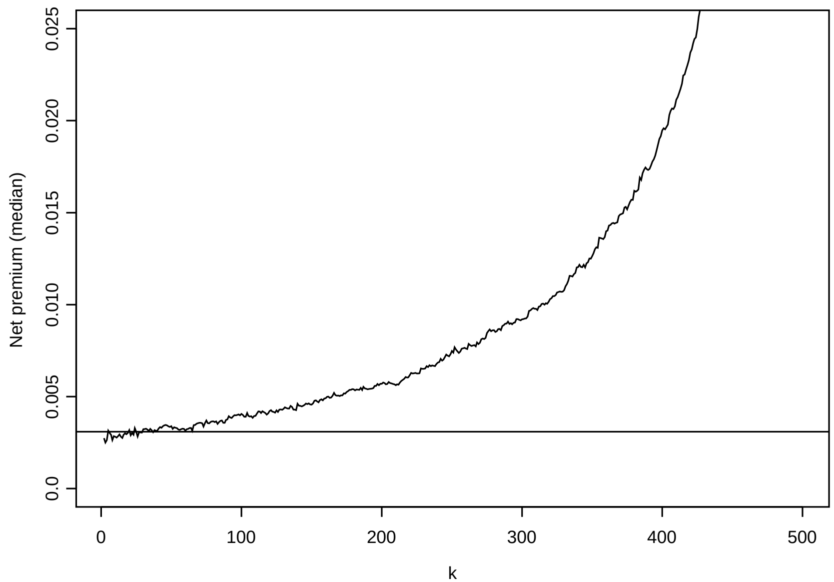
Fréchet distribution

We consider 100 simulated samples of size 500 from a Fréchet(η) distribution with $\eta = 4 \quad (\gamma = 1/4)$

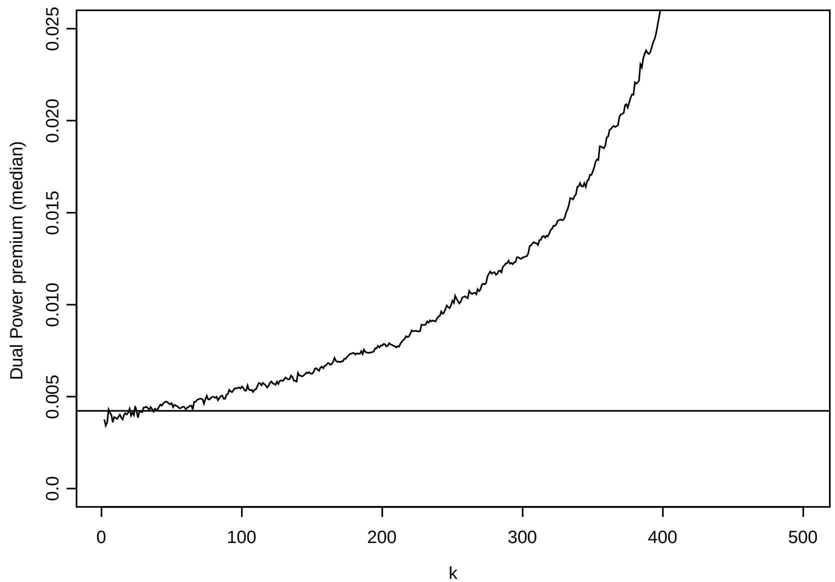
$$\bar{F}(x) = 1 - \exp(-x^{-\eta})$$

and retention level $R = U(x)$ with $x = 1.025n$

(Net)



(DP $\alpha = 1.366$)



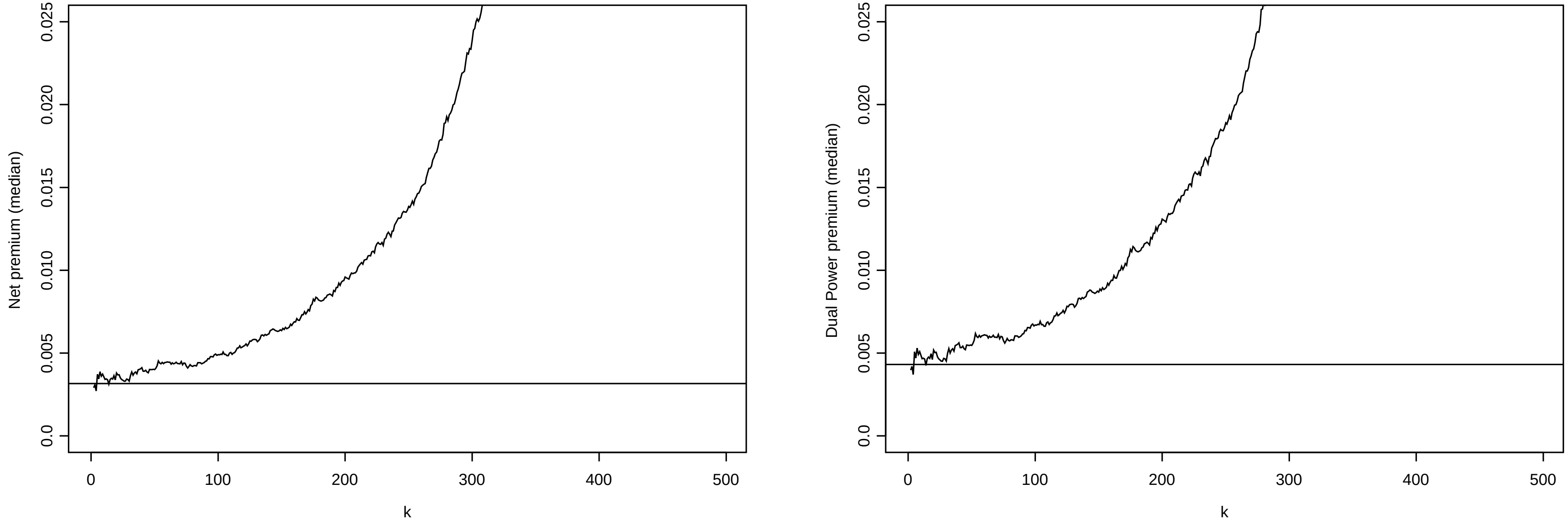
Burr distribution

We consider 100 simulated samples of size 500 from a Burr(β, τ, λ) distribution with $\beta = 1$, $\tau = 4$ and $\lambda = 1$ $\quad (\gamma = 1/4)$

$$\bar{F}(x) = \beta^{\lambda} \left(\beta + x^{\tau}\right)^{-\lambda}$$

and retention level $R = U(n)$

(Net) $\qquad$ (DP $\alpha = 1.366$)



Motivation

In order for these extreme value estimators of reinsurance premiums to become practically applicable, an optimal selection of the tail sample fraction k seems to be desirable

A bias-variance trade off through a minimizing asymptotic mean squared error criterion seems preferable in order to find an optimal tail sample fraction k_o

- asymptotic bias term due to estimating $\dot{\Pi}(R)$ by $\hat{\Pi}(R)$
- asymptotic error term due to approximating $\Pi(R)$ by $\dot{\Pi}(R)$

Extreme value theory (second order framework)

Most extreme value estimators suffer from bias due to a slow convergence rate in

$$\lim_{x\to\infty} \frac{l(\lambda x)}{l(x)} = 1$$

for all $\lambda > 0$

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$\rightsquigarrow$ Parametrization of the rate of convergence:

A slowly varying function l satisfies $(\mathcal{R}_{\rho,b})$ for some constant $\rho < 0$ and rate function b satisfying $b(x) \to 0$, as $x \to \infty$, if for all $\lambda > 1$

$$\frac{\left(\frac{l(\lambda x)}{l(x)} - 1\right)}{b(x)} \to \frac{\lambda^\rho - 1}{\rho}, \qquad \text{as } x \to \infty$$

Bingham et al. (1987)

Premium approximation error

LEMMA 1. Suppose that $l_U \in \mathcal{R}_{\tilde{\rho},\tilde{b}}$ and $l_G \in \mathcal{R}_{\rho^*,b^*}$, then as $R \to \infty$, the rate of convergence of the premium approximation $\dot{\Pi}(R)$ can be characterized as

$$\frac{\Pi(R)}{\dot{\Pi}(R)} = 1 - \frac{1}{\beta + \rho^* + \gamma}\, b^*\big(1/\bar{F}(R)\big) + \frac{\beta}{\gamma(\beta + \tilde{\rho} + \gamma)}\, \tilde{b}\big(1/\bar{F}(R)\big) + o\Big(r\big(1/\bar{F}(R)\big)\Big)$$

where the remainder function r is defined as

- $\rightsquigarrow$ $r = \tilde{b}$ in case $b^*(x) = o\big(\tilde{b}(x)\big)$
- $\rightsquigarrow$ $r = b^*$ in case $\tilde{b}(x) = o\big(b^*(x)\big)$
- $\rightsquigarrow$ $r = \tilde{b}$ or $r = b^*$ in case $\tilde{b}(x) \sim cb^*(x)$, with $c \neq 0$

Asymptotic normality and bias

THEOREM 1. Suppose that $l_U \in \mathcal{R}_{\tilde{\rho},\tilde{b}}$. Let $ABias(\hat{\gamma}_k) \sim I_{\gamma,\tilde{\rho}}\tilde{b}(n/k)$ and assume that as $k, n \to \infty$ such that $k/n \to 0$ and $\sqrt{k}\,\tilde{b}(n/k) \to 0$, $\sqrt{k}\,(\hat{\gamma}_k - \gamma) \to_d N(0, \sigma_\gamma^2)$ for some asymptotic variance σ_γ^2. Then the tail estimator $\hat{p}_{k,R_n}$ satisfies

- as $R_n \to \infty$ such that $\tilde{a}_n = \frac{k+1}{(n+1)p_{R_n}} \to \infty$ and $\frac{\log \tilde{a}_n}{\sqrt{k}} \to 0$, that

$$ABias(\hat{p}_{k,R_n}) \sim \frac{p_{R_n} \log \tilde{a}_n}{\gamma} I_{\gamma,\tilde{\rho}}\tilde{b}(n/k)$$

- and furthermore, as $k, n \to \infty$ such that $k/n \to 0$ and $\sqrt{k}\,\tilde{b}(n/k) \to 0$, that

$$\frac{\gamma}{\log \tilde{a}_n}\sqrt{k}\left(\frac{\hat{p}_{k,R_n}}{p_{R_n}} - 1\right) \to_d N(0, \sigma_\gamma^2)$$

Asymptotic normality and bias

THEOREM 2. Suppose that $l_U \in \mathcal{R}_{\tilde{\rho},\tilde{b}}$ and $l_G \in \mathcal{R}_{\rho^*,b^*}$. Moreover let $ABias(\hat{\gamma}_k) \sim I_{\gamma,\tilde{\rho}}\tilde{b}(n/k)$, and assume that as $k, n \to \infty$ such that $k/n \to 0$ and $\sqrt{k}\,\tilde{b}(n/k) \to 0$, $\sqrt{k}\,(\hat{\gamma}_k - \gamma) \to_d N(0, \sigma_\gamma^2)$ for some asymptotic variance σ_γ^2. Then, when $\gamma < -\beta$, the premium estimator $\hat{\Pi}(R)$ based on tail estimator $\hat{p}_{k,R_n}$ satisfies

- as $R_n \to \infty$ such that $\tilde{a}_n = \frac{k+1}{(n+1)p_{R_n}} \to \infty$ and $\frac{\log \tilde{a}_n}{\sqrt{k}} \to 0$, that

$$ABias(\hat{\Pi}(R)) \sim -\frac{\beta \log \tilde{a}_n \Pi(R)}{\gamma} I_{\gamma,\tilde{\rho}}\tilde{b}(n/k) + \frac{\Pi(R)}{\beta + \rho^* + \gamma}\, b_{\rho^*}\left(1/\bar{F}(R)\right)$$

- and furthermore, as $k, n \to \infty$ such that $k/n \to 0$, $\sqrt{k}\,\tilde{b}(n/k) \to 0$ and $\frac{\sqrt{k}}{\log \tilde{a}_n} b^*(1/p_R) \to 0$, that

$$-\frac{\gamma}{\beta \log \tilde{a}_n}\sqrt{k}\left(\frac{\hat{\Pi}(R)}{\Pi(R)} - 1\right) \to_d N(0, \sigma_\gamma^2)$$

Asymptotic normality and bias (AMSE)

In order to find an optimal threshold for $\hat{\Pi}(R)$, the asymptotic mean squared error that has to be minimized with respect to k then is proportional to

$$\frac{\beta^2(\log \tilde{a}_n)^2}{\gamma^2}\frac{\sigma_\gamma^2}{k} + \left\{\frac{1}{\beta+\rho^*+\gamma}\, b_{\rho^*}(1/p_R) - \frac{\beta \log \tilde{a}_n}{\gamma} I_{\gamma,\tilde{\rho}}\tilde{b}(n/k)\right\}^2$$

Example: Hill based estimator (Haeusler and Teugels, 1985)

- $\sigma_\gamma^2 = \gamma^2$
- $I_{\gamma,\tilde{\rho}} = \frac{1}{1-\tilde{\rho}}$

$$\Downarrow$$

Need are second order estimates of γ, $\tilde{b}(n/k)$, $\tilde{\rho}$ and p_R

Asymptotic normality and bias (second order estimates)

- $\gamma > 0$: Feuerverger and Hall (1999), Beirlant *et al.* (1999) with $\tilde{b}_{n,k} = \tilde{b}(n/k)$

$$Z_{jk} \approx_d \left(\gamma + \tilde{b}_{n,k} \left(\frac{j}{k+1} \right)^{-\tilde{\rho}} \right) g_{jk},$$

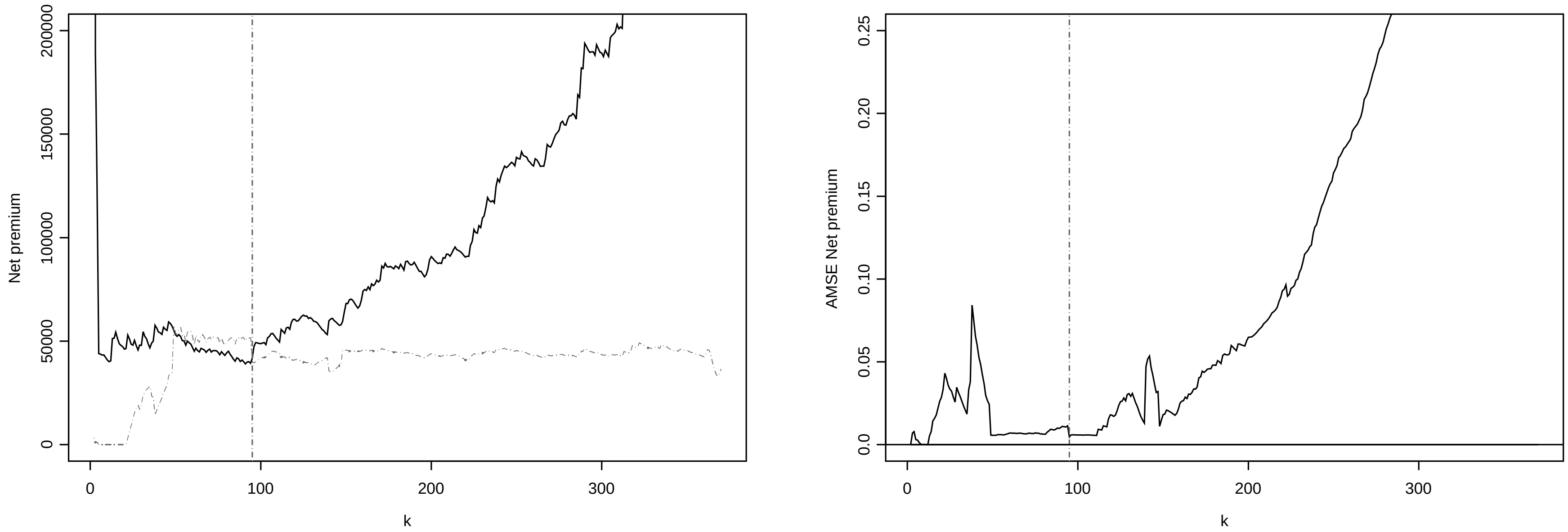
  Maximum likelihood estimator on the scaled log-spacings Z_{jk} for large threshold $X_{n-k,n}$ $(1 \leq j \leq k)$

- Matthys *et al.* (2004) numerical inversion of approximation for quantile function $x_p = Q(1-p)$ given by

$$x_p \approx X_{n-k,n} \left(\frac{k+1}{(n+1)p} \right)^{\gamma} \exp \left(\tilde{b}_{n,k} \frac{\left(\frac{k+1}{(n+1)p} \right)^{\tilde{\rho}} - 1}{\tilde{\rho}} \right)$$

Secura Belgian Re data

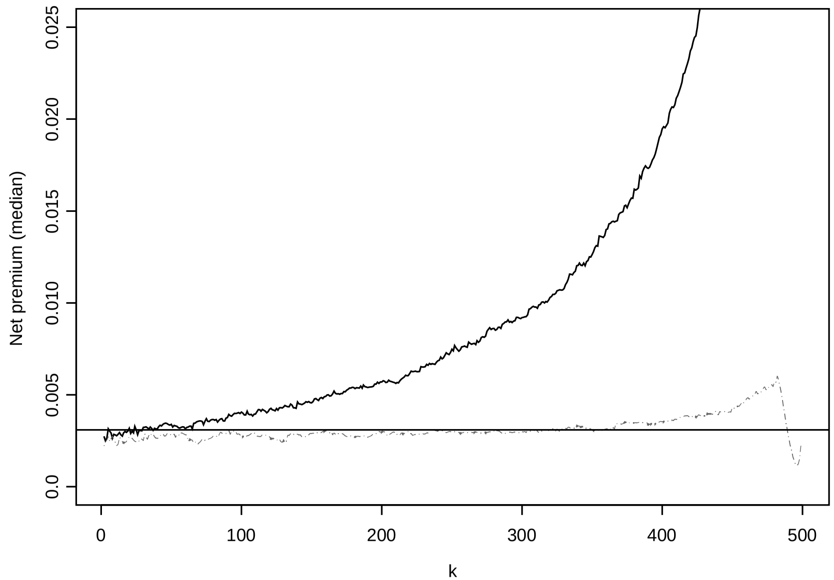
We consider the Secura Belgian Re data set on automobile claims from 1998 until 2001, as can be found in Beirlant et al. (2004). The data set consists of $n = 371$ claims which are at least as large as $1,200,000$ Euro.

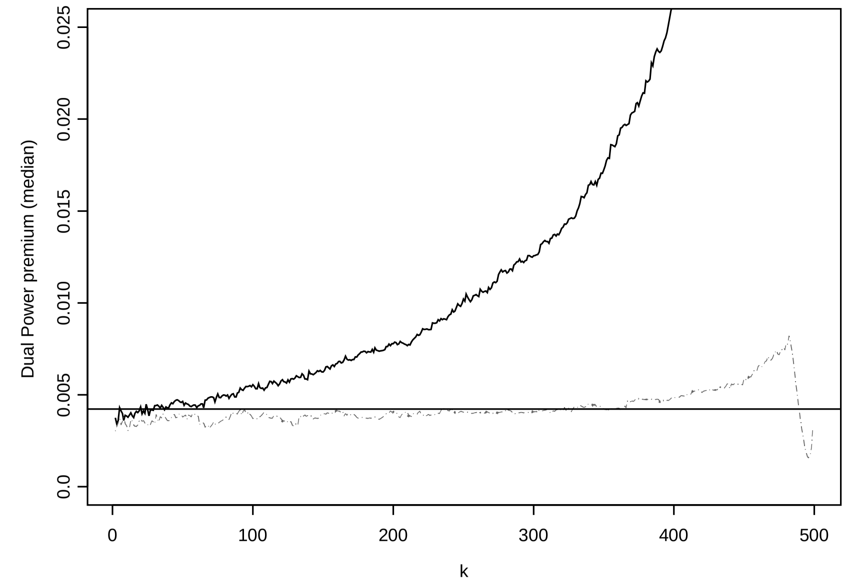


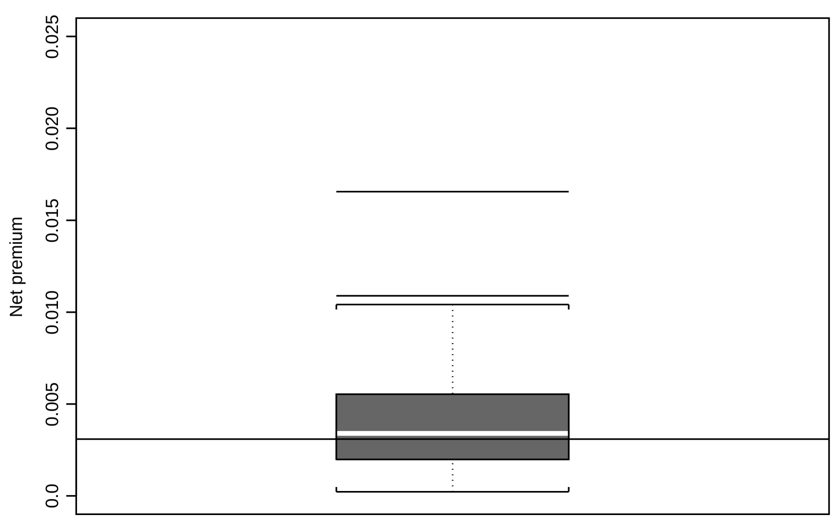
$R = 5,000,000$ Euro $\rightsquigarrow$ (Net) $k_o = 95$ $\hat{\Pi}_{k_o}(R) = 41,798.13$

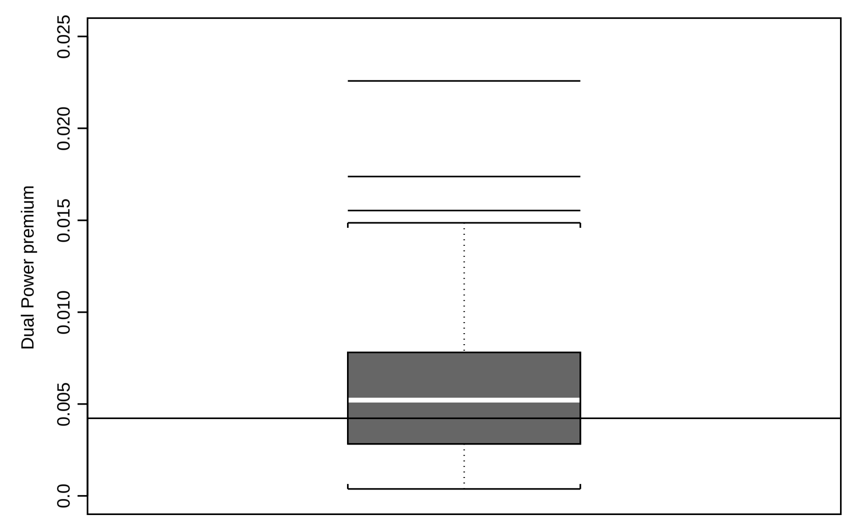
Fréchet(4) distribution

(Net) (DP $\alpha = 1.366$)



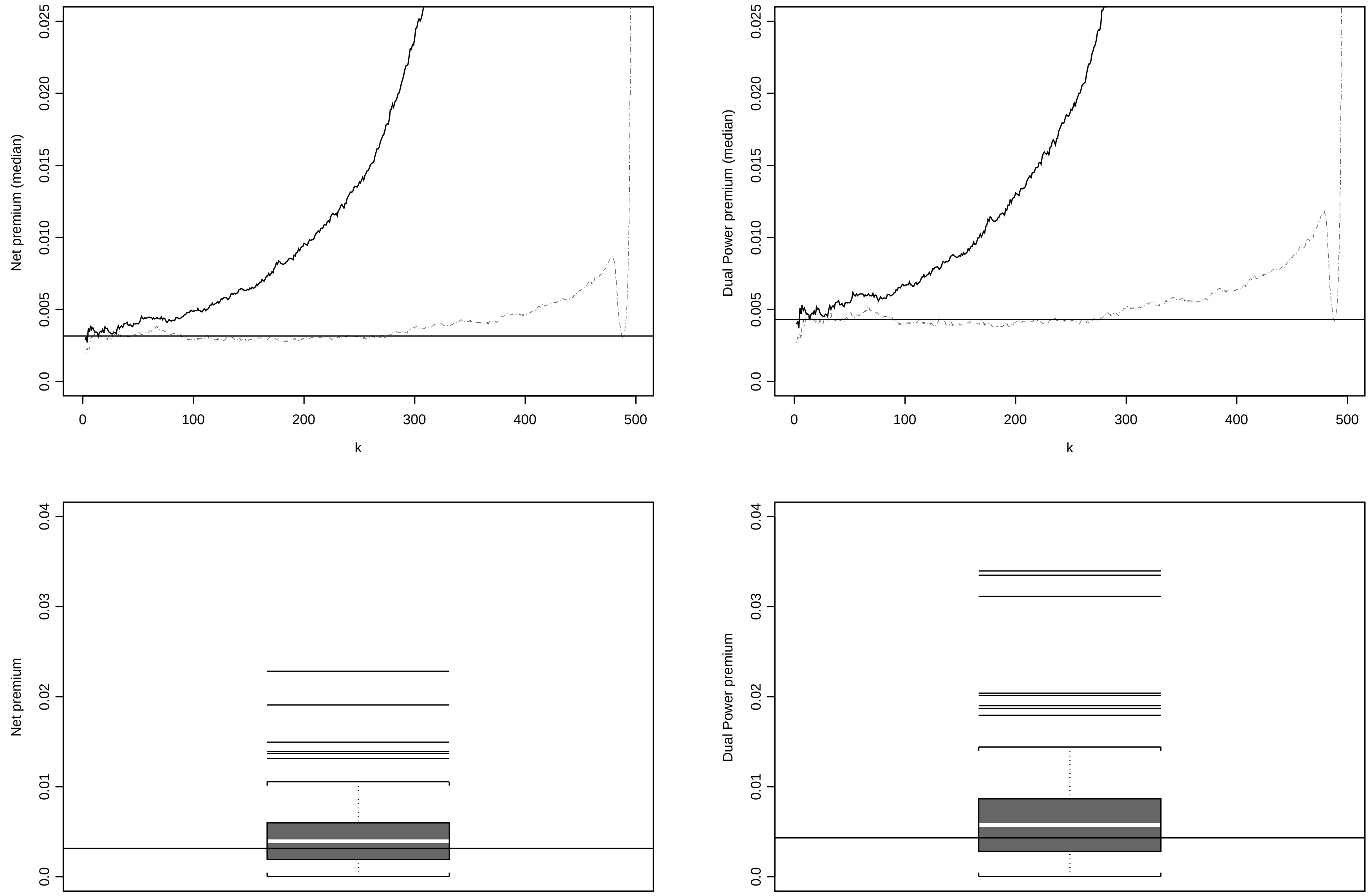






Burr(1,4,1) distribution

(Net) (DP $\alpha = 1.366$)



Conclusion

... on univariate extreme value statistics and the estimation of insurance premiums for excess-of-loss reinsurance policies in excess of a high retention level ... with special attention to heavy-tailed distributions and Wang's premium principle as a generalization to the net premium principle ...