GARCH processes – continuous counterparts (Part 2)

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Contents

Why continuous time models?

- Observations are quite often irregularly spaced.
- Observations quite often come in at a very high frequency.

Then a continuous time model may provide a better approximation to the discrete data than a discrete model.

Aim: Construct continuous time models with features of GARCH.

The diffusion approximation of Nelson

$$\begin{aligned}\sigma_n^2 &= \omega + \lambda\sigma_{n-1}^2\varepsilon_{n-1}^2 + \delta\sigma_{n-1}^2 \\ &= \omega + (\lambda\varepsilon_{n-1}^2 + \delta)\sigma_{n-1}^2 \\ Y_n &= \sigma_n\varepsilon_n\end{aligned}$$

where $(\varepsilon_n)_{n\in\mathbb{N}}$ i.i.d.: GARCH(1,1) process.

Set

$$G_n := \sum_{i=0}^{n} Y_i, \quad n \in \mathbb{N}_0$$

Then

$$G_n - G_{n-1} = Y_n,$$

so the increments of $(G_n)_{n\in\mathbb{N}_0}$ are a GARCH process, i.e. (G_n) is "accumulated GARCH".

Question: Can we find a sequence of processes, whose increments on finer becoming grids are GARCH processes, such that the processes converge in distribution to a non-trivial limit process?

Setup:

Take grid width $h > 0$

$$
\begin{aligned}
{}_hG_{nh} &= {}_hG_{(n-1)h} + {}_h\sigma_{nh} \cdot {}_h\varepsilon_{nh}, \quad n \in \mathbb{N}, \\
{}_h\sigma^2_{(n+1)h} &= \omega_h + \left(h^{-1}\lambda_h \cdot {}_h\varepsilon^2_{nh} + \delta_h\right) \cdot {}_h\sigma^2_{nh}, \quad n \in \mathbb{N}_0,
\end{aligned}
$$

$({}_h\varepsilon_{nh})_{n\in\mathbb{N}_0}$ i.i.d. $N(0,h)$
$({}_h\sigma^2_0, {}_hG_0)$ independent of $({}_h\varepsilon_{nh})_{n\in\mathbb{N}}$
$\omega_h > 0,\ \lambda_h \geq 0,\ \delta_h \geq 0.$

Then $({}_hG_{nh} - {}_hG_{(n-1)h})_{n\in\mathbb{N}}$ is GARCH(1,1) process

$$
{}_hG_t := {}_hG_{nh}, \qquad {}_h\sigma^2_t := {}_h\sigma^2_{nh}, \quad nh \leq t < (n+1)h
$$

defines $({}_hG_t, {}_h\sigma^2_t)$ for all $t \in \mathbb{R}_+$

Question: When does $({}_hG_t, {}_h\sigma^2_t)_{t\geq 0}$ converge weakly to a process (G, σ^2) as $h \downarrow 0$?

(weak convergence is in the space $D(\mathbb{R}_+, \mathbb{R}^2)$ of càdlàg functions, endowed with the Borel sets of the Skorohod topology; weak convergence of processes implies in particular convergence of finite dimensional distributions)

Theorem: (Nelson, 1990)
Suppose

$$({}_hG_0,\ {}_h\sigma_0^2) \xrightarrow{d} (G_0, \sigma_0^2), \quad h \downarrow 0$$

$$P(\sigma_0^2 > 0) = 1$$

$$\lim_{h\downarrow 0} h^{-1}\omega_h = \omega \geq 0$$

$$\lim_{h\downarrow 0} h^{-1}(1 - \delta_h - \lambda_h) = \theta$$

$$\lim_{h\downarrow 0} 2h^{-1}\lambda_h^2 = \lambda^2 > 0$$

Then $({}_hG,\ {}_h\sigma^2)$ converges weakly as $h \downarrow 0$ to the unique solution (G, σ^2) of the diffusion equation

$$dG_t = \sigma_t \, dB_t^{(1)}, \tag{1}$$

$$d\sigma_t^2 = (\omega - \theta\sigma_t^2)\, dt + \lambda\sigma_t^2 \, dB_t^{(2)}, \tag{2}$$

with starting value (G_0, σ_0^2), where $(B_t^{(1)})_{t\geq 0}$ and $(B_t^{(2)})_{t\geq 0}$ are *independent* Brownian motions, independent of (G_0, σ_0^2).

If $2\theta/\lambda^2 > -1$ and $\omega > 0$, then the solution $(\sigma_t^2)_{t\geq 0}$ is strictly stationary iff $\sigma_0^{-2} \stackrel{d}{=} \Gamma(1 + 2\theta/\lambda^2, 2\omega/\lambda^2)$

Example: $\omega_h = \omega h$, $\delta_h = 1 - \lambda\sqrt{h/2} - \theta h$, $\lambda_h = \lambda\sqrt{h/2}$

Interpretation:
The solution of (1) and (2) can be approximated by a GARCH process in discrete time.

Observe:

- The stationary limiting process σ^2 has Pareto like tails.
- The limit (G, σ^2) is driven by **two independent** Brownian motions. The GARCH process has only one source of randomness!
- The processes G and σ^2 are continuous. But empirical volatility can exhibit jumps.
- Estimation of the parameters of the diffusion limit and of the discrete GARCH processes may lead to significantly different results (Wang, 2002)

Further literature: Drost and Werker (1996), Duan (1997)

The COGARCH(1,1) process
Klüppelberg, Lindner, Maller (2004)

Recall discrete GARCH(1,1):

$$\begin{aligned}
\sigma_n^2 &= \omega + \lambda Y_{n-1}^2 + \delta\sigma_{n-1}^2 = \omega + (\delta + \lambda\varepsilon_{n-1}^2)\sigma_{n-1}^2 \\
&\vdots \\
&= \omega \sum_{i=0}^{n-1} \prod_{j=i+1}^{n-1} (\delta + \lambda\varepsilon_j^2) + \sigma_0^2 \prod_{j=0}^{n-1} (\delta + \lambda\varepsilon_j^2) \\
&= \Big(\omega \int_0^n \exp\Big\{ \underbrace{-\sum_{j=0}^{\lfloor s \rfloor} \log(\delta + \lambda\varepsilon_j^2)}_{\textbf{random walk}} \Big\} ds + \sigma_0^2 \Big) \\
&\qquad \times \exp\Big\{ \underbrace{\sum_{j=0}^{n-1} \log(\delta + \lambda\varepsilon_j^2)}_{\textbf{random walk}} \Big\} \\
Y_n &= \sigma_n \varepsilon_n = \sigma_n \underbrace{\Big(\sum_{j=0}^{n} \varepsilon_j - \sum_{j=0}^{n-1} \varepsilon_j\Big)}_{\textbf{increment of random walk}}
\end{aligned}$$

Both appearing random walks are linked

Idea: Replace appearing random walks by *Lévy processes*
(= continuous time analogue of random walk)
Replace ε_j by jumps of a Lévy process L.

Recall: A stochastic process $(L_t)_{t \geq 0}$ is a *Lévy process* iff

- it has *independent increments*:
  for $0 \leq a < b \leq c < d$: $L_d - L_c$ and $L_b - L_a$ are independent
- it has *stationary increments*:
  the distribution of $L_{t+s} - L_t$ does not depend on t
- it is stochastically continuous
- with probability one it has right-continuous paths with finite left-limits (càdlàg paths)
- $L_0 = 0$ a.s.

Examples

- *Brownian motion* (has normal increments)
- *Compound Poisson process*:
  $(\varepsilon_n)_{n\in\mathbb{N}}$ i.i.d. sequence, independent of $(v_n)_{n\in\mathbb{N}}$ i.i.d. with exponential distribution with mean c

$$T_n := \sum_{j=1}^{n} v_n$$

$$N_t := \max\{n \in \mathbb{N}_0 : T_n \leq t\}$$

$$L_t := \sum_{j=1}^{N_t} \varepsilon_j$$

Note: All Lévy processes apart from Brownian motion have jumps.

Lévy-Kintchine formula

$(L_t)_{t\geq 0}$ Lévy process:

$$Ee^{isL_t} = e^{t\chi_L(s)}, \quad s \in \mathbb{R},$$

$$\chi_L(s) = i\gamma_L s - \tau_L^2 \frac{s^2}{2} + \int_{\mathbb{R}} (e^{isx} - 1 - isx\mathbf{1}_{\{|x|<1\}})\Pi_L(dx), \quad s \in \mathbb{R}.$$

- $(\gamma_L, \tau_L, \Pi_L)$ *characteristic triplet*
- $\tau_L^2 \geq 0$ *Brownian part*
- Π_L *Lévy measure*:

$$\int_{|x|<1} x^2 \Pi_L(dx) < \infty, \quad \int_{|x|\geq 1} \Pi_L(dx) < \infty$$

- If $\int_{|x|<1} |x|\Pi_L(dx) < \infty$ and $\tau_L^2 = 0$: *finite variation* case.

$$\chi_L(s) = i\gamma_{L,0} s - \tau_L^2 \frac{s^2}{2} + \int_{\mathbb{R}} (e^{isx} - 1)\Pi_L(dx) \quad s \in \mathbb{R}.$$

  $\gamma_{L,0}$ *drift* of L

Jumps of Lévy processes

- $\Delta L_t := L_t - L_{t-}$ *jumps*
- totally described by the Lévy measure
- Π_L infinite (not compound Poisson) $\Longrightarrow$ almost surely, $(L_t)_{t\geq 0}$ has infinitely many jumps in finite time intervals
- $\sum_{0<s\leq t} |\Delta L_s| < \infty \Longleftrightarrow \int_{|x|<1} |x| \Pi_L(dx) < \infty$
- Always: $\sum_{0<s\leq t} |\Delta L_s|^2 < \infty$ almost surely.

COGARCH(1,1) - definition

For $\Pi_L \neq 0$, $\delta > 0$, $\lambda \geq 0$ define *auxiliary Lévy process*

$$X_t = -t \log \delta - \sum_{0<s\leq t} \log(1 + \frac{\lambda}{\delta}(\Delta L_s)^2), \quad t \geq 0.$$

For $\omega > 0$ and a finite random variable σ_0^2 independent of $(L_t)_{t\geq 0}$ define the *volatility process*

$$\sigma_t^2 = \left(\omega \int_0^t e^{X_s} ds + \sigma_0^2\right) e^{-X_{t-}}, \quad t \geq 0.$$

Define *COGARCH(1,1)* $(G_t)_{t\geq 0}$ by

$$G_0 = 0, \quad dG_t = \sigma_t dL_t, \quad t \geq 0.$$

Note:
G jumps at the same times as L with jump size $\Delta G_t = \sigma_t \Delta L_t$.

Properties

(1) $(X_t)_{t\geq 0}$ is spectrally negative Lévy process of finite variation with Brownian part 0, drift $(-\log\delta)$ and Lévy measure Π_X:

$$\Pi_X([0,\infty)) = 0$$

$$\Pi_X((-\infty,-x]) = \Pi_L(\{|y| \geq \sqrt{(e^x-1)\delta/\lambda}\}), \quad x > 0.$$

Proof By definition: $\gamma_X = -\log\delta$, $\tau_X^2 = 0$,

$$\begin{aligned}\Pi_X((-\infty,-x]) &= E\Big[\sum_{0<s\leq 1} I_{\{-\log(1+(\lambda/\delta)(\Delta L_s)^2)\leq -x\}}\Big] \\ &= E\Big[\sum_{0<s\leq 1} I_{\{|y|\geq\sqrt{(e^x-1)\delta/\lambda}\}}\Big], \quad x > 0.\end{aligned}$$

$$\int_{|x|<1} |x|\Pi_X(dx) = \int_{|y|\leq\sqrt{(e^x-1)\delta/\lambda}} \log(1+\frac{\lambda}{\delta}y^2)\,\Pi_L(dy) < \infty.$$

(2) $(\sigma_t^2)_{t\geq 0}$ satisfies the SDE

$$d\sigma_t^2 = \omega dt + \sigma_t^2 e^{X_{t-}} d(e^{-X_t}), \quad t \geq 0.$$

and

$$\sigma_t^2 = \sigma_0^2 + \omega t + \log\delta \int_0^t \sigma_s^2 ds + \frac{\lambda}{\delta} \sum_{0<s\leq t} \sigma_s^2 (\Delta L_s)^2, \quad t \geq 0. \quad (3)$$

Proof Itô's Lemma. □

Compare (3) to discrete-time GARCH(1,1):

$$\sigma_n^2 - \sigma_{n-1}^2 = \omega - (1-\delta)\sigma_{n-1}^2 + \lambda \sigma_{n-1}^2 \varepsilon_{n-1}^2$$

$$\Longrightarrow \quad \sigma_n^2 = \sigma_0^2 + \omega n - (1-\delta) \sum_{i=1}^{n-1} \sigma_i^2 + \lambda \sum_{i=1}^{n-1} \sigma_i^2 \varepsilon_i^2.$$

(3) Stationarity: Suppose

$$\int_{\mathbb{R}} \log\left(1+\frac{\lambda}{\delta}x^2\right)\Pi_L(dx) < -\log\delta \qquad (4)$$

Then $\sigma_t^2 \xrightarrow{d} \sigma^2 \stackrel{d}{=} \omega\int_0^\infty e^{-X_t}dt$. Otherwise, $\sigma_t^2 \xrightarrow{P} \infty$.

Proof Erickson & Maller (2004) □

Example:

L compound Poisson with rate c and jump distribution ε
$\Longrightarrow \Pi_L = cP_\varepsilon$

$$(4) \iff c\,E\log\left(1+\frac{\lambda}{\delta}\varepsilon^2\right) < -\log\delta$$
$$\iff -c\,\log\delta + E\log(\delta+\lambda\varepsilon^2) < -\log\delta$$

If $c=1$, then

$$(4) \iff E\log(\delta+\lambda\varepsilon^2) < 0$$

(4) $(\sigma_t^2)_{t\geq 0}$ is a Markov process, hence $(\sigma_t^2)_{t\geq 0}$ is strictly stationary for $\sigma_0^2 \stackrel{d}{=} \sigma^2$.

(5) $(\sigma_t^2, G_t)_{t\geq 0}$ is a bivariate Markov process.

(6) If $(\sigma_t^2)_{t\geq 0}$ is stationary, then $(G_t)_{t\geq 0}$ has stationary increments.

Second order properties of $(\sigma_t^2)_{t\geq 0}$

$$X_t = -t\log\delta - \sum_{s\leq t}\log(1+\frac{\lambda}{\delta}(\Delta L_s)^2), \quad t\geq 0$$

is a Lévy process of finite variation

$$\sigma_t^2 = \left(\omega\int_0^t e^{X_s}ds + \sigma_0^2\right)e^{-X_{t-}}, \quad t\geq 0$$

Define $Ee^{-cX_t} = e^{t\Psi_X(c)}$, then

$$\Psi_X(c) = \log Ee^{-cX_1} = c\log\delta + \int_{\mathbb{R}}\left((1+\frac{\lambda}{\delta}y^2)^c - 1\right)\Pi_L(dy)$$

For $c > 0$: $(\sigma_t)_{t\geq 0}$ stationary.

(1) $Ee^{-cX_1} < \infty \iff EL_1^{2c} < \infty$ and $|\Psi_X(c)| < \infty$

(2) $EL_1^2 < \infty$ and $\Psi_X(1) < 0 \implies \sigma_t^2 \xrightarrow{d} \sigma^2$ (finite random variable)

(3) $E(\sigma^2)^k < \infty \iff EL_1^{2k} < \infty$ and $\Psi_X(k) < 0$.
In that case

$$E\sigma^{2k} = \frac{k!\omega^k}{\prod_{l=1}^{k}(-\Psi_X(l))}$$

$$\mathrm{cov}(\sigma_t^2, \sigma_{t+h}^2) = \omega^2 \left(\frac{2}{\Psi_X(1)\Psi_X(2)} - \frac{1}{(\Psi_X(1))^2} \right) e^{-h|\Psi_X(1)|}$$

(4) $0 < \delta < 1$, $\lambda > 0 \implies \sigma$ has always infinite moments.

Second order properties of $(G_t)_{t\geq 0}$

$dG_t = \sigma_t dL_t$
$\Longrightarrow$ G jumps at the same times as L does: $\Delta G_t = \sigma_t \Delta L_t$
$\Longrightarrow$ $\forall t > 0$ fix : $E(\Delta G_t)^k = 0$

Define for $r > 0$ fix :

$$G_t^{(r)} = G_{t+r} - G_t = \int_{(t,t+r]} \sigma_s dL_s$$

Take $(\sigma_t^2)_{t\geq 0}$ stationary $\Longrightarrow$ $(G_t^{(r)})_{t\geq 0}$ stationary.

Theorem (ACF of $(G_t^{(r)})$)
$(L_t)_{t\geq 0}$ pure jump process ($\tau_L^2 = 0$),
$EL_1^2 < \infty$, $EL_1 = 0$, $\Psi_X(1) < 0$.
Then

$$\begin{aligned} EG_t^{(r)} &= 0 \\ E(G_t^{(r)})^2 &= \frac{\omega r}{-\Psi_X(1)} EL_1^2 \\ \operatorname{cov}(G_t^{(r)}, G_t^{(r+h)}) &= 0 \end{aligned}$$

If $EL_1^4 < \infty$, $\Psi(2) < 0$, then

$$\begin{aligned} &\operatorname{cov}((G_t^{(r)})^2, (G_t^{(r+h)})^2) \\ &= \left(e^{r|\Psi_X(1)|} - 1\right) \frac{EL_1^2}{|\Psi_X(1)|} \operatorname{cov}(\sigma_r^2, G_r^2) e^{-h|\psi_X(1)|}. \qquad (5) \end{aligned}$$

If $EL_1^8 < \infty$, $\Psi(4) < 0$, $\int_{\mathbb{R}} x^3 \Pi_L(dx) = 0 \quad \Longrightarrow \quad (5) > 0$.

Theorem (Tail behaviour):
Suppose for $D := \{d \in [0, \infty) : E|L_1|^{2d} < \infty\}$ we have

$$\sup D \notin D$$

$$\Longrightarrow \exists C > 0, \kappa \in D : \lim_{x\to\infty} x^{\kappa} P(\sigma^2 > x) = C$$

If furthermore: $\exists \alpha > 4\kappa : E|L_1|^{\alpha} < \infty$ and
$(L_t)_{t\geq 0}$ of finite variation and not negative of a subordinator

$$\Longrightarrow \ \forall\, t > 0\ \exists\, C_t > 0 : \lim_{x\to\infty} x^{2\kappa} P(G_t > x) = C_{1,t}$$

Proof:

$$\sigma_1^2 = e^{-X_{1-}}\sigma_0^2 + \omega \int_0^1 e^{X_s - X_{1-}}\, ds,$$

σ_0^2 independent of $\left(e^{-X_{1-}}, \omega \int_0^1 e^{X_s - X_{1-}}\, ds\right)$.

Hence for stationary solution σ^2 a *random fixed point equation*

$$\sigma^2 \stackrel{d}{=} M\sigma^2 + Q,$$

$$M \stackrel{d}{=} e^{-X_1}, \quad Q \stackrel{d}{=} \omega \int_0^1 e^{-X_s}\, ds.$$

Then apply Theorem of Goldie (1991).

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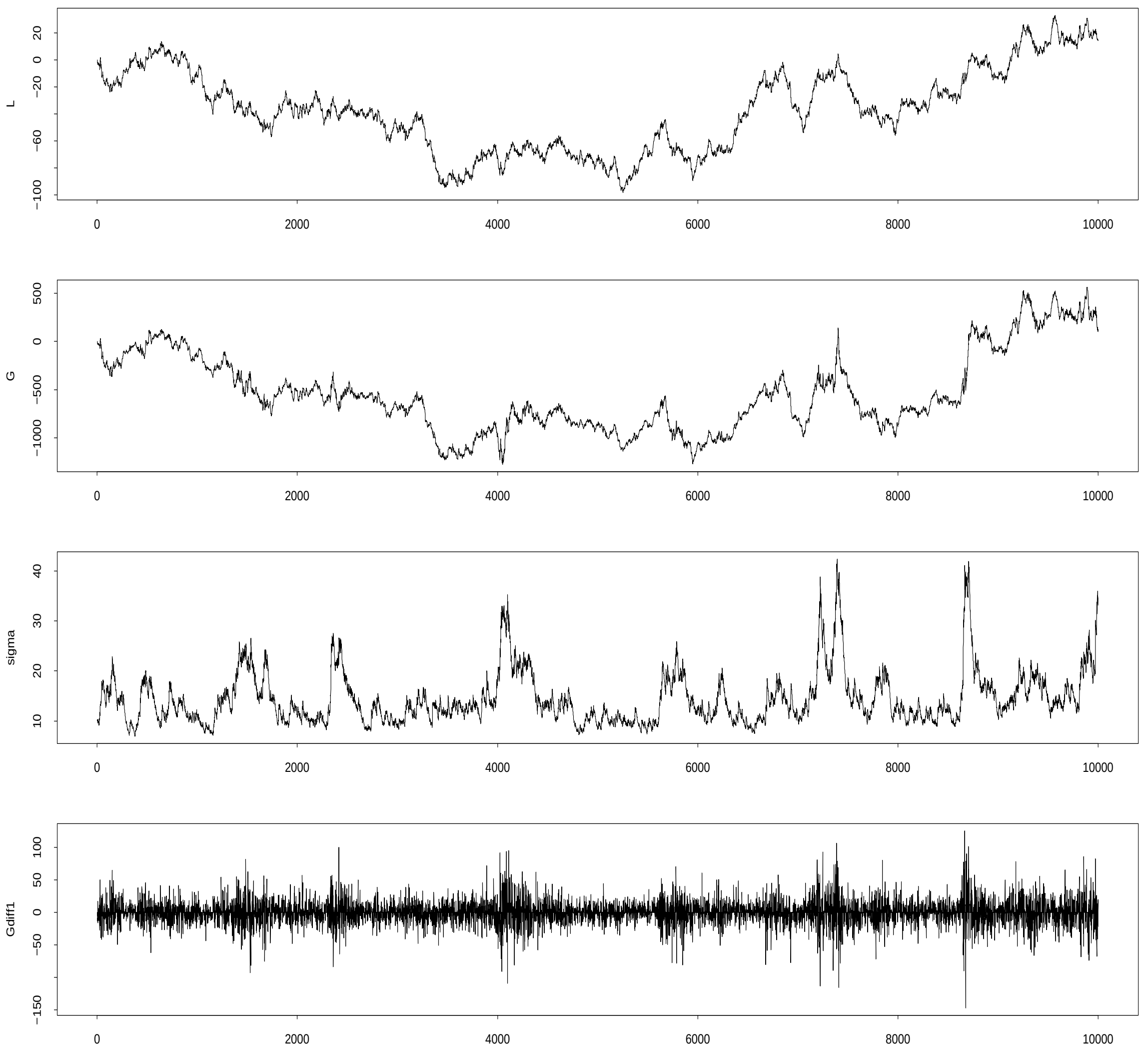

Figure 1: Simulated compound Poisson process $(L_t)_{0 \leq t \leq 10\,000}$ with rate 1 and standard normally distributed jump sizes (*first*) with corresponding COGARCH process (G_t) (*second*), volatility process (σ_t) (*third*) and differenced COGARCH process $(G_t^{(1)})$ of order 1, where $G_t^{(1)} = G_{t+1} - G_t$ (*last*). The parameters were: $\beta = 1$, $\delta = 0.95$ and $\lambda = 0.045$. The starting value was chosen as $\sigma_0 = 10$.

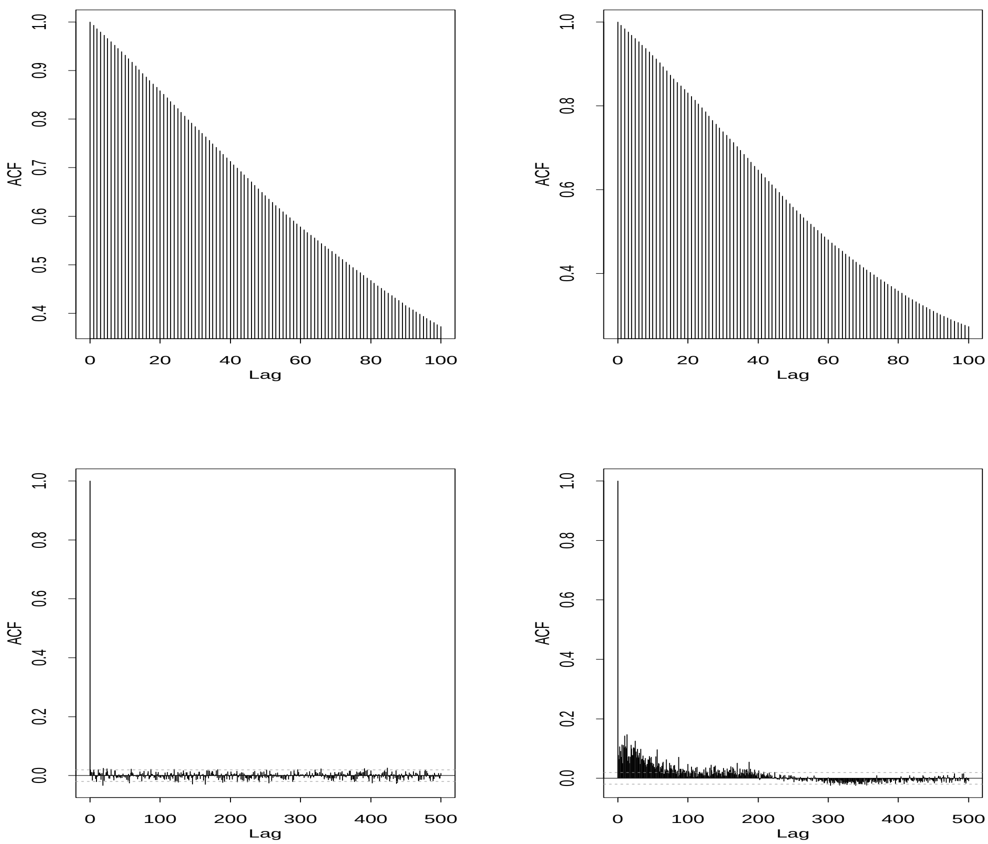

Figure 2: Sample autocorrelation functions of σ_t (*top left*), σ_t^2 (*top right*), $G_t^{(1)}$ (*bottom left*) and $(G_t^{(1)})^2$ (*bottom right*), for the process simulated in Figure 1. The dashed lines in the bottom graphs show the confidence bounds $\pm 1.96/\sqrt{9999}$.