Asymptotically (In)dependent Multivariate Maxima of Moving Maxima Processes

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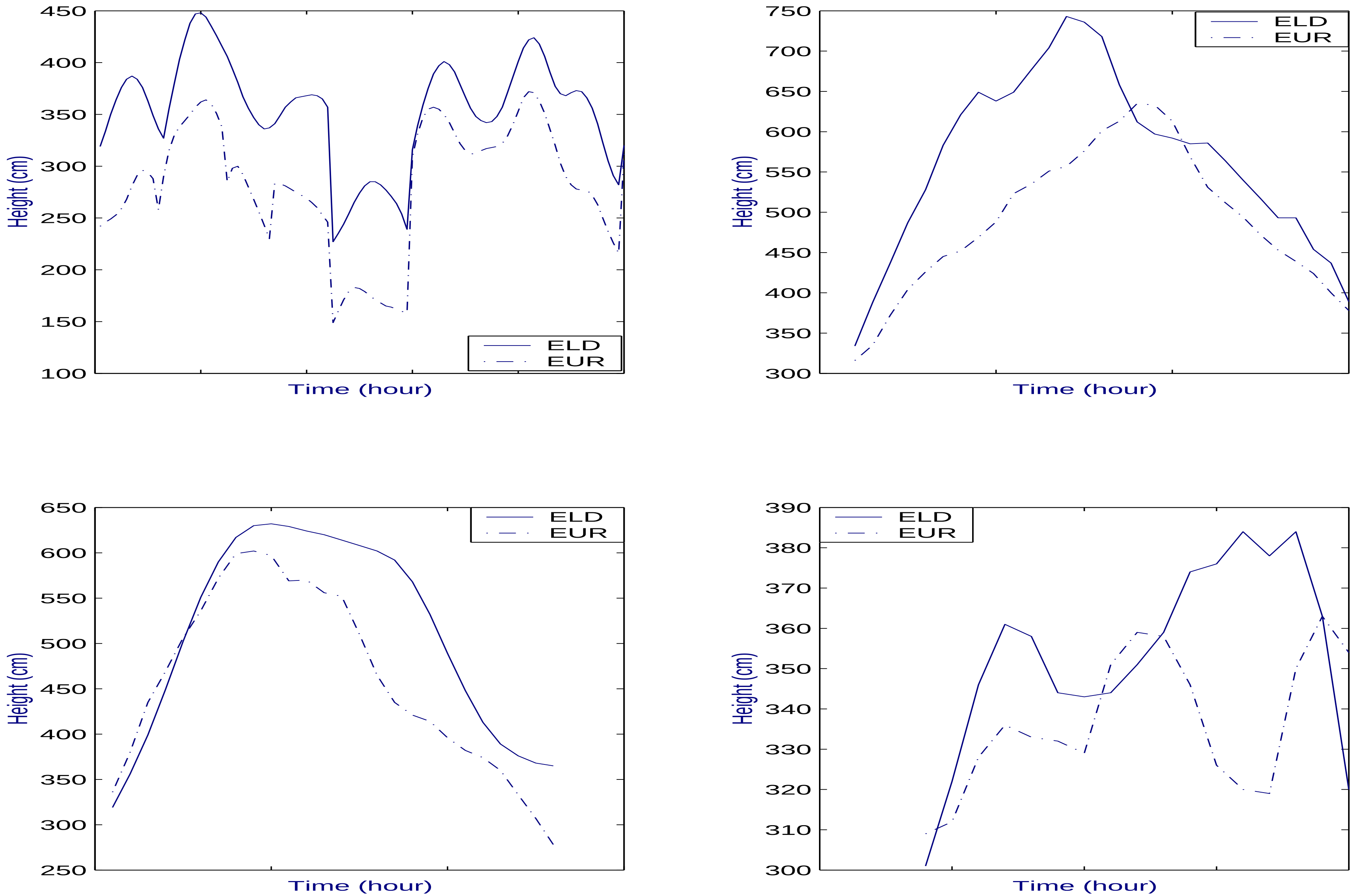
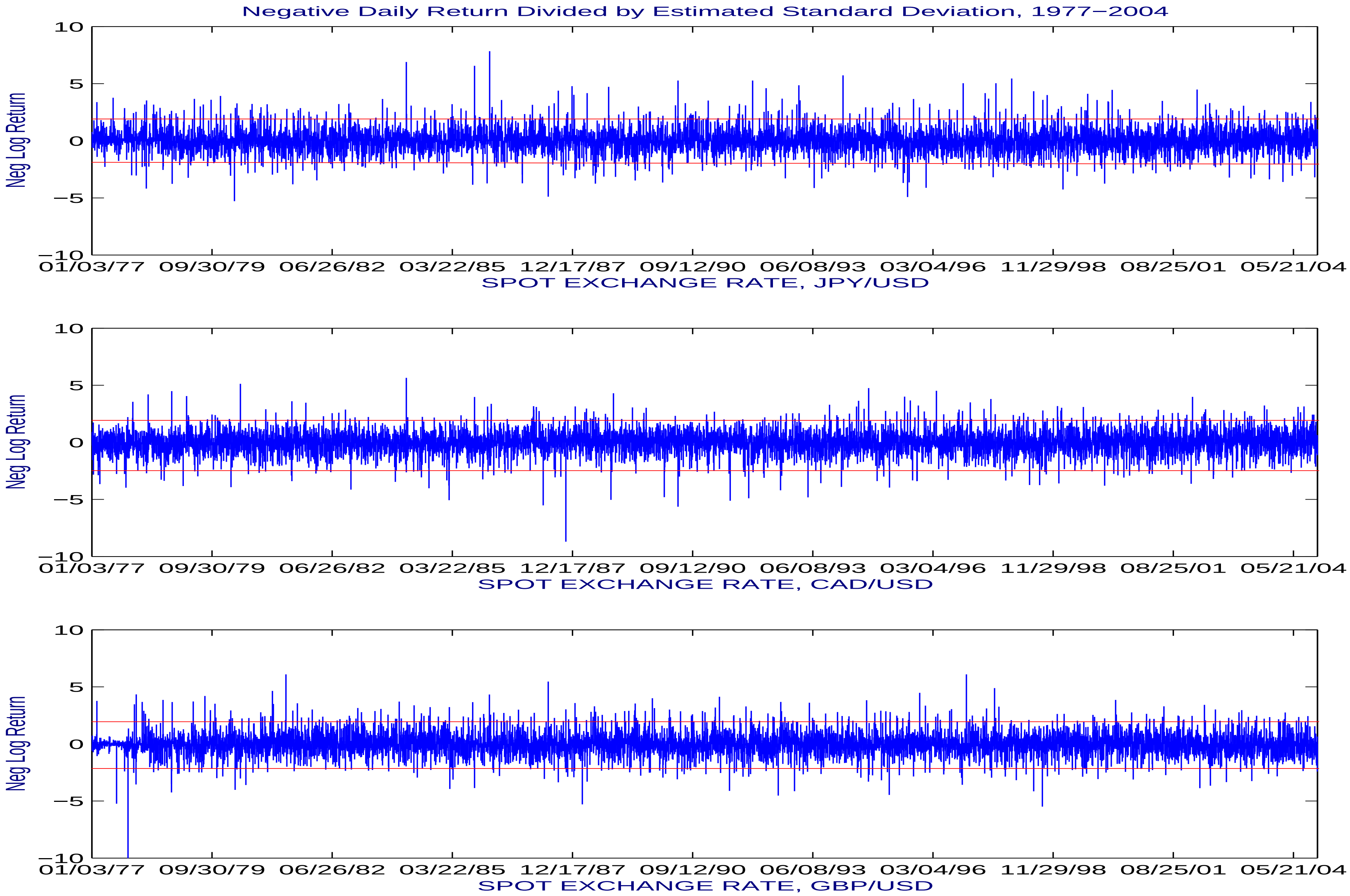
Wave heights at two different locations



Pseudo exchange returns



Challenges

- "Choose dependence model with desired dependence properties."
  - dependence in asymptotic independence
  - tail dependence
  - extremal co-movements
  - extremal impacts, etc.

Outline

1. Multivariate extremes
2. M4 class
3. Some dependence measures
4. Asymptotically independent processes

1. Multivariate extremes

Basic definitions

Suppose $\{\mathbf{X}_i = (X_{i1}, \ldots, X_{iD}), i = 1, 2, ...\}$ is a D-dimensional i.i.d. random vectors with distribution F. Let $M_{nd} = \max\{X_{id}, 1 \le i \le n\}$. If there exist normalizing constants $\mathbf{a}_n > 0, \mathbf{b}_n$ such that

$$\mathrm{P}\{M_{nd} \le a_{nd}x_d + b_{nd}, d = 1, \ldots, D\} \to H(\mathbf{x}),$$

then the distribution H is called a D-dimensional multivariate extreme value distribution and F is said to belong to the domain of attraction of H, which we write $F \in D(H)$.

- Representations: Pickands, de Haan and Resnick, Deheuvels (1970s) gave general representation formulae for MEVDs (see Resnicks (1987) book for full description). However these formulae are too general to be useful for statistics.
- Statistics: Much work on parametric subfamilies (Tawn, Coles, ...) and on nonparametric estimation methods but these work well only for small D.
- Problem 1: What to do about large D? (e.g. $D \approx 100$ for a typical portfolio)
- Problem 2: How to extend these methods to take into account also time-series dependence within each series?

2. M4 processes

Max-Stable Processes

- Suppose $\{Y_{id}, i = 0, 1, 2, d = 1, ..., D\}$ is a D-dimensional time series with discrete time index i.
- W.l.o.g. we may assume $\mathrm{P}\{Y_{id} \leq y\} = e^{-1/y}$ for $0 < y < \infty$ (unit Fréchet assumption).
- The process is max-stable if for any $i = i_1, i_1 + 1, ..., i_2$ and any positive set of values $\{y_{id},\ i = i_1, ..., i_2,\ d = 1, ..., D\}$, we have

$$\mathrm{P}\{Y_{id} \leq y_{id},\ i = i_1, ..., i_2,\ d = 1, ..., D\}$$
$$= \mathrm{P}^n\{Y_{id} \leq ny_{id},\ i = i_1, ..., i_2,\ d = 1, ..., D\}.$$

Representations

For univariate processes, such a characterization was provided by Deheuvels (1983), Davis and Resnick (1989), Hall, Peng and Yao (2002), Ferreira and de Haan (2005). This was generalized by Smith and Weissman (1996) to the following: any max-stable process with unit Fréchet margins may be approximated by a *multivariate maxima of moving maxima* process, or M4 for short, with the representation

$$Y_{id} = \max_{l=1,2,\ldots} \max_{-\infty<k<\infty} a_{l,k,d} Z_{l,i-k}, \ -\infty < i < \infty, \ d = 1, \ldots, D.$$

- Finite representations:

$$Y_{id} = \max_{1 \le l \le L_d} \max_{-K_{1ld} \le k \le K_{2ld}} a_{l,k,d} Z_{l,i-k}, \quad -\infty < i < \infty, \qquad (1)$$

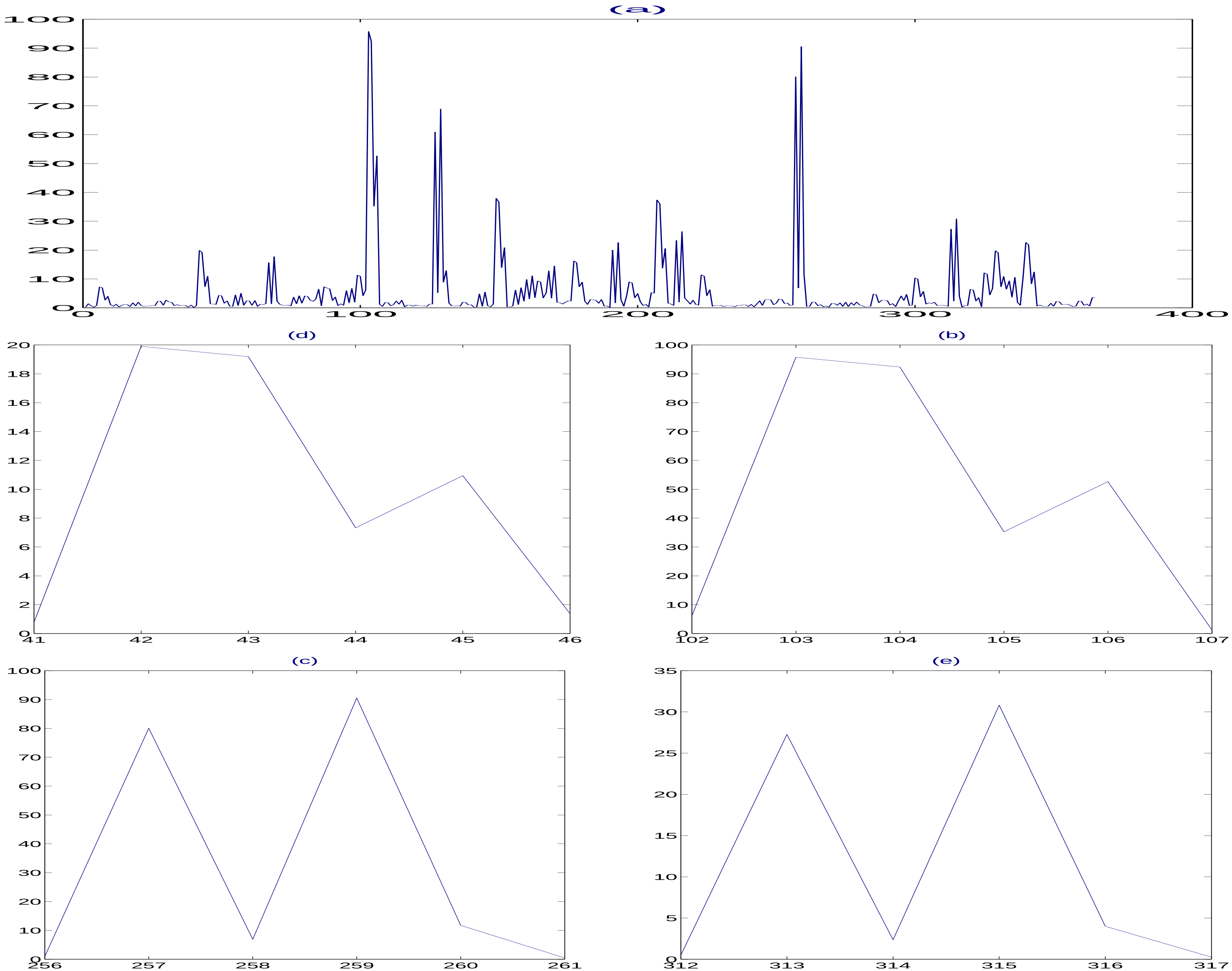
where $L_d,\ K_{1ld},\ K_{2ld}$ are finite and the coefficients satisfy $\sum_{l=1}^{L_d} \sum_{k=-K_{1ld}}^{K_{1ld}} a_{l,k,d} = 1$ for each d.

- Estimation: Based on bivariate joint distributions

- Applications in finance: Zhang and Smith (2003), Zhang (AIE vol 20, 2005), Zhang and Huang (2005).

- Application in wave height data: Zhang (2003?)

2. M4 class



3. Some tail dependence measures

- **Definition 1** *The bivariate asymptotic dependence index*

$$\lambda = \lim_{u \to x_F} \mathrm{P}(X > u | Y > u). \tag{2}$$

$\lambda > 0 \Rightarrow$ aysmpt. dep., otherwise asympt. indep.

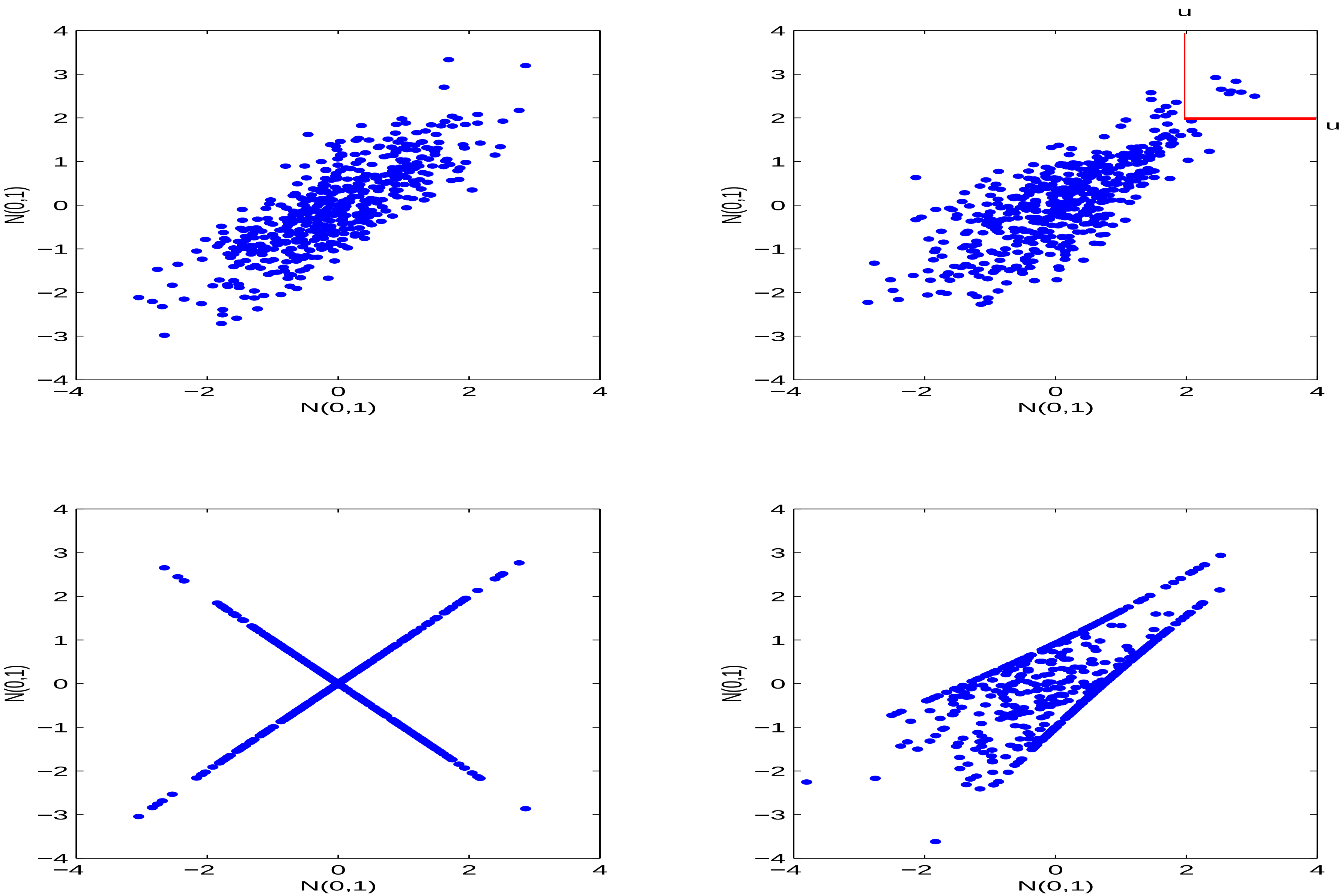
Sibuya (1960), de Haan and Resnick (1977), Embrechts, McNeil, and Straumann (2002), Zhang (2004).

- **Definition 2** *A sequence of variables* $\{X_1,\ X_2,\ \ldots,\ X_n\}$ *is called lag-k asympt. dep. if*

$$\lambda_k = \lim_{u \to x_F} \mathrm{P}(X_{k+1} > u | X_1 > u) > 0,$$

$$\lim_{u \to x_F} \mathrm{P}(X_{k+j} > u | X_1 > u) = 0,\ \ j > 1,$$

3. Some tail dependence measures



Asymptotic dependence in M4

Cross sections:

$$\lambda_{dd'} = 2 - \sum_{l=1}^{\max(L_d, L_{d'})} \sum_{m=1-\max(K_{2ld}, K_{2ld'})}^{1+\max(K_{1ld}, K_{1ld'})} \max\{a_{l,1-m,d}, a_{l,1-m,d'}\},$$

Lag-k in time:

$$\lambda_{d(k)} = 2 - \sum_{l=1}^{L_d} \sum_{m=1-K_{2ld}}^{1+k+K_{1ld}} \max\{a_{l,1-m,d}, a_{l,1+k-m,d}\}.$$

Coefficients of tail dependence

- Ledford and Tawn (1996, 1997) consider the following model:

$$P(X_1 > x,\ X_2 > x) \sim L(x)x^{-1/\eta} \qquad \text{as } x \to \infty, \tag{3}$$

- Connection of λ and η

$$P(X_2 > x | X_1 > x) \sim L(x)x^{1-1/\eta} \qquad \text{as } x \to \infty, \tag{4}$$

when margins are unit Fréchet.

Coefficients of tail dependence in M4

$$\eta = 1$$

or

$$\eta = 1/2$$

4. Asymptotic (In)dependent M4 processes

Extention I of M4

Replacing Z_{li} in M4 by independent GEV shock variables:

$$Y_{id} = \max_l \max_k a^{-1}_{l,k,d} W_{l,i-k}, \quad d = 1, \ldots, D, \ -\infty < i < \infty, \tag{5}$$

for $\{a_{l,k,d} > 0, l \geq 1, -\infty < k < \infty\}$, and $\{W_{lk}, l \geq 1, -\infty < k < \infty\}$ being an array of independent GEV shock variables which have a unified distribution form

$$H(x; \mu, \sigma, \xi) = \exp\{-[1 + \frac{\xi(x-\mu)}{\sigma}]^{-1/\xi}\} \tag{6}$$

Two conditions

C1 Suppose all moving coefficients a_{lkd} satisfy

$$\sum_l \sum_k a_{lk1}^{-1/\xi} = \sum_l \sum_k a_{lkd'}^{-1/\xi}, \text{ for all } d' = 2, \ldots, D. \quad (7)$$

C2 Suppose there are numbers $a^* > 0$ and n^* such that

$$a^* = \min_l \min_k a_{lkd}, \; n^* = \sum_l \sum_k \mathbf{1}_{(a_{lkd}=a^*)}, \text{ for all } d = 1, \ldots, D, \quad (8)$$

where $\mathbf{1}_{(.)}$ is an indicator function.

Result # 3 *Under* $\xi > 0$ *and the condition* **C1**, *we have*

$$\lambda_{dd'} = \frac{2\sum\limits_{l}\sum\limits_{k} a_{lkd}^{-1/\xi} - \sum\limits_{l=1}^{\infty}\sum\limits_{m=-\infty}^{\infty} \min(a_{l,1-m,d},\ a_{l,1-m,d'})^{-1/\xi}}{\sum_l \sum_k a_{lkd}^{-1/\xi}}, \tag{9}$$

$$\lambda_{d_r} = \frac{2\sum\limits_{l}\sum\limits_{k} a_{lkd}^{-1/\xi} - \sum\limits_{l=1}^{\infty}\sum\limits_{m=-\infty}^{\infty} \min(a_{l,1-m,d},\ a_{l,1+r-m,d})^{-1/\xi}}{\sum_l \sum_k a_{lkd}^{-1/\xi}}, \tag{10}$$

and

$$\lambda_{dd'_r} = \frac{2\sum\limits_{l}\sum\limits_{k} a_{lkd}^{-1/\xi} - \sum\limits_{l=1}^{\infty}\sum\limits_{m=-\infty}^{\infty} \min(a_{l,1-m,d},\ a_{l,1+r-m,d'})^{-1/\xi}}{\sum_l \sum_k a_{lkd}^{-1/\xi}}. \tag{11}$$

Result # 4 *Under* $\xi = 0$ *and the condition* **C2**, *we have*

$$C_s = \sum_l \sum_k a_{lks} \mathbf{1}_{(a_{lks}=a^*)}$$

$$\lambda_{dd'} = \frac{\sum_{s=d,d'} C_s - \sum_{l=1}^{\infty} \sum_{m=-\infty}^{\infty} \min(a_{l,1-m,d},\ a_{l,1-m,d'}) \mathbf{1}_{(\min(a_{l,1-m,d},\ a_{l,1-m,d'})=a^*)}}{C_d}, \tag{12}$$

$$\lambda_{d_r} = \frac{2C_d - \sum_{l=1}^{\infty} \sum_{m=-\infty}^{\infty} \min(a_{l,1-m,d},\ a_{l,1+r-m,d}) \mathbf{1}_{(\min(a_{l,1-m,d},\ a_{l,1+r-m,d})=a^*)}}{C_d}, \tag{13}$$

and

$$\lambda_{dd'_r} = \frac{\sum_{s=d,d'} C_s - \sum_{l=1}^{\infty} \sum_{m=-\infty}^{\infty} \min(a_{l,1-m,d},\ a_{l,1+r-m,d'}) \mathbf{1}_{(\min(a_{l,1-m,d},\ a_{l,1+r-m,d'})=a^*)}}{C_d}. \tag{14}$$

Result # 5 *Under* $\xi < 0$ *and the condition* **C2**, *we have*

$$\lambda_{dd'} = \frac{2\sum\limits_{l}\sum\limits_{k}\mathbf{1}_{\{a_{lkd}=a^*\}} - \sum\limits_{l}\sum\limits_{m}\mathbf{1}_{\{\min(a_{l,1-m,d},\ a_{l,1-m,d'})=a^*\}}}{\sum\limits_{l}\sum\limits_{k}\mathbf{1}_{\{a_{lkd}=a^*\}}}, \tag{15}$$

$$\lambda_{d_r} = \frac{2\sum\limits_{l}\sum\limits_{k}\mathbf{1}_{\{a_{lkd}=a^*\}} - \sum\limits_{l}\sum\limits_{m}\mathbf{1}_{\{\min(a_{l,1-m,d},\ a_{l,1+r-m,d})=a^*\}}}{\sum\limits_{l}\sum\limits_{k}\mathbf{1}_{\{a_{lkd}=a^*\}}}, \tag{16}$$

and

$$\lambda_{dd'_r} = \frac{\sum\limits_{s=d,d'}\sum\limits_{l}\sum\limits_{k}\mathbf{1}_{(a_{lks}=a^*)} - \sum\limits_{l=1}^{\infty}\sum\limits_{m=-\infty}^{\infty}\mathbf{1}_{(\min(a_{l,1-m,d},\ a_{l,1+r-m,d'})=a^*)}}{\sum_l\sum_k\mathbf{1}_{(a_{lkd}=a^*)}}. \tag{17}$$

Coeff. of tail dependence in Extension I

$$\eta = 1$$

or

$$\eta = 1/2$$

Extention II of M4

The new model is as follows:

$$Y_{id} = \max(W_{id}'^{1/\alpha}, \ \max_l \ \max_k a_{l,k,d}^{-1} W_{l,i-k}), \qquad d = 1, \cdots, D, \tag{18}$$

where $\alpha > 0$, $a_{l,k,d} > 0$, $\{W_{li}, l \geq 1, -\infty < i < \infty\}$ are an array of independent positive random variables; $\{W_{id}', -\infty < i < \infty, \ d = 1, \ldots, D\}$ are an array of independent positive random variables, and they are independent of W_{li}. These max-shock random variables are identically distributed.

Result # 6 *Under the conditions that max-shock variable being unit Fréchet and that* **C1** *with the value being 1, we have*

$$\lambda_{dd'} = \begin{cases} 0, & \textit{if } \alpha < 1; \\ 2 - \sum_l \sum_k \max(a_{lk1}^{-1}, a_{lk2}^{-1}), & \textit{if } \alpha \geq 1, \end{cases} \tag{19}$$

$$\lambda_{d_r} = \begin{cases} 0, & \textit{if } \alpha < 1; \\ 2 - \sum_l \sum_k \max(a_{lkd}^{-1}, a_{l,k+r,d}^{-1}), & \textit{if } \alpha \geq 1, \end{cases} \tag{20}$$

and

$$\lambda_{dd'_r} = \begin{cases} 0, & \textit{if } \alpha < 1; \\ 2 - \sum_l \sum_k \max(a_{lkd}^{-1},\ a_{l,k+r,d'}^{-1}), & \textit{if } \alpha \geq 1. \end{cases} \tag{21}$$

Result # 7 *The coefficients of tail dependence are given by*

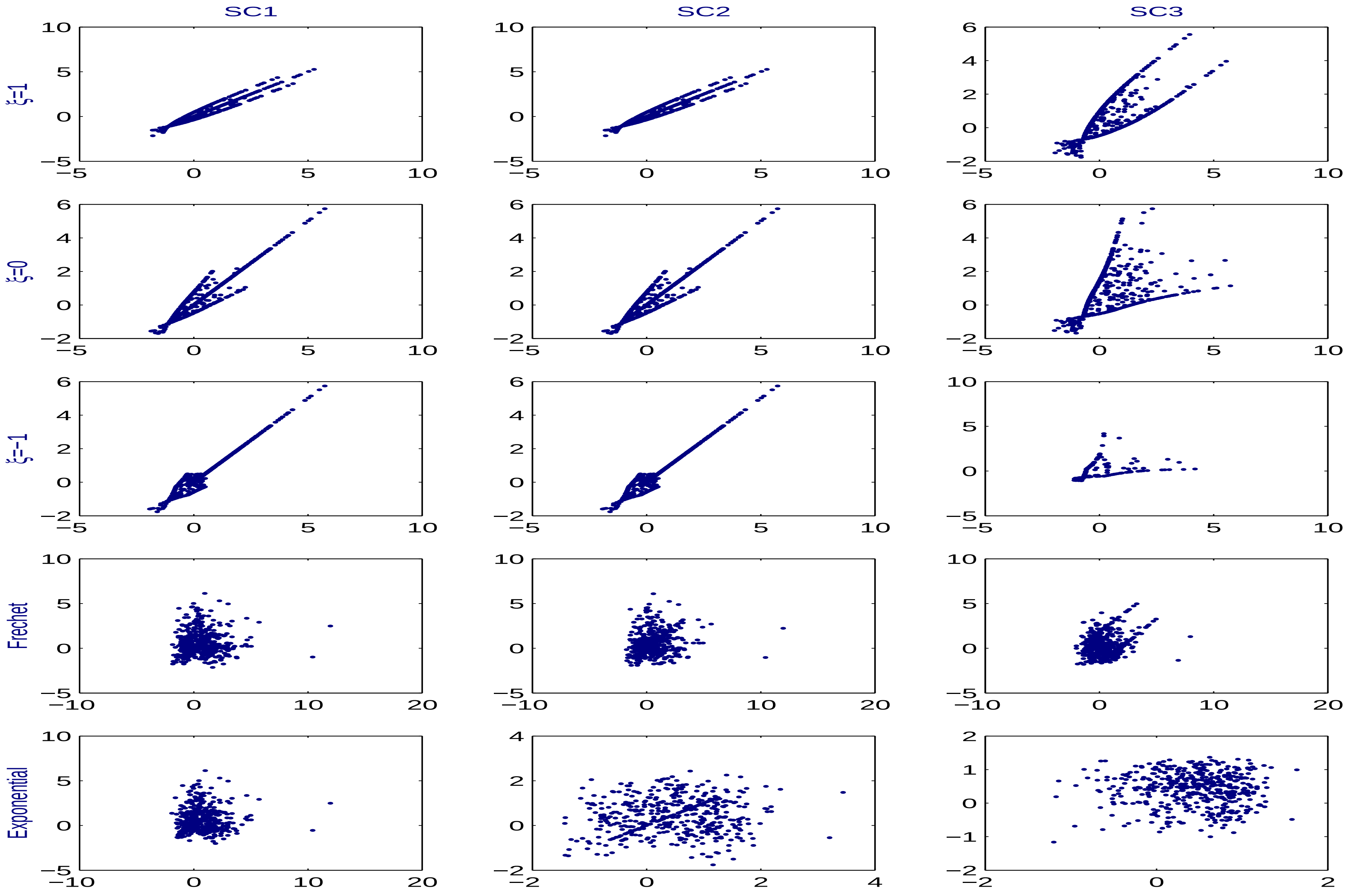
$$\eta = \begin{cases} \max(1/2, \alpha), & \text{if } \alpha < 1; \\ 1, & \text{if } \alpha \geq 1. \end{cases} \tag{22}$$

Result # 8 *Under the conditions that max-shock variable being unit exponential. In addition,* **C2** *holds, and* $a_{lkd} > 1$ *, we have*

$$\lambda_{dd'} = \begin{cases} 0, & \textit{if } \alpha \leq 1; \\ \frac{2n^* - \sum_{l,m} \mathbf{1}_{[\mathbf{1}_{(a_{l,1-m,d}=a^*)} + \mathbf{1}_{(a_{l,1-m,d'}=a^*)}=1]}}{n^*}, & \textit{if } \alpha > 1, \end{cases} \tag{23}$$

$$\lambda_{d_r} = \begin{cases} 0, & \textit{if } \alpha \leq 1; \\ \frac{2n^* - \sum_{l,m} \mathbf{1}_{[\mathbf{1}_{(a_{l,1-m,d}=a^*)} + \mathbf{1}_{(a_{l,1+r-m,d}=a^*)}=1]}}{n^*}, & \textit{if } \alpha > 1, \end{cases} \tag{24}$$

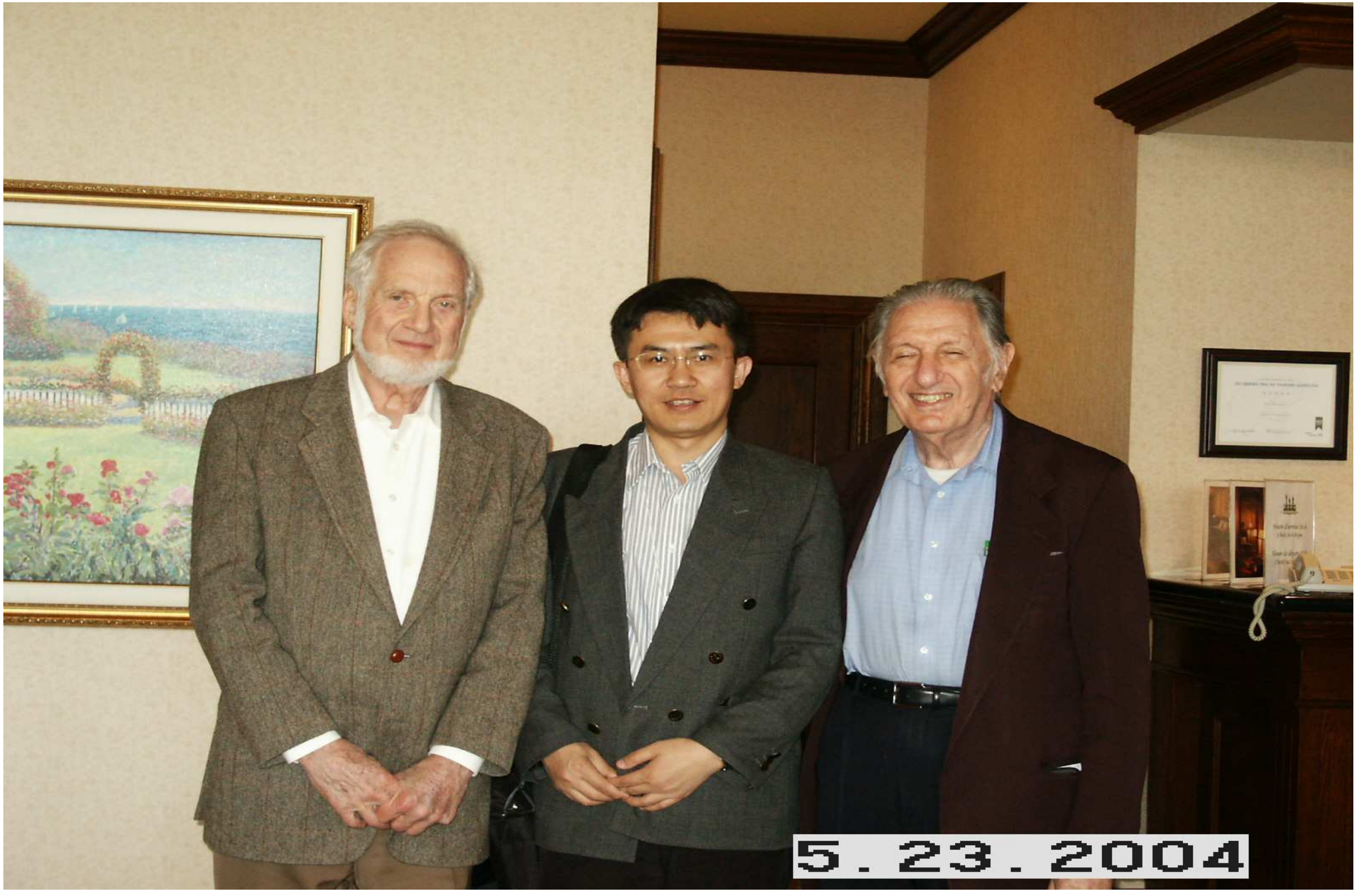
4. Asymptotic (In)dependent M4 processes



Concluding remarks

1. M4 class of processes are very flexible models for temporally dependent multivariate extreme value processes.
2. All variables in these M4 models are asymptotically dependent.
3. The purpose of this paper is to extend this M4 class in a number of ways to produce classes of models which are also asymptotically independent.
4. Ultimately, all of these models will be explored in modeling financial data, environmental data, etc.

The man who introduced the name of copula in statistics



Tack så mycket!