GARCH processes – continuous counterparts
(Part 2)

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Why continuous time models?

- Observations are quite often irregularly spaced.
- Observations quite often come in at a very high frequency.

Then a continuous time model may provide a better approximation to the discrete data than a discrete model.

**Aim:** Construct continuous time models with features of GARCH.
The diffusion approximation of Nelson

\[ \sigma_n^2 = \omega + \lambda \sigma_{n-1}^2 \varepsilon_{n-1}^2 + \delta \sigma_{n-1}^2 = \omega + (\lambda \varepsilon_{n-1}^2 + \delta) \sigma_{n-1}^2 \]

\[ Y_n = \sigma_n \varepsilon_n \]

where \((\varepsilon_n)_{n \in \mathbb{N}}\) i.i.d.: GARCH(1,1) process.

Set

\[ G_n := \sum_{i=0}^{n} Y_i, \quad n \in \mathbb{N}_0 \]

Then

\[ G_n - G_{n-1} = Y_n, \]

so the increments of \((G_n)_{n \in \mathbb{N}_0}\) are a GARCH process, i.e. \((G_n)\) is “accumulated GARCH”.

**Question:** Can we find a sequence of processes, whose increments on finer becoming grids are GARCH processes, such that the processes converge in distribution to a non-trivial limit process?
Setup:
Take grid width $h > 0$

\[ hG_{nh} = hG_{(n-1)h} + h\sigma_{nh} \cdot h\epsilon_{nh}, \quad n \in \mathbb{N}, \]
\[ h\sigma_{(n+1)h}^2 = \omega_h + (h^{-1}\lambda_h \cdot h\epsilon_{nh}^2 + \delta_h) \cdot h\sigma_{nh}^2, \quad n \in \mathbb{N}_0, \]

\[ (h\epsilon_{nh})_{n \in \mathbb{N}_0} \text{ i.i.d. } N(0, h) \]
\[ (h\sigma_{0}^2, hG_0) \text{ independent of } (h\epsilon_{nh})_{n \in \mathbb{N}} \]
\[ \omega_h > 0, \lambda_h \geq 0, \delta_h \geq 0. \]

Then \((hG_{nh} - hG_{(n-1)h})_{n \in \mathbb{N}}\) is GARCH(1,1) process

\[ hG_t := hG_{nh}, \quad h\sigma_t^2 := h\sigma_{nh}^2, \quad nh \leq t < (n + 1)h \]

defines \((hG_t, h\sigma_t^2)\) for all $t \in \mathbb{R}_+$

**Question:** When does \((hG_t, h\sigma_t^2)_{t \geq 0}\) converge weakly to a process \((G, \sigma^2)\) as $h \downarrow 0$?

(weak convergence is in the space $D(\mathbb{R}_+, \mathbb{R}^2)$ of càdlàg functions, endowed with the Borel sets of the Skorohod topology; weak convergence of processes implies in particular convergence of finite dimensional distributions)
Theorem: (Nelson, 1990)
Suppose
\[
(hG_0, h\sigma_0^2) \xrightarrow{d} (G_0, \sigma_0^2), \quad h \downarrow 0
\]
\[
P(\sigma_0^2 > 0) = 1
\]
\[
\lim_{h \downarrow 0} h^{-1}\omega_h = \omega \geq 0
\]
\[
\lim_{h \downarrow 0} h^{-1}(1 - \delta_h - \lambda_h) = \theta
\]
\[
\lim_{h \downarrow 0} 2h^{-1}\lambda_h^2 = \lambda^2 > 0
\]
Then \((hG, h\sigma^2)\) converges weakly as \(h \downarrow 0\) to the unique solution \((G, \sigma^2)\) of the diffusion equation
\[
dG_t = \sigma_t dB_t^{(1)}, \quad (1)
\]
\[
d\sigma_t^2 = (\omega - \theta\sigma_t^2) dt + \lambda\sigma_t^2 dB_t^{(2)}, \quad (2)
\]
with starting value \((G_0, \sigma_0^2)\), where \((B_t^{(1)})_{t \geq 0}\) and \((B_t^{(2)})_{t \geq 0}\) are independent Brownian motions, independent of \((G_0, \sigma_0^2)\).

If \(2\theta/\lambda^2 > -1\) and \(\omega > 0\), then the solution \((\sigma_t^2)_{t \geq 0}\) is strictly stationary iff \(\sigma_0^{-2} \overset{d}{=} \Gamma(1 + 2\theta/\lambda^2, 2\omega/\lambda^2)\)

Example: \(\omega_h = \omega h, \delta_h = 1 - \lambda\sqrt{h/2} - \theta h, \lambda_h = \lambda\sqrt{h/2}\)
**Interpretation:**
The solution of (1) and (2) can be approximated by a GARCH process in discrete time.

**Observe:**

- The stationary limiting process $\sigma^2$ has Pareto like tails.
- The limit $(G, \sigma^2)$ is driven by **two independent** Brownian motions. The GARCH process has only one source of randomness!
- The processes $G$ and $\sigma^2$ are continuous. But empirical volatility can exhibit jumps.
- Estimation of the parameters of the diffusion limit and of the discrete GARCH processes may lead to significantly different results (Wang, 2002)

**Further literature:** Drost and Werker (1996), Duan (1997)
The COGARCH(1,1) process
Klüppelberg, Lindner, Maller (2004)

Recall discrete GARCH(1,1):

$$\sigma_n^2 = \omega + \lambda Y_{n-1}^2 + \delta \sigma_{n-1}^2 = \omega + (\delta + \lambda \varepsilon_{n-1}^2) \sigma_{n-1}^2$$

\[\vdots\]

\[\begin{align*}
&= \omega \sum_{i=0}^{n-1} \prod_{j=i+1}^{n-1} (\delta + \lambda \varepsilon_j^2) + \sigma_0^2 \prod_{j=0}^{n-1} (\delta + \lambda \varepsilon_j^2) \\
&= \left( \omega \int_0^n \exp \left\{ - \sum_{j=0}^{[s]} \log(\delta + \lambda \varepsilon_j^2) \right\} ds + \sigma_0^2 \right) \\
&\quad \times \exp \left\{ \sum_{j=0}^{n-1} \log(\delta + \lambda \varepsilon_j^2) \right\}
\end{align*}\]

random walk

random walk

random walk

increment of random walk

$$Y_n = \sigma_n \varepsilon_n = \sigma_n \left( \sum_{j=0}^{n} \varepsilon_j - \sum_{j=0}^{n-1} \varepsilon_j \right)$$

Both appearing random walks are linked
Idea: Replace appearing random walks by Lévy processes
(= continuous time analogue of random walk)
Replace $\varepsilon_j$ by jumps of a Lévy process $L$.

Recall: A stochastic process $(L_t)_{t \geq 0}$ is a Lévy process iff

- it has independent increments:
  for $0 \leq a < b \leq c < d$: $L_d - L_c$ and $L_b - L_a$ are independent
- it has stationary increments:
  the distribution of $L_{t+s} - L_t$ does not depend on $t$
- it is stochastically continuous
- with probability one it has right-continuous paths with finite left-limits (càdlàg paths)
- $L_0 = 0$ a.s.
Examples

- *Brownian motion* (has normal increments)
- *Compound Poisson process:*
  
  \((\varepsilon_n)_{n\in\mathbb{N}}\) i.i.d. sequence, independent of \((v_n)_{n\in\mathbb{N}}\) i.i.d. with exponential distribution with mean \(c\)

\[ T_n := \sum_{j=1}^{n} v_n \]

\[ N_t := \max\{n \in \mathbb{N}_0 : T_n \leq t\} \]

\[ L_t := \sum_{j=1}^{N_t} \varepsilon_j \]

**Note:** All Lévy processes apart from Brownian motion have jumps.
Lévy-Kintchine formula

\( (L_t)_{t \geq 0} \) Lévy process:

\[
E e^{isL_t} = e^{t\chi_L(s)}, \quad s \in \mathbb{R},
\]

\[
\chi_L(s) = i\gamma_L s - \tau_L^2 \frac{s^2}{2} + \int_{\mathbb{R}} \left( e^{isx} - 1 - isx 1_{\{|x| < 1\}} \right) \Pi_L(dx), \quad s \in \mathbb{R}.
\]

\begin{itemize}
  \item \((\gamma_L, \tau_L, \Pi_L)\) characteristic triplet
  \item \(\tau_L^2 \geq 0\) Brownian part
  \item \(\Pi_L\) Lévy measure:
    \[
    \int_{|x| < 1} x^2 \Pi_L(dx) < \infty, \quad \int_{|x| \geq 1} \Pi_L(dx) < \infty
    \]
  \item If \(\int_{|x| < 1} |x| \Pi_L(dx) < \infty\) and \(\tau_L^2 = 0\): finite variation case.
    \[
    \chi_L(s) = i\gamma_{L,0} s - \tau_L^2 \frac{s^2}{2} + \int_{\mathbb{R}} (e^{isx} - 1) \Pi_L(dx), \quad s \in \mathbb{R}.
    \]
\end{itemize}

\(\gamma_{L,0}\) drift of \(L\)
Jumps of Lévy processes

- $\Delta L_t := L_t - L_{t-}$ jumps
- totally described by the Lévy measure
- $\Pi_L$ infinite (not compound Poisson) $\implies$ almost surely, $(L_t)_{t \geq 0}$ has infinitely many jumps in finite time intervals
- $\sum_{0 < s \leq t} |\Delta L_s| < \infty \iff \int_{|x| < 1} |x| \Pi_L(dx) < \infty$
- Always: $\sum_{0 < s \leq t} |\Delta L_s|^2 < \infty$ almost surely.
COGARCH(1,1) - definition

For $\Pi_L \neq 0$, $\delta > 0$, $\lambda \geq 0$ define auxiliary Lévy process

$$X_t = -t \log \delta - \sum_{0 < s \leq t} \log(1 + \frac{\lambda}{\delta} (\Delta L_s)^2), \quad t \geq 0.$$  

For $\omega > 0$ and a finite random variable $\sigma_0^2$ independent of $(L_t)_{t \geq 0}$ define the volatility process

$$\sigma_t^2 = \left( \omega \int_0^t e^{X_s} ds + \sigma_0^2 \right) e^{-X_t}, \quad t \geq 0.$$  

Define $COGARCH(1,1)$ $(G_t)_{t \geq 0}$ by

$$G_0 = 0, \quad dG_t = \sigma_t dL_t, \quad t \geq 0.$$  

**Note:**

$G$ jumps at the same times as $L$ with jump size $\Delta G_t = \sigma_t \Delta L_t$.  

Properties

(1) \((X_t)_{t \geq 0}\) is spectrally negative Lévy process of finite variation with Brownian part 0, drift \((-\log \delta)\) and Lévy measure \(\Pi_X\):

\[
\Pi_X([0, \infty)) = 0
\]

\[
\Pi_X((-, \infty, -x]) = \Pi_L([|y| \geq \sqrt{(e^x - 1)\delta/\lambda}]), \quad x > 0.
\]

Proof  By definition: \(\gamma_X = -\log \delta, \tau_X^2 = 0,\)

\[
\Pi_X((-, \infty, -x]) = E \left[ \sum_{0 < s \leq 1} I_{\{-\log(1 + (\lambda/\delta)(\Delta L_s)^2) \leq -x\}} \right]
\]

\[
= E \left[ \sum_{0 < s \leq 1} I_{\{|y| \geq \sqrt{(e^x - 1)\delta/\lambda}\}} \right], \quad x > 0.
\]

\[
\int_{|x| < 1} |x| \Pi_X(dx) = \int_{|y| \leq \sqrt{(e^x - 1)\delta/\lambda}} \log(1 + \frac{\lambda}{\delta}y^2) \Pi_L(dy) < \infty.
\]
(2) \((\sigma^2_t)_{t \geq 0}\) satisfies the SDE
\[
d\sigma^2_t = \omega dt + \sigma^2_t e^{X_t - d(e^{-X_t})}, \quad t \geq 0.
\]
and
\[
\sigma^2_t = \sigma^2_0 + \omega t + \log \delta \int_0^t \sigma^2_s ds + \frac{\lambda}{\delta} \sum_{0<s\leq t} \sigma^2_s (\Delta L_s)^2, \quad t \geq 0. \tag{3}
\]

**Proof**  Itô’s Lemma. \(\square\)

Compare (3) to discrete-time GARCH(1,1):

\[
\sigma^2_n - \sigma^2_{n-1} = \omega - (1 - \delta)\sigma^2_{n-1} + \lambda \sigma^2_{n-1} \epsilon^2_{n-1}
\]

\[\implies \quad \sigma^2_n = \sigma^2_0 + \omega n - (1 - \delta) \sum_{i=1}^{n-1} \sigma^2_i + \lambda \sum_{i=1}^{n-1} \sigma^2_i \epsilon^2_i. \]
(3) **Stationarity:** Suppose

\[
\int_{\mathbb{R}} \log \left( 1 + \frac{\lambda}{\delta} x^2 \right) \Pi_L(dx) < -\log \delta
\]  

(4)

Then \( \sigma_t^2 \xrightarrow{d} \sigma^2 = \omega \int_0^\infty e^{-X_t} dt \). Otherwise, \( \sigma_t^2 \xrightarrow{P} \infty \).

**Proof** Erickson & Maller (2004)

**Example:**

\( L \) compound Poisson with rate \( c \) and jump distribution \( \varepsilon \)

\( \Rightarrow \Pi_L = cP_\varepsilon \)

(4) \( \iff c \mathbb{E} \log \left( 1 + \frac{\lambda}{\delta} \varepsilon^2 \right) < -\log \delta \)

\( \iff -c \log \delta + \mathbb{E} \log(\delta + \lambda \varepsilon^2) < -\log \delta \)

If \( c = 1 \), then

(4) \( \iff \mathbb{E} \log(\delta + \lambda \varepsilon^2) < 0 \)
(4) \((\sigma^2_t)_{t \geq 0}\) is a Markov process, hence \((\sigma^2_t)_{t \geq 0}\) is strictly stationary for \(\sigma^2_0 \stackrel{d}{=} \sigma^2\).

(5) \((\sigma^2_t, G_t)_{t \geq 0}\) is a bivariate Markov process.

(6) If \((\sigma^2_t)_{t \geq 0}\) is stationary, then \((G_t)_{t \geq 0}\) has stationary increments.
Second order properties of \((\sigma^2_t)_{t \geq 0}\)

\[ \begin{align*} X_t &= -t \log \delta - \sum_{s \leq t} \log(1 + \frac{\lambda}{\delta} (\Delta L_s)^2), \quad t \geq 0 \\ \end{align*} \]

is a Lévy process of finite variation

\[ \sigma^2_t = \left( \omega \int_0^t e^{X_s} ds + \sigma^2_0 \right) e^{-X_t}, \quad t \geq 0 \]

Define \(E e^{-cX_t} = e^{t \Psi_X(c)}\), then

\[ \begin{align*} \Psi_X(c) &= \log E e^{-cX_1} = c \log \delta + \int_{\mathbb{R}} \left( (1 + \frac{\lambda}{\delta} y^2)^c - 1 \right) \Pi_L(dy) \end{align*} \]
For $c > 0$: $(\sigma_t)_{t \geq 0}$ stationary.

(1) $E e^{-cX_1} < \infty \iff EL_1^{2c} < \infty$ and $|\Psi_X(c)| < \infty$

(2) $EL_1^2 < \infty$ and $\Psi_X(1) < 0 \implies \sigma_t^2 \overset{d}{\to} \sigma^2$ (finite random variable)

(3) $E(\sigma_t^2)^k < \infty \iff EL_1^{2k} < \infty$ and $\Psi_X(k) < 0$.

In that case

$$E\sigma^{2k} = \frac{k! \omega^k}{\prod_{l=1}^{k} (-\Psi_X(l))}$$

$$\text{cov}(\sigma_t^2, \sigma_{t+h}^2) = \omega^2 \left( \frac{2}{\Psi_X(1)\Psi_X(2)} - \frac{1}{(\Psi_X(1))^2} \right) e^{-h|\Psi_X(1)|}$$

(4) $0 < \delta < 1, \lambda > 0 \implies \sigma$ has always infinite moments.
Second order properties of $\( (G_t)_{t \geq 0} \)$

\[ dG_t = \sigma_t dL_t \]

\[ \implies G \text{ jumps at the same times as } L \text{ does: } \Delta G_t = \sigma_t \Delta L_t \]

\[ \implies \forall t > 0 \text{ fix : } E(\Delta G_t)^k = 0 \]

Define for \( r > 0 \) fix :

\[ G_t^{(r)} = G_{t+r} - G_t = \int_{(t,t+r]} \sigma_s dL_s \]

Take \( (\sigma_t^2)_{t \geq 0} \text{ stationary } \implies (G_t^{(r)})_{t \geq 0} \text{ stationary.} \)
Theorem (ACF of \((G_t^{(r)})\))

\((L_t)_{t \geq 0}\) pure jump process \((\tau^2_L = 0)\),

\(EL_1^2 < \infty, EL_1 = 0, \Psi_X(1) < 0\).

Then

\[
EG_t^{(r)} = 0
\]

\[
E(G_t^{(r)})^2 = \frac{\omega r}{-\Psi_X(1)} EL_1^2
\]

\[
\text{cov}(G_t^{(r)}, G_t^{(r+h)}) = 0
\]

If \(EL_1^4 < \infty, \Psi(2) < 0\), then

\[
\text{cov}((G_t^{(r)})^2, (G_t^{(r+h)})^2)
\]

\[
= \left(e^{r|\Psi_X(1)|} - 1\right) \frac{EL_1^2}{|\Psi_X(1)|} \text{cov}(\sigma^2_r, G^2_r)e^{-h|\Psi_X(1)|}.
\] (5)

If \(EL_1^8 < \infty, \Psi(4) < 0\), \(\int_{\mathbb{R}} x^3 \Pi_L(dx) = 0 \implies (5) > 0\).
Theorem (Tail behaviour):
Suppose for $D := \{d \in [0, \infty) : E|L_1|^{2d} < \infty\}$ we have
\[
\sup D \not\in D
\]
\[
\implies \exists C > 0, \kappa \in D : \lim_{x \to \infty} x^\kappa P(\sigma^2 > x) = C
\]
If furthermore: $\exists \alpha > 4\kappa : E|L_1|^\alpha < \infty$ and
$(L_t)_{t \geq 0}$ of finite variation and not negative of a subordinator
\[
\implies \forall t > 0 \exists C_t > 0 : \lim_{x \to \infty} x^{2\kappa} P(G_t > x) = C_{1,t}
\]
Proof:

\[ \sigma_1^2 = e^{-X_1} \sigma_0^2 + \omega \int_0^1 e^{X_s - X_1} \, ds, \]

\( \sigma_0^2 \) independent of \( \left( e^{-X_1}, \omega \int_0^1 e^{X_s - X_1} \, ds \right) \).

Hence for stationary solution \( \sigma^2 \) a random fixed point equation

\[ \sigma^2 \overset{d}{=} M \sigma^2 + Q, \]

\[ M \overset{d}{=} e^{-X_1}, \quad Q \overset{d}{=} \omega \int_0^1 e^{-X_s} \, ds. \]

References


Figure 1: Simulated compound Poisson process \((L_t)_{0 \leq t \leq 10000}\) with rate 1 and standard normally distributed jump sizes (first) with corresponding COGARCH process \((G_t)\) (second), volatility process \((\sigma_t)\) (third) and differenced COGARCH process \((G_t^{(1)})\) of order 1, where \(G_t^{(1)} = G_{t+1} - G_t\) (last). The parameters were: \(\beta = 1, \delta = 0.95\) and \(\lambda = 0.045\). The starting value was chosen as \(\sigma_0 = 10\).
Figure 2: Sample autocorrelation functions of $\sigma_t$ (top left), $\sigma_t^2$ (top right), $G_t^{(1)}$ (bottom left) and $(G_t^{(1)})^2$ (bottom right), for the process simulated in Figure 1. The dashed lines in the bottom graphs show the confidence bounds $\pm 1.96/\sqrt{9999}$. 