GARCH processes – probabilistic properties (Part 1)

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Contents

GARCH(p,q) on $\mathbb{N}_0$

$$(\varepsilon_t)_{t\in\mathbb{N}_0} \quad \text{i.i.d.}, \quad P(\varepsilon_0 = 0) = 0 \tag{1}$$

$$\alpha_0 > 0, \quad \alpha_1, \ldots, \alpha_p \geq 0, \quad \beta_1, \ldots, \beta_q \geq 0.$$

$$(\sigma_0^2, \ldots, \sigma^2_{\max(p,q)-1}) \text{ independent of } \{\varepsilon_t : t \geq \max(p,q) - 1\}$$

$$Y_t = \sigma_t \varepsilon_t, \tag{2}$$

$$\sigma_t^2 = \underbrace{\underbrace{\alpha_0 + \sum_{i=1}^{p} \alpha_i Y_{t-i}^2}_{\text{ARCH}} + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2}_{\text{GARCH}}, \quad t \geq \max(p,q). \tag{3}$$

$(Y_t)_{t\in\mathbb{N}_0}$ GARCH(p,q) process

GARCH(p,q) on $\mathbb{Z}$

(1), (2) and (3) for $t \in \mathbb{Z}$, and

ε_t independent of

$$\underline{Y_{t-1}} := \{Y_s : s \leq t-1\}.$$

ARCH(p): Engle (1981)

GARCH(p,q): Bollerslev (1986)

Generalized **A**uto**R**egressive **C**onditional **H**eteroscedasticity

Why conditional volatility?

Suppose $\sigma_t^2 \in \underline{Y_{t-1}}$,

or equivalently $\sigma_t^2 \in \underline{\varepsilon_{t-1}}$, $\forall\, t \in \mathbb{Z}$

(the process is "*causal*")

Suppose $E\varepsilon_t = 0$, $E\varepsilon_t^2 = 1$, $EY_t < \infty$. Then

$$\begin{aligned} E(Y_t|\underline{Y_{t-1}}) &= E(\sigma_t\varepsilon_t|\underline{Y_{t-1}}) = \sigma_t\, E(\varepsilon_t|\underline{Y_{t-1}}) = 0, \\ V(Y_t|\underline{Y_{t-1}}) &= E(\sigma_t^2\varepsilon_t^2|\underline{Y_{t-1}}) = \sigma_t^2 E(\varepsilon_t^2|\underline{Y_{t-1}}) = \sigma_t^2 \end{aligned}$$

Hence σ_t^2 is the *conditional variance*

Markov property

(a) GARCH(1,1):

$(\sigma_t^2)_{t\in\mathbb{Z}}$ is a Markov process, since

$$\sigma_t^2 = \alpha_0 + (\alpha_1 \varepsilon_{t-1}^2 + \beta_1)\sigma_{t-1}^2$$

$(\sigma_t^2, Y_t)_{t\in\mathbb{Z}}$ is Markov process, but $(Y_t)_{t\in\mathbb{Z}}$ is not.

(b) GARCH(p,q), $\max(p,q) > 1$

$(\sigma_t^2)_{t\in\mathbb{Z}}$ is not Markov process

$(\sigma_t^2, \ldots, \sigma_{t-\max(p,q)+1}^2)_{t\in\mathbb{Z}}$ is Markov process.

Strict stationarity - GARCH(1,1)

$$\begin{aligned}\sigma_t^2 &= \alpha_0 + \alpha_1 \sigma_{t-1}^2 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \\ &= (\alpha_1 \varepsilon_{t-1}^2 + \beta_1)\sigma_{t-1}^2 + \alpha_0 \\ &=: A_t \sigma_{t-1}^2 + B_t \end{aligned} \tag{4}$$

$(A_t, B_t)_{t\in\mathbb{Z}}$ is i.i.d. sequence.

(4) is called a *random recurrence equation.*

$$\begin{aligned}\sigma_t^2 &= A_t \sigma_{t-1}^2 + B_t \\ &= A_t A_{t-1} \sigma_{t-1}^2 + A_t B_{t-1} + B_t \\ &\vdots \\ &= A_t \cdots A_{t-k} \sigma_{t-k-1}^2 + \sum_{i=0}^{k} A_t \cdots A_{t-i+1} \underbrace{B_{t-i}}_{=\alpha_0} . \end{aligned} \tag{5}$$

Let $k \to \infty$ and hope that (5) converges.

Strict stationarity - continued

Suppose $\gamma := E \log A_1 < 0$

$$\frac{1}{k} \underbrace{(\log A_t + \log A_{t-1} + \ldots + \log A_{t-k+1})}_{\textbf{random walk!}} \stackrel{k\to\infty}{\longrightarrow} E \log A_1 \quad \text{a.s.}$$

$$\Longrightarrow \text{a.s.} \forall\, \omega\ \exists n_0(\omega) : \frac{1}{k} \log \left(A_t \cdots A_{t-k+1}\right) \le \gamma/2 < 0 \quad \forall k \ge n_0(\omega)$$

$$\Longrightarrow |A_t \cdots A_{t-k+1}| \le e^{k\gamma/2} \quad \forall k \ge n_0(\omega)$$

Hence (5) converges almost surely.

Theorem: (Nelson, 1990)
If

$$E \log(\alpha_1 \varepsilon_1^2 + \beta_1) < 0, \tag{6}$$

then GARCH(1,1) has a strictly stationary solution.
Its marginal distribution is given by

$$\alpha_0 \sum_{i=0}^{\infty} (\alpha_1 \varepsilon_{-1}^2 + \beta_1) \cdots (\alpha_1 \varepsilon_{-i}^2 + \beta_1) \tag{7}$$

If $\beta_1 > 0$ (i.e. GARCH but not ARCH) and a strictly stationary solution exists, then (6) holds.

Remark:
If $E \log^+ |A_1| < \infty$, then

$$\gamma = \inf \left\{ E \left(\frac{1}{n+1} \log |A_0 A_{-1} \cdots A_{-n}| \right) : n \in \mathbb{N} \right\}$$

is called the *Lyapunov exponent* of the sequence $(A_n)_{n \in \mathbb{Z}}$.

When is $EY_t^2 < \infty$?

$$\sigma_0^2 = \alpha_0 \sum_{i=0}^{\infty} (\alpha_1 \varepsilon_{-1}^2 + \beta_1) \cdots (\alpha_1 \varepsilon_{-i}^2 + \beta_1)$$

$$E\sigma_0^2 = \alpha_0 \sum_{i=0}^{\infty} \left(E(\alpha_1 \varepsilon_1^2 + \beta_1) \right)^i < \infty$$

$$\iff \alpha_1 E\varepsilon_1^2 + \beta_1 < 1$$

$$EY_t^2 = E\sigma_t^2 \; E\varepsilon_t^2$$

So stationary solution $(Y_t)_{t \in \mathbb{Z}}$ has finite variance iff

$$\alpha_1 E\varepsilon_1^2 + \beta_1 < 1$$

Tail behaviour of stochastic recurrence equations

Theorem (Goldie 1991, Kesten 1973, Vervaat 1979)
Suppose $(Z_t)_{t\in\mathbb{N}}$ satisfies the stochastic recurrence equation

$$Z_t = A_t Z_{t-1} + B_t, \quad t \in \mathbb{N},$$

where $((A_t, B_t))_{t\in\mathbb{N}}$, (A, B) i.i.d. sequences. Suppose $\exists \kappa > 0$ such that
(i) The law of $\log|A|$, given $|A| \neq 0$, is not concentrated on $r\mathbb{Z}$ for any $r > 0$
(ii) $E|A|^\kappa = 1$
(iii) $E|A|^\kappa \log^+|A| < \infty$
(iv) $E|B|^\kappa < \infty$
Then $Z \stackrel{d}{=} AZ+B$, where Z independent of (A, B), has a unique solution in distribution which satisfies

$$\lim_{x\to\infty} x^\kappa P(Z > x) = \frac{E[((AZ+B)^+)^\kappa - ((AZ)^+)^\kappa]}{\kappa E|A|^\kappa \log^+|A|} =: C \geq 0 \tag{8}$$

Tail behaviour of GARCH(1,1)

Corollary: Suppose ε_1 is continuous, $P(\varepsilon_1 > 0) > 0$ and all of its moments exist.
Further, suppose that

$$E \log(\alpha_1 \varepsilon_1^2 + \beta_1) < 0.$$

Then $\exists \kappa > 0$ and $C_1 > 0, C_2 > 0$ such that for the stationary solutions of (σ_t^2) and (Y_t):

$$\begin{aligned} \lim_{x\to\infty} x^{\kappa} P(\sigma_0^2 > x) &= C_1 \\ \lim_{x\to\infty} x^{2\kappa} P(Y_0 > x) &= C_2 \end{aligned}$$

Mikosch, Stărică (2000): GARCH(1,1)

de Haan, Resnick, Rootzén, de Vries: ARCH(1)

Extremal behaviour of GARCH(1,1)

Mikosch and Stărică (2000) and de Haan et al. (1989) gave extreme value theory for GARCH(1,1) and ARCH(1).

Suppose a stationary solution $(Y_t)_{t\in\mathbb{N}_0}$ exists.

Let $(\widetilde{Y}_t)_{t\in\mathbb{N}_0}$ be an i.i.d. sequence with $\widetilde{Y}_0 \stackrel{d}{=} Y_0$.

Then under the previous assumptions, $\exists\, \theta \in (0,1)$ such that for a certain sequence $c_t > 0$, $t \in \mathbb{N}$:

$$\begin{aligned}
\lim_{t\to\infty} P\left(\max_{i=1,\dots,t} Y_i \geq c_t x\right) &= \exp(-\theta x^{-2\kappa}), \quad x \in \mathbb{R} \\
\lim_{t\to\infty} P\left(\max_{i=1,\dots,t} \widetilde{Y}_i \geq c_t x\right) &= \exp(-x^{-2\kappa}), \quad x \in \mathbb{R}.
\end{aligned}$$

The GARCH(1,1) process has an *extremal index* θ, i.e. exceedances over large thresholds occur in clusters, with an average cluster length of $1/\theta$

What can be done for GARCH(p,q)?

$$\begin{aligned}\tau_t &:= (\beta_1 + \alpha_1 \varepsilon_t^2, \beta_2, \ldots, \beta_{p-1}) \in \mathbb{R}^{p-1} \\ \xi_t &:= (\varepsilon_t^2, 0, \ldots, 0) \in \mathbb{R}^{p-1} \\ \alpha &:= (\alpha_2, \ldots, \alpha_{q-1}) \in \mathbb{R}^{q-2}\end{aligned}$$

I_{p-1}: $(p-1) \times (p-1)$ identity matrix
M_t: $(p+q-1) \times (p+q-1)$ matrix

$$M_t := \begin{pmatrix} \tau_t & \beta_p & \alpha & \alpha_q \\ I_{p-1} & 0 & 0 & 0 \\ \xi_t & 0 & 0 & 0 \\ 0 & 0 & I_{q-2} & 0 \end{pmatrix}$$

$$\begin{aligned}N &:= (\alpha_0, 0, \ldots, 0)' \in \mathbb{R}^{p+q-1} \\ X_t &:= (\sigma_{t+1}^2, \ldots, \sigma_{t-p+2}^2, Y_t^2, \ldots, Y_{t-q+2}^2)'\end{aligned}$$

Then $(Y_t)_{t\in\mathbb{Z}}$ solves the GARCH(p,q) equation if and only if

$$X_{t+1} = M_{t+1} X_t + N, \quad t \in \mathbb{Z} \tag{9}$$

GARCH(p,q) - continued

(9) is a random recurrence equation. Theory for existence of stationary solutions can be applied.

For example, if $E\varepsilon_1 = 0$, $E\varepsilon_1^2 = 1$, then a necessary and sufficient condition for existence of a strictly stationary solution with finite second moments is

$$\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1. \tag{10}$$

(Bougerol and Picard (1992), Bollerslev (1986))

But stationary solutions can exist also if

$$\sum_{i=1}^{p} \alpha_1 + \sum_{j=1}^{q} \beta_j = 1.$$

A characterization for the existence of stationary solutions is achieved via the Lyapunov exponent (whether it is strictly negative or not).

GARCH(p,q) is White Noise

Suppose $E\varepsilon_t = 0$, $E\varepsilon_t^2 = 1$ and (10). Then for $h \geq 1$,

$$\begin{aligned} EY_t &= E(\sigma_t \varepsilon_t) = E(\sigma_t E(\varepsilon_t | \underline{Y_{t-1}})) = 0 \\ E(Y_t Y_{t+h}) &= E(\sigma_t \sigma_{t+h} \varepsilon_t \underbrace{E(\varepsilon_{t+h} | \underline{Y_{t+h-1}})}_{=0}) = 0 \end{aligned}$$

But $(Y_t^2)_{t\in\mathbb{Z}}$ is not White Noise.

$$u_t := Y_t^2 - \sigma_t^2 = \sigma_t^2(\varepsilon_t^2 - 1)$$

Suppose $Eu_t^2 < \infty$ and let $h \geq 1$:

$$\begin{aligned} Eu_t &= 0 \\ E(u_t u_{t+h}) &= E(\sigma_t^2(\varepsilon_t^2 - 1)\sigma_{t+h}^2)\ E(\varepsilon_{t+h}^2 - 1) = 0 \end{aligned}$$

Hence $(u_t)_{t\in\mathbb{Z}}$ is White Noise

The ARMA representation of Y_t^2

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i Y_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2$$

Substitute $\sigma_t^2 = Y_t^2 - u_t$. Then

$$Y_t^2 - \sum_{i=1}^{p} \alpha_i Y_{t-i}^2 - \sum_{j=1}^{q} \beta_j Y_{t-j}^2 = \alpha_0 + u_t - \sum_{j=1}^{q} \beta_j u_{t-j}$$

Hence $(Y_t^2)_{t\in\mathbb{Z}}$ satisfies an ARMA$(\max(p,q),q)$ equation.

Hence autocorrelation function and spectral density of $(Y_t^2)_{t\in\mathbb{Z}}$ is that of ARMA$(\max(p,q),q)$ process with the given parameters.

Drawbacks of GARCH

- Black (1976): Volatility tends to rise in response to "bad news" and to fall in response to "good news"
- The volatility in the GARCH process is determined only by the magnitude of the previous return and shock, not by its sign.
- The parameters in GARCH are restricted to be positive to ensure positivity of σ_t^2. When estimating, however, sometimes best fits are achieved for negative parameters.

Exponential GARCH (EGARCH) (Nelson, 1991)

$$(\varepsilon_t)_{t\in\mathbb{Z}} \quad \text{i.i.d.}(0,1)$$

$$Y_t = \sigma_t \varepsilon_t$$

$$\log(\sigma_t^2) = \alpha_0 + \sum_{i=1}^{p} \alpha_i g(\varepsilon_{t-i}) + \sum_{j=1}^{q} \beta_j \log(\sigma_{t-j}^2)$$

$$g(\varepsilon_t) = \theta\varepsilon_t + \gamma(|\varepsilon_t| - E|\varepsilon_t|), \quad \theta^2 + \gamma^2 \neq 0$$

Most often, ε_t i.i.d. normal.

EGARCH models often fit the data nicely, **but**

- $\log(\sigma_t^2)$ has tails not much heavier than Gaussian tails (like $x^c e^{-dx^2}$, $x \to \infty$, $c, d > 0$)
- tails of Y_t are approximately like

$$P(Y_0 > x) \approx e^{-(\log x)^2} = x^{-\log x}, \quad x \to \infty$$

- Neither $(\log \sigma_t^2)_{t\in\mathbb{Z}}$ nor $(Y_t)_{t\in\mathbb{Z}}$ show cluster behaviour.

(Lindner, Meyer (2001))

Similar results for stochastic volatility models by Breidt and Davis (1998)

Other GARCH type models

Many many other GARCH type models exist, like

- MGARCH (multiplicative GARCH)
- NGARCH (non-linear asymmetric GARCH)
- TGARCH (threshold GARCH)
- FIGARCH (fractional integrated GARCH)
- $\cdots$
- $\cdots$

See Gouriéroux (1997) or Duan (1997) for some of them.

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