We deal with the introduction of life insurance and pension decisions in the personal financial problem of optimal lifetime consumption of lifetime income. We introduce in Section 2 the classical notion of reserves and present well-known differential equations characterizing these. We start with the survival model and discuss also the case where pension saving takes place in a bank. We then analyze the disability model and the multistate model that are generalizations of the survival model. This structure is repeated in all sections of the paper. In Section 3 we introduce the notion of utility reserves that makes it possible to compare the different contracts offered by the insurance company, and present differential equations characterizing these. The utility reserve is the basis for static optimization of payment streams in Section 4 and for dynamic optimization of payment streams in Section 5. In particular in the case of dynamic optimization, the differential equation characterizing the utility reserve plays a crucial role since the so-called Hamilton-Jacobi-Bellman equation characterizing the optimal solution is based on it. Sections 2-5 are ended by a continued numerical example illustrating our findings for the survival model. We conclude by further remarks on generalizations and applications.


1 Introduction

There are basically three categories of decision problems connected with life and pension insurance. First, the society or the state needs to decide on the arrangement of the pension system. Who are
Merton (1969,1971) initiated the continuous time approach to the financial discipline nowadays spoken of as personal finance where the individual’s personal financial decision making is in focus. He also worked with the individual’s uncertain lifetime but other authors introduced the life insurance choice in the decision problem. Yaari (1965), Hakansson (1969), Fischer (1973), and Campbell (1980) modelled the life insurance decision and discussed the demand for life insurance in a discrete time setting. Richard (1975) merged the continuous time stochastic control approach taken by Merton with the life insurance decision making.

Although all these papers, of course, contain a variety of modelling, decision making, and conclusions, the very essence of these papers are here reduced to the following two bullets:

- In asset allocation, the human capital should be added to the financial wealth to form the individual’s total wealth. Here we use the term ‘human capital’ for the financial value of future labor income. This terminology is often used in personal finance in contrasts to corporate finance and economics where ‘human capital’ can be used in other ways. From here the optimal asset allocation of the financial wealth should be calculated. If labor income is uncertain due to an uncertain lifetime and there is no possibility of protection against death risk, the future labor income has no unique price. But life insurance completes the market, and gives a unique value to future labor income such that the human capital can be determined. The human capital approach leads to the life cycle investment advice: Allocation to risky assets should decrease with age since, as time goes by, the ratio of human capital over total wealth goes down.

- The individual buys life insurance in order to protect his human capital. Thus at younger ages where the individual’s human capital is large, he needs a strong protection. As he grows old, the demand goes down. It may even go negative in the sense that, at old ages, he should sell life insurance, thereby earning a premium for the risk of losing (parts of) his financial wealth when he dies. This all depends on preferences over bequest and consumption.

During the subsequent 20-25 years the contributions have been sparse. The publications have primarily repeated the common knowledge in the two bullets above under small alterations of the setting, some of them in new house-hold economic and microeconomic disguise. However, to the authors knowledge there has been no substantial developments in non of the following two directions: 1) Implementation of strategies in an increasingly varied and integrated financial and insurance market. 2) Generalizations to other life course risks that may (or may not) influence the financial status and may (or may not) be traded in the insurance market, e.g. disability and unemployment risk.

However, during the last years the area has gained renewed attention. A series of recent papers by Piggott and Purcal (2005a,2005b,2005c) and Purcal (2005) search for new solutions under more realistic assumptions (e.g. non-hedgeable income) and search for new assumptions under more realistic solutions (can e.g. tax effects and public pensions explain the realized demand for life annuities?). Kraft and Steffensen (2008) work with saving in the insurance company by directly
modelling the reserve of the insurance contract in the way this is done by life insurance actuaries. Their main contribution is, however, to model life course risk by a multistate model such that e.g. disability and unemployment risk can be dealt with. Applications also include various types of health insurance decision making and insurance decision making of a married couple, possibly with correlated residual lifetimes. However, they do not address the problem of asset allocation but focus on the insurance and consumption decisions.

Common for the works mentioned in the previous paragraph is that they present Richard (1975) as primary reference and thereby revives his elegant presentation of the essence of life insurance decision making from a time where continuous time personal finance was still in its infancy.

The framework needed for a multistate life course model is the multistate Markov chain that is studied intensively and very well understood in the actuarial literature. Hoem (1969) demonstrated its inevitability as the tool for construction of general life insurance products and modelling of general life insurance risk. Since then it has been studied in the context of life insurance and vice versa by Hoem (1988), Norberg (1991) and many others. Steffensen (2004,2006) has approached the decision making of the life insurance company by modelling life insurance risk by a multistate model. He demonstrates how the stochastic control machinery works in such a model and that many known results in the area carry over introducing state-dependent coefficients ‘everywhere’. Kraft and Steffensen (2008) use the same multistate Markov chain for modelling the dynamic decision making of the individual.

This article contains results concerning a static optimization problem where the individual chooses at time 0 (or any other time point) saving rate and term insurance sum for the remaining time horizon under the assumption that he is not allowed to reverse this decision. We also demonstrate how this static problem and its solution generalizes, first to a three state disability model where the individual also make static decisions, e.g. about a disability annuity, and second to a multistate case.

To give a survey over the different possibilities of problem formulation, solution methods, and results, we also cite the results and sketch of arguments in the dynamic decision making case. These parts of the paper are partly based on Kraft and Steffensen (2008), although the presentation differs a lot: First, we go from the simple two-state survival model via the three state disability model to the general multistate model. Second, we skip deep mathematical arguments and focus on intuition and convenient patterns of thinking. As Kraft and Steffensen (2008), we also focus in general on insurance decision making and reduce the usual asset allocation decision to a small remark in the very last section.

The paper is structured as follows: The four Sections 2, 3, 4, and 5 present ideas and results related to classical reserves, utility reserves, static optimization, and dynamic optimization, respectively. Each of these sections has four subsections, where the survival model with saving in the insurance company, the survival model with saving in the bank, the disability model, and a multistate model are treated (the last two with saving in the insurance company). The last section briefly discusses some aspects that are not addressed in the core of the article.

Throughout we assume that the reader is familiar with traditional actuarial notation but present also in the appendix the relevant formulas. The appendix also serves as a natural place to present new notation used in the article, that is inspired by this tradition.


2 Classical Reserves

2.1 Saving in an insurance company

Consider a policy holder with a deferred temporary life annuity and a temporary life insurance.

The contract starts at time 0 where the policy holder is x years old. The life annuity starts at
time m and pays out an annuity benefit b until time m + n, i.e. the annuity runs for n years. The
term insurance pays out the death sum S if the policy holder dies before time m. The package
of coverages is paid for by a level premium $\pi$ paid until death or time m whatever occurs first.

We assume that the insurance company for valuation uses a deterministic interest rate r and a
deterministic mortality rate $\mu^*$. We denote by $\mu$ the so-called objective mortality intensity, possibly
different from the valuation mortality intensity $\mu^*$.

We can now put up the so-called (prospective) reserve, i.e. a value for the future payments
given that the policy holder is alive. This reserve becomes

$$Y(t) = \int_t^{m+n} e^{-\int_t^s r + (\mu^* (s) S - \pi) 1_{(0 < s \leq m)} + b1_{(m < s \leq m+n)}} ds$$

(1)

using traditional actuarial notation in the last equation, see the appendix. This reserve is actually
the result of taking a conditional expectation of future discounted payments. We can specify it
as such by formalizing the underlying stochastic phenomenons in play. Let $N$ denote a counting
process counting the number of deaths (equals 0 or 1) such that $N(t)$ equals the number of deaths
at time t. Let I(t) indicate whether the policy holder is alive at time t. (Obviously, $N(t) = 1 - I(t)$
and one of them is therefore redundant, but later we generalize these processes to a situation where
this is not true.) Then we can also write

$$Y(t) = E_{t,0}^* \left[ \int_t^{m+n} e^{-\int_t^s r + (\mu^* (s) S - \pi) 1_{(0 < s \leq m)} + b1_{(m < s \leq m+n)}} ds \right],$$

(2)

dB(t) = (SdN(t) - \pi I(t) dt) 1_{(0 < t \leq m)} + bI(t) dt 1_{(m < t \leq m+n)},

where $E_{t,0}^*$ denotes expectation given that I(t) = 1 and using the mortality intensity $\mu^*$, and B is
the process of accumulated payments. Using that

$$E^* [dN(t) | I(t) = 1] = e^{-\int_t^s \mu^* (s) ds},$$

$$E^* [I(s) | I(t) = 1] = e^{-\int_t^s \mu^* ds},$$

directly gives (1).

The reserve (1) fulfills the backward Thiele differential equation with pasting and terminal
conditions,

$$\frac{d}{dt} Y(t) = \begin{cases} rY(t) - b + \mu^* (t) Y(t), & t \geq m, \\ rY(t) + \pi - \mu^* (t) (S - Y(t)), & t < m, \end{cases}$$

(3)

$Y(m-) = Y(m),$

$Y(m+n-) = 0.$

The solution to this differential equation is then (1). The differential equation (3) can also be
viewed as a forward equation for $Y$ if we fix $Y(s) = y$, in the sense that

$$Y^* y (t) = Y(t) | Y(s) = y$$
fulfills the forward Thiele differential equation (equal to the differential equation above) with initial condition \( Y_{x:y}(s) = y \). The solution to the forward differential equation with initial condition \( Y(s) = y \) is

\[
Y_{x:y}(t) = ye^{\int_s^t r + \mu^* \, d\tau} + \int_s^t (\pi - \mu^*(\tau)) S e^{\int_s^\tau r + \mu^* \, d\tau} \, d\tau
\]

\[
= y + \pi a^*_x + s:1 - S A^*_1 \frac{e^{\int_0^t r + \mu^* \, d\tau}}{t-s}.
\]

Such an expression is sometimes spoken of as a retrospective reserve although this is nothing but \( Y(t) \) given that \( Y(s) = y \).

The equivalence principle states that the elements \( b, S, \pi \), and a single premium at time 0, \( y_0 \), are determined to fulfill the condition \( Y(0) = y_0 \). If this condition is fulfilled \( Y \) coincides with the solution to the forward equation above with initial condition \( Y^0,y_0(0) = y_0 \). This is the equivalence principle performed at time 0. More generally, one can for a given reserve \( y \) at time \( t \) consider the equation \( Y(t) = y \) as an equivalence principle that determines the elements \( b, S, \pi \) such that the prospective reserve at time \( t \) equals \( y \). Under this condition \( Y \) coincides with the solution to the initial condition \( Y^{t:y}(t) \).

The condition \( Y(t) = y \) reads

\[
y = SA^*_1 + b\, m-ta^*_x + t:m - \pi a^*_x + t:m - t,
\]

such that we can isolate \( b \) and get, for a given \( t, y, \pi, \) and \( S \), the equivalence annuity

\[
b(t, y, \pi, S) = \frac{y + \pi a^*_x + t:m - t - SA^*_1}{m-ta^*_x + t:n}.
\]

In particular we get the equivalence annuity at time 0,

\[
b(0, y_0, \pi, S) = \frac{y_0 + \pi a^*_x + m:0 - S A^*_1}{m a^*_x + m:n}.
\]

After retirement the equivalence annuity is given by

\[
b(t, y) = \frac{y}{a^*_x + t+m-n-t}.
\]

Note that the same annuity as (4) is also obtained by buying an immediate annuity at time \( m \) for the retrospective reserve at time \( m \),

\[
Y_{x:y}^t(m) = b \ a^*_x + m:n \Rightarrow b(t, y, \pi, S) = \frac{Y_{x:y}^t(m)}{a^*_x + m:n}.
\]

The two benefits are seen to coincide by noting the relation \( m-ta^*_x + t:n = t-sE^*_x + s a^*_x + m:n \).

The presentation of the annuity rate as a function of the premium rate and the life insurance sum is important. This shows that decisions about the annuity rate are indirectly made via decisions about the premium rate and the life insurance sum. When we formalize our decision problems below, we use this to work with the premium rate and the life insurance sum as decision variables although the decision maker may have primary focus on consumption as retiree.

**Example 1** Throughout the paper we follow an example with the following parameters: \( S = 250 \) (all amounts are to be thought of as in thousand Euro); \( r = 5\% \); \( \mu^*(t) = 0.0005 + 10^{7.88+0.0381-10} \) (the Danish G82M mortality law); \( x = 30 \) (age at time 0); \( m = 35 \) (retirement age 65); \( n = 35 \) (the life annuity terminates at age 100). Figure 1 shows the reserve as a function of age for the (premium;annuity) pair (4.6;41.1) which fulfills the equivalence criterion at time 0.
2.2 Saving in a bank

The formulas in Subsection 2.1 represent the situation where the saving for retirement takes place in an insurance company. By this we mean that the policy holder makes a long term agreement with an insurance institution about premiums and sum insured and the insurance institution puts up an individual reserve for this policy holder. This reserve increases by interest, premium payments and decreases by the risk premium, that may be positive or negative in the saving period, \(t < m\), and definitely is negative in the retirement period, \(t > m\).

An alternative construction is to let all the saving take place in a bank account. The insurance part of the contract is then constructed by buying continuously infinitesimal short-term term insurance contracts that cover the risk of death in the next instant. This continuum of insurance contracts is paid for by its natural premium such that no saving takes place in the insurance company and this natural premium is simply withdrawn from the bank account.

This approach is often taken by financial economists when introducing insurance decisions in personal finance, see Richard (1975). If there are no constraints on the reserve process and the bank account, one can realize that saving in the two different institutions are just two ways of parametrizing the essentially same situation from a mathematical point of view. However, as we see below, this fact only holds true for dynamic decision making. In the case of static decision making, there are genuine differences between saving in the two institutions. The reason is that preventing unconstrained dynamic decision making, e.g. by allowing static decision making only, also prevents efficient coordination of different financial activities like buying of insurance protection and saving. Therefore we find it appropriate to state all resulting formulas for both cases in this survey on both static and dynamic optimization.

We denote by \(X^{s,x}(t)\) the cash balance of this bank account at time \(t\) given that this bank account initiates by \(x\) at time \(s\). Then, denoting by \(S(t)\) the death sum insured at time \(t\), we have the following dynamics of the bank account for \(t < m\),

\[
    dX^{s,x}(t) = rX^{s,x}(t) \, dt + \pi dt - \mu^x(t) \, S(t) \, dt,
\]

\[
    X^{s,x}(s) = x.
\]

This should be compared with the differential equation for \(Y\) for \(t < m\), (3). We see that if one
can choose \( S(t) = S - X^{s,x}(t) \) the two differential equations coincide such that if \( x = y \) then also the two reserves \( X \) and \( Y \) coincide. This is very reasonable: By saving in the bank, the wealth upon death is the sum paid out by the insurance company plus the cash balance in the bank, \( S(t) + X^{s,x}(t) \), and if you can choose \( S(t) \) such that this sum equals \( S \) then it really does not matter whether saving takes place in the insurance company or the bank. This is of course based on the critical assumption that the insurance account and the bank account earn the same interest. The solution to the differential equation (5) is
\[
X^{s,x}(t) = xe^{\int_0^t (\pi - \mu^a(\tau)) d\tau} + \int_0^t (\pi - \mu^a(\tau)) e^{\int_0^\tau d\tau} d\tau.
\]

After retirement the bank saver may have the possibility to buy an immediate temporary life annuity which would put him in the same situation as the insurance saver. His benefit rate is then
\[
b(t, x, \pi, S) = \frac{X^{t,x}(m)}{a^+_{x+m:n}}.
\]

But he could, in principle, also continue his bank saving and then withdraw from the bank account continuous benefits at rate \( b \). The insurance part of the contract is again constructed by buying short-term insurance contracts. Then the dynamics of the bank account for \( t > m \) becomes
\[
dX^{s,x}(t) = rX^{s,x}(t) dt - bdt - \mu^*(t) S(t) dt.
\]

The insurance contract is copied by choosing \( S(t) = -X^{s,x}(t) \) since then the dynamics of \( X \) and \( Y \) are the same also for \( t > m \). Upon death the saver has to pay out the insurance sum \( X^{s,x}(t) \) to the insurance company but this is exactly the cash balance on the bank account as it should be.

Choosing \( S(t) = S - X^{s,x}(t) \) for \( t < m \) is not a big problem as long as \( S \geq X^{s,x}(t) \). But choosing \( S(t) = S - X^{s,x}(t) \) for \( S < X^{s,x}(t) \) and, in particular, choosing \( S(t) = -X^{s,x}(t) \) for \( t > m \) and \( X^{s,x}(t) > 0 \) means that the individual is a seller of life insurance on his own life without owning a reserve in the insurance company. This is presumably not allowed anywhere in the world.

The word without is so important since also the insurance saver can be said to be a seller of life insurance in the retirement period. A life annuitant sells life insurance! But since the insurance sum \( Y^{s,x}(t) \) is smaller or equal to (actually it is equal to in our case) the reserve this is acceptable for the insurance company.

### 2.3 Introducing a disability annuity

In this subsection we generalize the insurance contract to include also a disability annuity. We assume that the policy holder while being disabled until retirement receives a disability annuity of rate \( \beta \). We work with the same death sum \( S \) no matter whether death occurs as active or disabled and the same retirement annuity \( b \) no matter whether this annuity is received as active or disabled. This assumption could easily be relaxed. We model the life course of the policy holder by a three state model where \( \sigma \) is the disability intensity of going from \( a = \text{active} \) to \( i = \text{disabled} \), \( \rho \) is the recovery intensity of going from \( i \) to \( a \), and \( \mu \) and \( \nu \) are the death intensities of going from \( a \) and \( i \), respectively, to \( d = \text{dead} \).

We need two reserves corresponding to being active and being disabled, respectively. For \( j \in \{a, i\} \), we have that
\[
Y^j(t) = E_{t,j} \left[ \int_t^{m+n} e^{-\int_0^s d\tau} dB(s) \right],
\]
\[
dB(t) = (S(dN^{ad}(t) + dN^{id}(t)) - \pi I^a(t) dt + \beta I^i(t) dt) 1_{(0 < t \leq m)} + bI(t) dt I_{(m < t \leq m+n)}
\]
\]
where $E^*_{t,j}$ denotes conditional expectation given that the policy holder is in state $j$ at time $t$, $N^{ad}$ and $N^{id}$ count the number of transitions from $a$ to $d$ and $i$ to $d$, respectively, $I^a$ and $I^i$ indicate sojourn in state $a$ and $i$, respectively, at time $t$, and we put $I = I^a + I^i$. By the general Thiele differential equation, see Hoem (1969), we have the following differential equation for $Y^a$ with pasting and terminal condition,

$$\frac{d}{dt} Y^a (t) = \begin{cases} r Y^a (t) - b + \mu^* (t) Y^a (t), & t \geq m, \\ r Y^a (t) + \pi - \mu^* (t) (S - Y^a (t)) - \sigma^* (t) (Y^i (t) - Y^a (t)), & t < m, \end{cases}$$

$Y^a (m-) = Y^a (m)$, $Y^a (m+n-) = 0$.

The differential equation for $Y^i$ is the same with $\pi, \mu^*, \sigma^*$, and topscript $a$ replaced by $-\beta, \nu^*, \rho^*$, and topscript $i$, respectively. These differential equations have no explicit solution. But we can represent their solution by means of the (non-explicit) transition probabilities, $p_{aa}^* (t, s), p_{ai}^* (t, s), p_{ia}^* (t, s)$, and $p_{ii}^* (t, s)$. Adopting the notation presented in the Appendix we get

$$Y^j (t) = S A_{x+tm-t}^{1sj} t + b_{m-t} a_{x+tm:n}^{sji} t + \beta a_{x+tm-m-t}^{sji} t - \pi a_{x+tm-m-t}^{sji} t,$$

The general equivalence principle for a given initial reserve $y$ and a given policy state $j$ at time $t$ is now

$$Y^j (t) = y.$$

From this we can calculate the general equivalence annuity

$$b^j (t, y, \pi, \beta, S) = \frac{y + \pi a_{x+tm-m-t}^{sja} t - S A_{x+tm-m-t}^{1ja} t - \beta a_{x+tm-m-t}^{sji} t}{m-t a_{x+tm:n}^{sji} t}. \quad (7)$$

In particular, the equivalence relation performed at time 0, given that the policy holder starts as active, reads

$$b^a (0, y_0, \pi, \beta, S) = \frac{y_0 + \pi a_{x+tm}^{saa} t - S A_{x+tm}^{1sa} t - \beta a_{x+tm}^{sai} t}{m a_{x+tm}^{saa} t}.$$

**Remark 1** If $\rho^* = 0$, all the formulas can be made explicit since we can then calculate $Y^j (t)$ explicitly and plug in our result in the differential equation for $Y^a (t)$ in order to obtain an explicit solution.

### 2.4 Beyond the disability model

In the general setup of a multistate Markov chain one considers the reserve and payment process,

$$Y^j (t) = E^*_{t,j} \left[ \int_t^{t+m+n} e^{-\int_s^t r dB (s)} \right],$$

$$dB (t) = \sum_j (I^j (t) (b^j (t) dt + \Delta B^j (t)) + \sum_{j \neq k} b^{jk} (t) dN^{jk} (t),$$

where $b^j (t)$ and $\Delta B^j (t)$ are the payment rate and lump sum payment, respectively, given that the policy holder is in state $j$ at time $t$, and $b^{jk} (t)$ is the lump sum payment given that the policy holder jumps from state $j$ to state $k$ at time $t$. The processes $I^j$ and $N^{jk}$ indicate sojourn in state $j$ and counts the number of jumps from state $j$ to state $k$ respectively. The differential equation for the reserve is the general Thiele differential equation with pasting and terminal conditions,

$$\frac{d}{dt} Y^j (t) = r Y^j (t) - b^j (t) - \sum_{k, k \neq j} \mu^{jks} (t) (b^{jk} (t) + Y^k (t) - Y^j (t)), \quad Y^j (t-) = Y^j (t) + \Delta B^j (t),$$

$$Y^j (m+n-) = 0.$$
One can then, similarly to (7), for a given reserve at time $t$ and given all other payments coefficients calculate one residual payment coefficient in accordance with the equivalence principle $Y^j (t) = y$.

3 Utility Reserves

3.1 Saving in an insurance company

Recall the definition of the reserve as stated in (2). This expresses the reserve as a conditional expected present value of future payments. We now introduce a different conditional expected present value which we call the utility reserve. Instead of measuring the future payments themselves it measures the utility of these payments.

There are several roads to take when measuring utility of a stream of payments. The starting point is a so-called utility function $u$ that measures the utility of a given payment $b$ such that $u(b)$ is the utility of $b$. For discrete series of payments $b(1), b(2), \ldots, b(n)$ one typically works with time-additive and time-dependent utility such that the utility of this payment stream equals $\sum_{t=1}^{n} u(t, b(t))$ where $u(t, b(t))$ is the utility of the payment $b(t)$ at time $t$. But this is not necessarily the ‘right thing’ to do. One could also define the utility of the payment stream by the utility of the present value of the payments, $u \left( \sum_{t=1}^{n} e^{-rt} b(t) \right)$. However, we work with the time-additive and time-dependent utility.

Furthermore, since we work with continuous payments in the sense of payment rates the question is how one should measure utility of the infinitesimal payment $bdt$ due over the time interval $[t, t+dt)$. The plain proposal assigning this payment the utility $u(bdt)$ gives problems when adding (integrating) these utilities over time. The mathematically more tractable solution is to separate the utility into a utility of the payment rate and the $dt$ factor, i.e. assigning the payment the utility $u(b) dt$. This approach may very well be criticized, in particular if one now starts to add utility of continuous payment measured by the utility of the rate to utility of discrete payments measured by the actual payment. Nevertheless, this is the approach we take here.

Before we replace the payments in the classical reserve by utility of payments in the utility reserve we need one more discussion. What is the utility of paying the premium rate $\pi$? It does not make sense to work with utility of paying payments so we need to say more. If the premium payment is taken out of a salary then, until retirement, one can measure utility of the consumption rate defined as the salary rate $a$ minus the premium rate $\pi$, i.e. $a - \pi$. This is what is left to consumption after the insurance premium has been paid.

We are now almost ready to replace the death sum $S$ paid at time $t$ by $u(t, S)$, the premium rate $\pi$ paid at time $t$ by $u(t, a - \pi)$, and the annuity benefit $b$ paid at time $t$ by $u(t, b)$. However, having realized in Equation (4) that the annuity benefit is essentially a function of $y$, we need also to add the reserve as a state variable. Note that we also replace expectation with $\mu^*$ with expectation with $\mu$ and that we skip discounting since time-dependence is already present in the utility function. Thus, we define

$$V(t, y) = E_{t,a,y} \left[ \int_1^{m+n} dU(s) \right],$$

$$dU(t) = (u(t, S) dN(t) + u(t, a - \pi) I(t) dt) 1_{0 < t \leq m} + u(t, b) I(t) dt 1_{m < t \leq m+n}.$$ 

where $E_{t,a,y}$ denotes expectation given that the insured is in state $a$ = ’alive’ with the reserve $y$ at
time \( t \), i.e. \( I(t) = 1, Y(t) = y \). We then get that

\[
V(t, y) = \int_t^{m+n} e^{-\int_t^s u'((\mu(s)u(s,S) + u(s,a-\pi))1_{t<s\leq m} + u(s,b(t,y,\pi,S))1_{m<s\leq m+n})} \, ds,
\]

In order to move on we need to specify a utility function. Considering \( u(t, b) \), one typically separates the time-dependence and the \( b \)-dependence into a time-weight function \( w \) and a utility function \( u \), i.e.

\[
u(t, b) = w(t)u(b).
\]

As time-weight function one often chooses the exponential function \( w(t) = e^{-\rho t} \) and speak of \( \rho \) as a subjective utility discount factor since it relates the individual’s utility of payments at different points in time to each other. As utility function the most popular ones for further calculations are the exponential utility, \( u(b) = e^{-\theta b} \), and power utility, \( u(b) = \frac{1}{\gamma} b^\gamma \) (with the special case logarithmic utility as the limit when \( \gamma \to 0 \)). We work with power utility throughout as it is usual in connection with financial decision making.

We introduce \( w_a \) as weight function for consumption while being alive and \( w_{ad} \) as weight function for consumption upon death. Furthermore, it turns out convenient to reparametrize the time-weight function \( w(t) \) into \( w^{1-\gamma}(t) \) which we do without loss of generality. Finally, replacing \( u(t, b) \) by \( w_a^{1-\gamma}(t) \frac{1}{\gamma} b^\gamma \), \( u(t, a-\pi) \) by \( w_a^{1-\gamma}(t) \frac{1}{\gamma} (a-\pi)^\gamma \), and \( u(t, S) \) by \( w_{ad}^{1-\gamma}(t) \frac{1}{\gamma} S^\gamma \) in the utility reserve above, we get

\[
\gamma V(t, y) = \begin{cases} 
S^\gamma A_{x+t:m-t}^{w'}(a-\pi)^\gamma a_{x+t:m-t}^{w'} + b(t, y, \pi,S)^\gamma a_{x+t:m+n-t}^{w'}, & t < m, \\
S^\gamma a_{x+t:m+n-t}^{w'} b(t, y)^\gamma, & t > m,
\end{cases}
\]

The utility reserve is also characterized by a certain partial differential equation with pasting and terminal condition,

\[
\begin{align*}
V_t(t, y) &= \\
&= \begin{cases} 
-\frac{1}{\gamma} w_a^{1-\gamma}(t) (a-\pi)^\gamma - \mu(t) \left( \frac{1}{\gamma} w_a^{1-\gamma}(t) S^\gamma - V(t, y) \right), & t < m, \\
-\gamma V_y(t, y) (r \pi + \pi - \mu^*(t) (S - y)), & t = m, \\
\frac{1}{\gamma} w_a^{1-\gamma}(t) (b(t, y))^\gamma + \mu(t) V(t, y) - V_y(t, y) (\mu^*(t) y - b(t, y)), & t > m,
\end{cases}
\end{align*}
\]

\[
V(m, y) = V(m, y),
\]

\[
V(m+n, y) = 0.
\]

This is easily verified by calculating the partial derivatives of \( V(t, y) \) and plugging them into the partial differential equation.

**Example 2** For the utility reserve we have chosen the following parameters: \( \gamma = 0.5; a = 50; w_a^{1-\gamma}(t) = w_a^{1-\gamma}(t) = e^{-\rho t} \) with \( \rho = 0.051 \) (the difference between \( \rho \) and \( r \) is essential and parametrizes the impatience of the policy holder. Here it equals 0.001), \( \mu = \mu^* \) (no (financial) risk premium for bearing mortality risk). Figure 2 shows the utility reserve as a function of age for the (premium,annuity) pair (4.6; 41.1). Note that in contrast to the (financial) reserve, the utility reserve is never increasing since the policy holder always 'earns' positive utility from consumption, both as a worker and as a retiree. The utility reserve at time 0 equals 228,763. This value has no unit.


3.2 Saving in a bank

We also define a utility reserve in the case where saving takes place in the bank. Assuming that the policy holder annuitizes the cash balance at the bank account at time \( m \), we get the utility reserve,

\[
V(t; x) = \int_t^m e^{-\int_s^m \mu(s) \, w_{ad}^{-\gamma}(s) \, ds} \left( S(s) + X_{t,x}(s) \right)^{\gamma} \, ds + (a - \pi)^\gamma \, a_{x+t;m-t}^{w'}
\]

\[
+ e^{-\int_t^m \mu_b(t, x, \pi, S) \, a_{x+t;m:n}^{w'}}
\]

that differs from (8) in two respects: First, the sum insured is replaced by the wealth upon death equal to \( S + X_{t,x} \) and the time dependence of this quantity prevents us from making use of the notation introduced in the Appendix. Second, the reserve \( Y_{t,y}(m) \) used for annuitization is replaced by the cash balance \( X_{t,x}(m) \). Concentrating on the pre-retirement period from 0 to \( m \), this utility reserve fulfills the differential equation with terminal condition,

\[
V_t(t, x) = -\frac{1}{\gamma} \, w_{ad}^{-\gamma}(t) (a - \pi)^\gamma - \mu(t) \left( \frac{1}{\gamma} \, w_{ad}^{-\gamma}(t) (S(t) + x)^\gamma - V(t, x) \right)
\]

\[
- V_x(t, x) \left( r x + \pi - \mu S(t) \right),
\]

\[
V(m, x) = a_{x+m:n}^{w} \, \frac{1}{\gamma} \left( \frac{x}{a_{x+m:n}^{w}} \right)^\gamma.
\]

Again, this is easily checked by recalling (6) and inserting the partial derivatives of \( X_{t,x}(m) \) with respect to \( t \) and \( x \). This differential equation is, of course, in general different from the differential equation for \( V(t, y) \) so the utility reserves are in general different. In the previous section we showed, however, that if one can choose \( S(t) = S - X_{t,x}(t) = S - x \), then the individual experiences the same payments no matter whether he saves in the bank or the insurance company. This is reflected also here, since with \( S(t) = S - x \) the two differential equations coincide. So, with the
same starting point \( x = y \), also the utility reserves coincide. It is of course no surprise that, since the policy holder has no other preferences over institutions, he is indifferent between saving in the two institutions provided that they bring him the same payments.

### 3.3 Introducing a disability annuity

We also generalize the concept of utility reserve to the disability model. Assuming that the income rate falls away during periods of disability, we introduce

\[
V^j (t, y) = \mathbb{E}_{t, j, y} \left[ \int_{t}^{m+n} dU (s) \right],
\]

\[
dU (t) = (u(t, S) \left(dN^{ad} (t) + dN^{id} (t)\right)) + u(t, a - \pi) I^a (t) dt + u(t, \beta) I^i (t) dt \mathbb{1}_{(0 < t \leq m)} + u(t, b) I (t) dt \mathbb{1}_{(m < t \leq m+n)}.
\]

We introduce \( w_i \) and \( w_i \) as weight functions corresponding to consumption as disabled and upon death from the disability state, respectively. Again, concentrating on the pre-retirement period, we can now represent the statewise utility reserves either by their partial differential equations and their solutions. We have the following differential equation for \( V^a \) with terminal condition,

\[
V^a_i (t, y) = -\frac{1}{\gamma} u^{1-\gamma} (t) \left((a - \pi)^\gamma - \mu (t) \left(\frac{1}{\gamma} u^{-\gamma} (t) S^\gamma - V^a (t, y)\right) - \sigma (t) (V^i (t, Y^i (t)) - V^a (t, y)) - V^a_j (t, y) \right) (ry + \pi \mu (t) (S - y) - \sigma^* (t) (Y^i (t) - y)),
\]

\[
V^a (m, y) = a^{w^a} (x+m:n) \frac{1}{\gamma} \left( \frac{y}{a^{x+m:n}} \right) \gamma.
\]

The differential equation for \( V^i \) is the same with \( a - \pi, \mu, \sigma, \pi, \mu^*, \sigma^* \), and top- and subscript \( a \) and \( i \) replaced by \( \beta, \nu, \rho, -\beta, \nu^*, \rho^* \), and top- and subscript \( i \) and \( a \), respectively. The (non-explicit) solution to these differential equations are

\[
\gamma V^j (t, y) = S^\gamma A_{x+t+m-t}^{w^j} + \left(a - \pi\right)^\gamma a_{x+t+m-t}^{w^j} + \beta^\gamma a_{x+t+m-t}^{w^j} + b(t, y, \pi, S)^\gamma m-t a_{x+t+m-n}^{w^j}.
\]

**Remark 2** If \( \rho = \rho^* = 0 \), all the formulas can be made explicit since we then can calculate \( V^i (t, y) \) explicitly and plug in our result in the differential equation for \( V^a (t, y) \) in order to obtain an explicit solution.

### 3.4 Beyond the disability model

In the general setup of a multistate Markov chain we define the utility reserve by

\[
V^j (t, y) = \mathbb{E}_{t, j, y} \left[ \int_{t}^{m+n} dU (t) \right],
\]

\[
dU (t) = \sum_j I^j (t) \left(u^j \left(t, (a + b)^{j} (t)\right) dt + \Delta U^j \left(t, \Delta (A + B)^j (t)\right)\right) + \sum_{j \neq k} u^j \left(t, (a + b)^{jk} (t)\right) dN^{jk} (t),
\]

where we, in general, measure utility of the (assumed-to-be) positive net payments coming from income plus net insurance benefits. The general labor income payment stream is modelled in the same way as the insurance payment process by

\[
dA (t) = \sum_j I^j (t) \left(a^j (t) dt + \Delta A^j (t)\right) + \sum_{j \neq k} a^{jk} (t) dN^{jk} (t),
\]
thereby allowing for rather general income patterns.

We introduce the weighted power utility functions

\[
  w^j \left( t, (a + b)^j (t) \right) = \frac{1}{\gamma} w^j (t)^{1 - \gamma} (a + b)^j (t)^{\gamma},
\]

\[
  \Delta U^j \left( t, \Delta (A + B)^j (t) \right) = \frac{1}{\gamma} \Delta W^j (t)^{1 - \gamma} \Delta (A + B)^j (t)^{\gamma},
\]

\[
  w^{jk} \left( t, (a + b)^{jk} (t) \right) = \frac{1}{\gamma} w^{jk} (t)^{1 - \gamma} (a + b)^{jk} (t)^{\gamma},
\]

where \( w^j (t) \), \( \Delta W^j (t) \), and \( w^{jk} (t) \) are weight functions corresponding to different types of payments. Although these are just weight coefficients, it is convenient to think of these as coming from a particular weight process which can be presented in the same form as the payment processes \( A \) and \( B \).

\[
dW(t) = \sum_j I^j (t) (w^j (t) dt + \Delta W^j (t)) + \sum_{j \neq k} w^{jk} (t) dN^{jk} (t).
\]

Finally, the differential equation for the utility reserve with general utility functions reads

\[
  V^j_t (t, y) = -w^j (t, a^j (t) - b^j (t)) - \sum_{k \neq j} w^{jk} (t, (a + b)^{jk} (t)) + V^k (t, Y^k (t) - V^j (t, y)) - V^j_y (t, y) \left( r_y - b^j (t) - \sum_{k \neq j} w^{jk} (t, b^{jk} (t) + Y^k (t) - Y^j (t)) \right),
\]

\[
  V^j (t-, y) = \Delta U^j \left( t, \Delta (A - B)^j (t) \right) + V^j (t, y - \Delta B^j (t)).
\]

4 Static Optimization

4.1 Saving in an insurance company

We now consider a static optimization problem. Optimization in the sense that the individual decides the premium level and/or the term insurance sum in order to maximize the expected value of future utility. Static in the sense that he decides at a fixed time point \( t \) a constant level of these payment coefficients over \( [t, m] \) and then 'sits on his hands', i.e. makes no changes to these hereafter. Thus we solve a static optimization problem for each \( t \). We can formalize the problem by

\[
  \sup E_{t,0,y} \left[ \int_t^{m+n} dU (s) \right],
\]

where the sup is taken over a constant \( \pi \) and/or \( S \). Recalling (8) we need to find the constant \( \pi \) and/or \( S \) to maximize

\[
  \gamma V (t, y) = S^\pi A_{x+t:m-t}^\pi + (a - \pi)^\gamma a_{x+t:m-t}^\pi + b (t, y, \pi, S)^\gamma a_{x+t:m-t}^\pi.
\]

Of particular interest is of course the decision of payments at time 0, i.e. maximization of \( V (0, y_0) \), but this is just a special case of the calculations below.

When calculating the first order conditions

\[
  \frac{\partial}{\partial \pi} V (t, y) \bigg|_{\pi^* (t)} = 0,
\]

\[
  \frac{\partial}{\partial S} V (t, y) \bigg|_{S^* (t)} = 0,
\]

13
we need the partial derivatives of $b(t, y, x, S)$ with respect to $x$ and $S$,

$$
\frac{\partial}{\partial x} b(t, y, x, S) = \frac{a_{x+y}^w}{a_{x+y}^w - a_{x+y}^x},
$$

$$
\frac{\partial}{\partial S} b(t, y, x, S) = -\frac{A_{x+y}^{1+x}}{m-ta_{x+y}^x}.
$$

We then get the first order conditions

$$
a - \pi^*(t) = b(t, y, \pi^*(t), S) \left( \frac{a_{x+y}^w}{a_{x+y}^w - a_{x+y}^x} \right)^\frac{1}{\theta}, \tag{13}
$$

$$
S^*(t) = b(t, y, S^*(t)) \left( \frac{A_{x+y}^{1+x}}{A_{x+y}^{1+y} - m-ta_{x+y}^x} \right)^\frac{1}{\theta}. \tag{14}
$$

The second order conditions securing that the solutions to these equations really form a global maximum, e.g. $\frac{\partial^2 V}{\partial x^2} < 0$, are easily seen to be fulfilled.

Equation (13). is a simple linear equation since $b$ is linear in $x$, that solves for an optimal $x$ given a death sum $S$. Equation (14) is a simple linear equation since $b$ is also linear in $S$, that solves for an optimal $S$ given a level premium $x$. If we want to solve for the optimal $x$ and $S$ simultaneously, we need to solve the two linear equations with the two unknowns,

$$
\left( \frac{\partial}{\partial x} V(t, y) \right)_{\pi^*(t), S^*(t)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
$$

This simply results in the two equations (13) and (14) with $x$ and $x^*(t)$ replaced by $x^{**}(t)$, and $S$ and $S^*(t)$ replaced by $S^{**}(t)$. We could of course solve these equations with respect to $x^{**}(t)$ and $S^{**}(t)$ but the resulting quantities are lengthy and add no immediate insight. However, by dividing the two equations we immediately get the following relation between the optimal $x$ and the optimal $S$,

$$
\frac{a - \pi^{**}(t)}{S^{**}(t)} = \left( \frac{A_{x+y}^{1+y}}{A_{x+y}^{1+y} - m-ta_{x+y}^x} \right)^\frac{1}{\theta}. \tag{15}
$$

The resulting controls can be used to calculate the optimal utility reserves $V(x(t), S(t, y), V(x(t), S^*(t, y), V(x^{**}(t), S^{**}(t), Y(t), y), and $V(x^{**}(t), S^{**}(t), Y(t), y)$. These can be compared in order to study e.g. utility loss coming from fixing the death sum at a given level. Often such a utility loss is translated into an indifference sum that the policy holder (with fixed death sum) would need at time $t$ to be indifferent between having the death sum fixed or not. Such an indifference sum $\theta(t)$ is the solution to the indifference equation,

$$
V(x(t), S(t, y + \theta(t)) = V(x^{**}(t), S^{**}(t), t, y). \tag{36}
$$

**Example 3** In Figure 3 we show the utility reserve at time 0 as a function of premium rate. We fix the death sum at 250 and all points in the graph correspond to combinations of premium and annuity rates that fulfill the equivalence principle. Clearly, all combinations are not equally good in an expected utility sense. The maximum point $(4.6; 228.763)$ gives the highest utility reserve at time 0 and is therefore the best static choice. This corresponds to the premium and annuity rate used in the illustrations above.
4.2 Saving in a bank

We can also search for the static optimal decision in the case where the saving takes place in a bank. Thus we seek for $\pi$ and/or $S$ so as to maximize the utility reserve given in (10).

We can continue in different ways. Recall that the insurance saving case is reconstructed by choosing $S(t) = S - X^{\pi,X}(t)$. If we do this we have parametrized a problem by $\pi$ and the constant $S$ from which the dynamic $S(t)$ is given. Given this choice of death sum we ask for the optimal static choices of $\pi$ and $S$. Since this situation is essentially the same as with saving in the insurance company, its optimal solution, i.e. $\pi^*, S^*, S^{**}$, and $S^{***}$ are the same no matter in which institution saving takes place, and so are the optimal utility reserves.

An alternative approach which is more interesting at least from a mathematical point of view, is to require that the death sum bought in the insurance company is constant, i.e. $S(t) = S$ for some $S$. Then we are in an essentially new situation because we are not allowed to copy the insurance saving situation. We cannot force the wealth upon death to be constant since it consists of a now constant death sum plus the non-constant savings in the bank. This is an important lesson to learn from the static formulation of the problem: When we put restrictions on the decision processes, like here the restriction that the payment coefficients are not allowed to be changed, this makes an essential difference between saving in different institutions. The institutional explanation is that saving in the bank separates two financial activities (buying insurance protection and saving) that are integrated when saving in an insurance company. Hereby we miss an opportunity to coordinate effectively these activities.

One can now look for solutions to the first order conditions

$$
\frac{\partial}{\partial \pi} V(t, x) \bigg|_{\pi^*(t)} = 0,
$$

$$
\frac{\partial}{\partial S} V(t, x) \bigg|_{S^*(t)} = 0,
$$

but one will realize that, in contrast to the case of insurance saving, this gives highly non-linear
and unattractive equations in $\pi$ and $S$. These equations can now be solved numerically one at a time, holding the other payment coefficient fixed or simultaneously solving for an optimal pair $(\pi^* (t), S^* (t))$. Again we can calculate and compare $V_{\pi^* (t), S^* (t)} (t, x)$, $V_{\pi (t), S^* (t)} (t, x)$, and $V_{\pi^* (t), S^* (t)} (t, x)$.

Even more interesting would it be to compare the two different optimal utility reserves in Subsections 4.1 and 4.2. Then one deals with questions about preferences concerning saving institutions. Our hypothesis, left to the reader to check, is that it is preferable to save in the insurance company, at least for reasonable choices of parameters. But what is the indifference e.g. as a function of risk aversion? We emphasize that these questions can only be asked under the awkward assumption that the policy holder cannot change his death sum although he continuously enters into a new infinitesimally short-termed contract.

4.3 Introducing a disability annuity

If we add a disability annuity we get an additional first order condition for the decision variable $\beta$. In general we can think of the policy holder deciding optimally on the set of future payment coefficients either given that he is active or given that he is disabled. Here we just present the formulas corresponding to the case where the policy holder is active at time $t$ and add superscripts accordingly. The first order conditions become

$$
\frac{\partial}{\partial \pi} V^a (t, y)|_{\pi^a (t)} = 0,
$$

$$
\frac{\partial}{\partial S} V^a (t, y)|_{S^a (t)} = 0,
$$

$$
\frac{\partial}{\partial \beta} V^a (t, y)|_{\beta^a (t)} = 0,
$$

where, in the utility reserve $V^a (0, y)$, we use the equivalence annuity benefit

$$
b^a (t, y, \pi, \beta, S) = \frac{y + \pi a^{s\pi a}_{x + t: m - t} - S A^{1\pi a}_{x + t: m - t} - \beta a^{s\pi a}_{x + t: m - t}}{m - t a^{s\pi a}_{x + t: n}}. \tag{16}
$$

Again, the second order conditions are easily verified. By means of the partial derivatives of $b^a (t, y, \pi, \beta, S)$ with respect to $\pi$, $\beta$, and $S$, we get the first order conditions,

$$
a - \pi^a (t) = b (t, y, \pi, \beta, S) \left( \frac{a^{s\pi a}_{x + t: m - t} - t a^{w\pi a}_{x + t: n}}{a^{w\pi a}_{x + t: m - t} - t a^{s\pi a}_{x + t: n}} \right) \frac{1}{\pi^a (t)},
$$

$$
\beta^a (t) = b (t, y, \pi, \beta, S) \left( \frac{a^{s\pi a}_{x + t: m - t} - t a^{w\pi a}_{x + t: n}}{a^{w\pi a}_{x + t: m - t} - t a^{s\pi a}_{x + t: n}} \right) \frac{1}{\pi^a (t)}, \tag{16}
$$

$$
S^a (t) = b (t, y, \pi, \beta, S) \left( \frac{A^{1\pi a}_{x + t: m - t} - t a^{w\pi a}_{x + t: n}}{A^{w\pi a}_{x + t: m - t} - t a^{s\pi a}_{x + t: n}} \right) \frac{1}{\pi^a (t)}.
$$

We easily see the systematic development from the case without disability. An important thing to notice is that we still have linear equation since $b (0, y, \pi, \beta, S)$ is an affine function of $\pi$, $\beta$, and $S$. These three linear equations can be solved separately, two at a time, or simultaneously, depending on the number of fixed payment coefficients. Comparisons between optimal utility reserves for different coefficients fixed and comparisons to the case without disability can be made.
4.4 Beyond the disability model

If we instead have a multistate Markov chain we simply generalize the formulas above. Considering the statewise utility reserve

\[ V_j^x(t, y) = \mathbb{E}_{t, j, y} \left[ \int_t^{m+n} dU(s) \right], \]

we are interested in maximizing this. The set of first order conditions depend on the number of free payment coefficients we can choose. The first order condition with respect to the benefit rate \( b^k \), the lump sum \( \Delta B^k \) and the lump sum \( b^{k^l} \), respectively, become

\[ \frac{\partial}{\partial b^k} V_j^x(t, y) \bigg|_{b^k} = 0, \]
\[ \frac{\partial}{\partial \Delta B^k} V_j^x(t, y) \bigg|_{\Delta B^k} = 0, \]
\[ \frac{\partial}{\partial b^{k^l}} V_j^x(t, y) \bigg|_{b^{k^l}} = 0. \]

Again, equations of these types can be solved separately or simultaneously depending on decision parameters. In concrete cases one can produce simple relations similar to (16).

5 Dynamic Optimization

5.1 Saving in an insurance company

We now consider a dynamic optimization problem. Optimization in the sense that the individual decides the premium level and/or the term insurance sum in order to maximize the expected value of future utility. Dynamic in the sense that he decides at any given point in the state space \((t, y)\) on which level of payment coefficients due in the next infinitesimally small time interval. We formalize the problem by introducing the optimal utility reserve

\[ V(t, y) = \sup_{\pi, S} \mathbb{E}_{t, 0, y} \left[ \int_t^{m+n} dU(s) \right], \]

where the supremum is taken over adapted processes \( \pi \) and \( S \) from time \( t \) to time \( m \). This means that the dynamic optimization are concentrated in the saving period. Once the annuity is bought at time \( m \) for the reserve available at that time, the policy holder receives this constant annuity benefit until time \( m + n \) or until he dies.

It is the exciting result of the so-called dynamic programming principle that the solution to this problem can be characterized by the solution to a certain system of differential equations embedded in a so-called Hamilton-Jacobi-Bellman (HJB) equation. Recall the differential equation for the utility reserve (9). The system of differential equations characterizing the solution to the dynamic optimization problem can be written by taking infimum over the control variables on the right hand side of (9). This gives what we could call a pointwise optimization in the deterministic differential equation in contrast to the previous section where we worked with a global optimization. The HJB equation with terminal condition becomes
This results in a set of first order conditions in terms of \((\text{the partial derivative of}) \) the optimal utility reserve
\[
V_i(t, y) = \inf_{\pi, S} \left[ -\frac{1}{\gamma} w_{a}^{\frac{1}{\gamma}}(t) (a - \pi)^{\gamma} - \mu(t) \left( \frac{1}{\gamma} w_{ad}^{\frac{1}{\gamma}}(t) S^{\gamma} - V(t, y) \right) \right. \\
\left. - V_y(t, y) (ry + \pi - \mu^*(t) (S - y)) \right],
\]
\[
V(m, y) = a_{x+m;n}^{w} \frac{1}{\gamma} \left( \frac{y}{a_x^{\gamma} m} \right)^{\gamma'},
\]
(17)

From there, one proceeds by calculating the \(\pi\) and \(S\) that minimizes the square bracket contents. This results in a set of first order conditions in terms of (the partial derivative of) the optimal utility reserve
\[
a - \pi = w_{a}(t) V_y^{\frac{1}{\gamma}}(t, y),
\]
\[
S = w_{ad}(t) h(t) V_y^{\frac{1}{\gamma}}(t, y),
\]
where
\[
h(t) = \left( \frac{\mu(t)}{\mu^*(t)} \right)^{\frac{1}{\gamma}}.
\]

The second order conditions are easily verified. Plugging the first order conditions on \(\pi\) and \(S\) into the HJB equation gives the differential equation,
\[
V_i(t, y) = -\frac{1}{\gamma} w_{a}(t) V_y^{\frac{1}{\gamma}}(t, y) - \mu(t) \left( \frac{1}{\gamma} w_{ad}(t) h^{\gamma}(t) V_y^{\frac{1}{\gamma}}(t, y) - V(t, y) \right)
\]
\[
- V_y(t, y) \left( \frac{1}{\gamma} w_{ad}(t) h^{\gamma}(t) V_y^{\frac{1}{\gamma}}(t, y) - V(t, y) \right) (r + \mu^*(t)) y + a - (w_{a}(t) - \mu^*(t) w_{ad}(t) h(t)) V_y^{\frac{1}{\gamma}}(t, y).
\]
(19)

We should now try to find a solution to this differential equation and assume the guess (including its partial derivatives)
\[
V(t, y) = \frac{1}{\gamma} f(t) \left( \frac{y + g(t)}{f(t)} \right)^{\gamma},
\]
\[
V_i(t, y) = \frac{1}{\gamma} f_i(t) \left( \frac{y + g(t)}{f(t)} \right)^{\gamma} + \left( \frac{y + g(t)}{f(t)} \right)^{\gamma - 1} g_i(t),
\]
\[
V_y(t, y) = \left( \frac{y + g(t)}{f(t)} \right)^{\gamma - 1}.
\]
(20)

Plugging in this guess in the differential equation yields (after some rearrangements)
\[
g_i(t) \left( \frac{y + g(t)}{f(t)} \right)^{\gamma - 1} + \frac{1 - \gamma}{\gamma} f_i(t) \left( \frac{y + g(t)}{f(t)} \right)^{\gamma}
\]
\[
= \left( g(t) (r + \mu^*) - a \right) \left( \frac{y + g(t)}{f(t)} \right)^{\gamma - 1}
\]
\[
+ \left( \mu(t) \frac{1}{\gamma} f(t) + \frac{1}{\gamma} w_{a}(t) + \frac{1}{\gamma} w_{ad}(t) \tilde{\mu}(t) - (r + \mu^*) f(t) \right) \left( \frac{y + g(t)}{f(t)} \right)^{\gamma},
\]

where
\[
\tilde{\mu}(t) = \mu(t) h \gamma(t) = \mu^*(t) h(t) = \mu(t)^\frac{1}{\gamma'} \mu^*(t)^\frac{1}{\gamma'}.
\]
Gathering terms with and dividing by \( \left( \frac{y + g(t)}{f(t)} \right)^{\gamma - 1} \) and \( \frac{1 - \gamma}{\gamma} \left( \frac{y + g(t)}{f(t)} \right)^{\gamma} \), respectively, gives differential equations for \( f \) and \( g \), respectively,

\[
\begin{align*}
  f_t (t) &= \tilde{r} (t) f (t) - w_a (t) - \tilde{\mu} (t) (w_{ad} (t) - f (t)), \\
  f (m) &= \left( a_{x+m:n}^{w'} \right) \tilde{\nu} \left( a_{x+m:n}^* \right) \tilde{\nu}, \\
  g_t (t) &= g (t) (r + \mu^* (t)) - a, \\
  g (m) &= 0,
\end{align*}
\]

where

\[
\tilde{r} = \frac{1}{1 - \gamma} \mu (t) - \frac{\gamma}{1 - \gamma} (r + \mu^*) - \tilde{\mu} (t).
\]

The functions \( f \) and \( g \) have some very nice interpretations. The differential equation for \( f \) has the solution

\[
f (t) = a_{x+t;m-t}^{w} + A_{x+t;m-t}^{w} + m-t \tilde{E}_{x+t} \left( a_{x+m:n}^{w'} \right) \tilde{\nu} \left( a_{x+m:n}^* \right) \tilde{\nu}, \quad (21)
\]

and is thus an artificial expected present value of the future utility weights. Artificial in the sense that the artificial discounting and mortality rates \( \tilde{r} \) and \( \tilde{\mu} \) are determined by the risk aversion and model parameters. The differential equation for \( g \) has the solution

\[
g (t) = a a_{x+t;m-t}^*,
\]

and is thus the financial value of the future earnings, also called the human capital.

We can now find the optimal controls by plugging the guess (20) into (18). We get the controls,

\[
\begin{align*}
  a - \pi (t, y) &= \frac{w_a (t)}{f (t)} (y + g (t)), \\
  S (t, y) &= \frac{w_{ad} (t)}{f (t)} h (t) (y + g (t)).
\end{align*}
\]

We see the that optimal consumption rate \( a - \pi \) is a fraction \( w_a / f \) of the total wealth consisting of the reserve plus the human capital. The fraction is some kind of an impatience factor measuring how important it is to consume now, represented by \( w_a \), compared to how important it is to consume later, represented by \( f \).

The optimal insurance sum \( S \) serves as a protection of wealth. The wealth that should be protected consists again of reserve plus human capital. This wealth is again multiplied by an impatience factor \( w_{ad} / f \) measuring how important it is (for the inheritors) to consume upon death, represented by \( w_{ad} \), compared to how important it is to consume later, represented by \( f \). To this impatience factor is multiplied a price factor \( h \) that decreases/increases the protection depending on how expensive the protection is. The risk aversion also appears in the price factor.

The solution above corresponds to the situation where all control stops at retirement. One could also formulate a problem where the control continues such that the annuitant decides in the annuity payment period continuously the annuity level. Then one would replace the terminal condition (17) with a pasting condition, add a piece from \( m \) to \( m + n \) to the HJB equation, and add a new terminal condition at time \( m + n \)

\[
\begin{align*}
  V (m-, y) &= V (m, y), \\
  V_t (t, y) &= \inf_b \left[ -\frac{1}{\gamma} w_a^{1-\gamma} (t) b^\gamma + \mu (t) V (t, y) - V_y (t, y) ((r + \mu^* (t)) y - b) \right], \\
  V (m + n-, y) &= 0.
\end{align*}
\]
We then solve the HJB equation piece by piece starting with the interval from $m$ to $m+n$. The optimal utility reserve and the optimal annuity benefit become

$$V(t, y) = \frac{1}{\gamma} f(t) \left( \frac{y}{f(t)} \right)^{\gamma},$$

$$b(t, y) = w_a(t) \left( V_g(t, y) \right)^{\frac{1}{\gamma}} = \frac{w_a(t)}{f(t)} y,$$

with

$$f(t) = \bar{a}_x^{w} + A^{1w}_{x+m:n} + m-tE_x a_{x+m:n}.$$

This function evaluated at time $m$, $f(m) = \bar{a}_x^{w} + A^{1w}_{x+m:n}$, and $g(m) = 0$, now form, by use of the pasting condition, the terminal condition for the next piece from 0 to $m$. Apart from the terminal condition for $f$, there are no changes to the HJB equation from 0 to $m$. The solution to the differential equation for $f$ becomes the even simpler (compared to (21)),

$$f(t) = \bar{a}_x^{w} + A^{1w}_{x+m:n} + m-tE_x a_{x+m:n}.$$

**Example 4** We have solved both cases where the control ends at retirement combined with a fixed annuity level and where the control continues until termination, respectively. We expect that the utility reserve at time 0 is higher in the partly dynamic case compared with the static case and even higher in the fully dynamic case. Indeed, the utility reserve at time 0 is exponentially decreasing, see Kraft and Steffensen (2008). For the case (and $\mu = \mu^*$ as in our case), all controls are the same namely corresponding to the fair (premium, annuity) pair (4.9; 45.1) giving a consumption rate of 45.1 both in the saving period ($50 - 4.9 = 45.1$) and in the retirement period.

### 5.2 Saving in a bank

We can also search for the dynamic optimal decision in the case where the saving takes place in a bank. We formalize the problem by introducing the optimal utility reserve

$$V(t, x) = \sup E_{t,a,x} \left[ \int_t^{m+n} dU(s) \right],$$

where again the supremum is taken over adapted processes $\pi$ and $S$ from $t$ to $m$. We already have an idea about what the solution should be. As a dynamic strategy we can always choose $S(t) = S - X^{t,x}(t)$ and thereby reproduce the construction with saving in the insurance company. So the optimal choice must be to consume as above and buy insurance protection corresponding to the insurance sum $S(t) = S^*(t) - X^{t,x}(t)$, where $S^*(t)$ is the optimal insurance sum under saving in the insurance company derived in Subsection 5.1. Anyway, to demonstrate the methodology we
perform roughly the calculations that verifies this intuition. The results are in principle identical
to the results obtained by Richard (1975).

As in the previous section the solution to the problem can be characterized by the solution to
an HJB equation. This system comes from taking infimum on the right hand side of (11), i.e. we
get the HJB equation,

$$
V_t(t, x) = \inf_{\pi, S} \left[ -\frac{1}{\gamma} w_a^{1-\gamma} (t) (a - \pi)^\gamma - \mu (t) \left( \frac{1}{\gamma} w_{ad}^{1-\gamma} (t) (S (t) + x)^\gamma - V(t, x) \right) \right. \\
- V_x (t, x) \left( (r x + \pi - \mu^* S (t)) \right),
$$

$$
V(m, x) = a_{x+m:n}^{\nu} \frac{1}{\gamma} \left( \frac{x}{a_{x+m:n}} \right)^\gamma.
$$

From there we proceed by calculating the $\pi$ and $S$ that minimizes the square bracket contents.
This results in the first order conditions

$$
a - \pi(t, x) = w_a(t) V_x^{-\gamma} (t, x),
$$

$$
S(t, x) + x = w_{ad}(t) h(t) V_x^{-\gamma} (t, x).
$$

Plugging these first order conditions on $\pi$ and $S$ into the HJB equation gives the differential
equation,

$$
V_t(t, x) = -\frac{1}{\gamma} w_a (t) V_x^{\gamma - \gamma} (t, x) - \mu (t) \left( \frac{1}{\gamma} w_{ad} (t) h^{\gamma} (t) V_x^{\gamma - \gamma} (t, x) - V(t, x) \right) \\
- V_x (t, x) \left( (r + \mu^* (t)) x + a - (w_a (t) - \mu^* (t) w_{ad} (t) h (t)) V_x^{\gamma - \gamma} (t, x) \right).
$$

But this is exactly the same differential equation as we found in (19) and therefore their solutions
must be the same since also the terminal conditions coincide. We conclude from repeating the
arguments in the previous section that the optimal utility reserve equals

\[ V(t, x) = \frac{1}{\gamma} f(t) \left( \frac{x + g(t)}{f(t)} \right)^\gamma, \]

with \( f \) and \( g \) defined as above. The optimal controls are given by

\[ a - \pi(t, x) = \frac{w_a(t)}{f(t)} (x + g(t)), \]
\[ S(t, x) + x = \frac{w_{ad}(t)}{f(t)} h(t) (x + g(t)), \]

and the interpretation from Section 5.1 applies. Only one thing has to be added: After having found the optimal protection of wealth upon death \( h(t) (x + g(t)) \) \( w_{ad}(t) / f(t) \) one should subtract the wealth that is already there, namely \( x \). Upon death this is just the cash balance of the bank account and this is already carried over to the inheritors upon death. Thus, one should buy insurance to protect wealth net of the bank account cash balance.

### 5.3 Introducing a disability annuity

We generalize the optimization problem to the disability model. We now define the statewise optimal utility reserve as

\[ V^j(t, y) = \sup_{E_{t,j,y}} \left[ \int_t^{m+n} dU(s) \right], \]

where we now need briefly to discuss over which quantities the supremum is taken. It turns out convenient to work with the reserve jump as a control variable. Being in state \( a \), the reserve jump from \( a \) to \( i \) is taken to be a control variable and we denote this reserve jump \( Y^i(t) - Y^a(t) \) as a control variable by \( y_{ai} \). Being in state \( i \), the reserve jump from \( i \) to \( a \) is taken to be a control variable and we denote this reserve jump \( Y^i(t) - Y^a(t) \) as a control variable by \( y_{ia} \). In practice, of course, the individual does not choose these reserve jumps directly. This is something that the insurance company calculates on the bases of all payments from all states in the future that has been agreed upon. But this just means that for a given optimal reserve jump, the individual has to set all payments such that this reserve jump is realized upon transition. We specify this action in the concrete case below.

As usually, we can characterize the statewise optimal utility reserve by the HJB equation, here presented for \( V^a \),

\[ V^a_i(t, y) = \inf_{\pi, S_{ad}, y_{ai}} \left[ -\frac{1}{\gamma} w_1^{1-\gamma} (t) (a - \pi)^\gamma - \mu \left( \frac{1}{\gamma} w_1^{1-\gamma} (t) S_\gamma^{ad} - V^a(t, y) \right) \right. \]
\[ -\sigma \left( V^i(t, y + y_{ai}) - V^a(t, y) \right) \]
\[ \left. -V^a_y(t, y) (y \pi - \mu^* (S_{ad} - y) - \sigma^* y_{ai}) \right], \]
\[ V^a(m, y) = w_{ia}^a \left( \frac{y}{a_{x+m:n}^a} \right)^\gamma. \]

The differential equation for \( V^i \) is the same with \( a - \pi, \mu, \pi, \mu^*, \sigma^* \), and top- and subscript \( a \) and \( i \) replaced by \( \beta, \nu, \rho, -\beta, \nu^*, \rho^* \), and top- and subscript \( i \) and \( a \), respectively. Furthermore, the infimum is take over \((\beta, S_{id}, y_{ia})\) instead of \((\pi, S_{ad}, y_{ai})\). The optimal controls are first presented
in terms of the optimal utility reserve

\[
\begin{align*}
    a - \pi(t, y) &= w_a(t) \left( V_{y}^a(t, y) \right)^{\frac{1}{\gamma}}, \\
    S^a(t, y) &= w_{ad}(t) h_{ad}(t) \left( V_{y}^a(t, y) \right)^{\frac{1}{\gamma}}, \\
    S^i(t, y) &= w_{id}(t) h_{id}(t) \left( V_{y}^i(t, y) \right)^{\frac{1}{\gamma}}, \\
    \beta(t, y) &= w_i(t) \left( V_{y}^i(t, y) \right)^{\frac{1}{\gamma}}, \\
    V_{y}^i(t, y + g^{ia}(t, y)) &= \frac{\sigma}{\sigma} V_{y}^a(t, y), \\
    V_{y}^a(t, y + g^{ia}(t, y)) &= \frac{\rho}{\rho} V_{y}^i(t, y).
\end{align*}
\]

Following the procedure from Subsection 5.1, we plug in these controls and the guess

\[
V^j(t, y) = \frac{1}{\gamma} f^j(t) \left( \frac{y + g^j(t)}{f^j(t)} \right)^{\gamma}.
\]

Gathering terms in the resulting partial differential equation, as in the survival model just taking a bit more time, leads to differential equations for \(f^j\) and \(g^j\) verifying that the guess is correct. These differential equations for \(f\) and \(g\) have the following representations

\[
\begin{align*}
    f^j(t) &= \bar{a}_{x+t}^{uj} A_{x+t}^{uj} + \bar{c}_{x+t}^{ia} \left( a_{x+m:n}^{u} \right)^{\frac{1}{\gamma}} \left( a_{x+m:n}^{a} \right)^{-\frac{1}{\gamma}} + m^{-t} \bar{f}_{x+t}^{ja} \left( a_{x+m:n}^{u} \right)^{\frac{1}{\gamma}} \left( a_{x+m:n}^{a} \right)^{-\frac{1}{\gamma}}, \\
    g^j(t) &= a_{x+t}^{ja} \bar{a}_{x+t}^{uj}.
\end{align*}
\]

The optimal controls are then

\[
\begin{align*}
    a - \pi(t, y) &= \frac{w_a(t)}{f^a(t)} (y + g^a(t)), \\
    S_{ad}(t, y) &= \frac{w_{ad}(t)}{f^a(t)} h_{ad}(t) (y + g^a(t)), \\
    S_{id}(t, y) &= \frac{w_{id}(t)}{f^i(t)} h_{id}(t) (y + g^i(t)), \\
    \beta(t, y) &= \frac{w_i(t)}{f^i(t)} (y + g^i(t)), \\
    g^{aj}(t, y) &= \frac{f^i(t)}{f^a(t)} h^{aj}(t) (y + g^a(t)) - (y + g^i(t)), \\
    g^{ia}(t, y) &= \frac{f^a(t)}{f^i(t)} h^{ia}(t) (y + g^i(t)) - (y + g^a(t)).
\end{align*}
\]

These controls have very nice interpretations. Let us focus on the optimal reserve jump from active to disabled, \(g^{aj}\). This reserve jump serves as a protection of wealth in connection with disability. The wealth that should be protected consists of reserve plus human capital, \(y + g^a\). This wealth is multiplied by an impatience factor \(f^i/f^a\) measuring how important it is to consume in the future as disabled, represented by \(f^i\), compared to how important it is to consume in the future as active, represented by \(f^a\). To this impatience factor is multiplied a price factor \(h^{aj}\) that decreases/increases the protection depending on how expensive the protection is. The risk aversion also appears in the price factor. However, from this optimal protection is finally subtracted the
wealth already owned upon transition. This wealth is the reserve plus the human capital in the disability state, \( y + g^i \). The result is the optimal reserve jump which should be demanded.

This demand is produced by agreeing upon a specific disability annuity rate. We see that the optimal reserve jump is obtained by demanding the disability annuity given by the equivalence relation

\[
y + y^{01} (t, y) = Y^i (t)
\]

This solution corresponds to the case where all control stops at retirement. If the annuity benefit control continues, then we get a solution similar to the one in Section 5.1. We add a pasting condition, a piece of a HJB equation and a terminal condition at time \( m+n \). The optimal utility reserve and the optimal annuity benefit become

\[
V^j (t, y) = \frac{1}{\gamma} f^j (t) \left( \frac{y}{f^j (t)} \right) ^\gamma, \\
b^j (t, y) = w^j (t) \left( \frac{V^j (y)}{y} (t, y) \right) ^{\gamma-1} = \frac{w^j (t)}{f^j (t)} y,
\]

with

\[
f^j (t) = \tilde{a}^w_{x+m:t:n}.
\]

All other results in this subsection hold true except for that the new terminal condition on \( f \) from 0 to \( m \), \( f^j (m) = \tilde{a}^w_{x+m:n} \), leads to the even simpler (compared to (22)),

\[
f^j (t) = \tilde{a}^w_{x+m:t-n} + \tilde{A}^1_{x+m:t-n} + m-t \tilde{a}^w_{x+t:n}.
\]

### 5.4 Beyond the disability model

In the general setup of a multistate Markov chain we introduce the optimal utility reserve

\[
V^j (t, y) = \sup_{E_{t,j,y}} \left[ \int_t^{m+n} dU (t) \right],
\]

where, again we need to discuss over what the supremum is taken. In general we take all elements in the insurance payment process as decision processes and this is then the processes over which the supremum is taken. In the HJB equation, however, this set of decision processes reduce to the payment coefficients for the state space point \( (t, j) \) and then all the reserve jumps from there. In this general subsection we disregard the special consideration of the retirement period from \( m \) to \( m+n \) where the policy holder may or may not be able to control the annuity benefit. We solve here the problem for the case of full control at all times. Similarly to the HJB equations for the survival and disability models, we form the HJB equation for the general model by embracing the left hand side of (12) by an infimum. This infimum is taken over the control processes given that the policy holder is in state \( j \), i.e. \( (b^i, b^{ik}, y^{ik}) \).

One can the perform exactly the same calculations as presented in Section 5.3. Although they become considerably more messy, the result is impressively simple. The optimal utility reserve can again be written in the form

\[
V^j (t, y) = \frac{1}{\gamma} f^j (t) \left( \frac{y + g^i (t)}{f^j (t)} \right) ^\gamma.
\]
The optimal controls are given by

\[ a^j + b^j (t, y) = \frac{w_j (t)}{f^j (t)} (y + g^j (t)), \]
\[ S^{jk} (t, y) = \frac{w_{jk} (t)}{f^j (t)} h_{jk} (t) (y + g^j (t)), \]
\[ y^{jk} (t, y) = \frac{f^k (t)}{f^j (t)} h_{jk} (t) (y + g^j (t)) - (a^{ik} (t) + y + g^k (t)). \]

Here \( f \) and \( g \) are just generalizations of the similar quantities formalized and interpreted before. The artificial expected present value of future utility weights \( f \) and the human capital \( g \) are in this general problem and by use of the income payment process \( A \) and the weight process \( W \) given by

\[ g^j (t) = E^* \left[ \int_t^{m+n} e^{-\int_s^t r \, ds} dA (s) \right], \]
\[ f^j (t) = \tilde{E} \left[ \int_t^{m+n} e^{\int_s^t \tilde{r} \, ds} dW (s) \right]. \]

**Remark 3** One may get the demoniac thought that the optimal static control which is a function of \( t \) (but constant over \([t, m]\)) and the optimal dynamic control which is also a function of \( t \) coincide. However this based on a completely wrong perception of ideas. Under static optimization the individual chooses a control from time \( t \) to time \( m \) under the requirement that he is not allowed to change his mind. The fact that we may allow him to anyway does not affect this. Under dynamic optimization the individual knows that he may adapt his choice to whatever information he may get along the way.

### 6 Further Remarks

#### 6.1 Investment

So far we have completely disregarded the asset allocation problem of the individual. Both in the pension institution and in the bank his financial wealth earns the interest rate \( r \). One could easily generalize this to a situation where a proportion of the financial wealth is invested in a risky asset modelled by a geometric Brownian motion. We could even have a multidimensional model with several risky assets. We would then introduce the proportions of wealth invested in different risky assets as decision processes.

The result is the following: It is possible to form a fund of risky assets such that the optimal mutual proportional distribution of risky capital in this fund is constant. Thus, only the proportion allocated to this fund is time and wealth dependent. Denote by \( \alpha \) and \( \sigma \) the drift and the volatility of this fund. Then the optimal proportion of financial wealth for a given state \( j \) and a given financial wealth \( y \) at time \( t \) invested in this fund is given by the relation

\[ \pi^j (t, y) = \frac{1}{1 - \gamma} \frac{\alpha - r \, y + g^j (t)}{\gamma \sigma^2 \, y}. \]

This is just a generalized version of the common knowledge in the first bullet in the introduction. The amount put in stocks \( \pi^j (t, y) \) should be a constant proportion \( \frac{1}{1 - \gamma} \frac{\alpha - r \, y + g^j (t)}{\gamma \sigma^2 \, y} \) of the total wealth consisting of financial wealth \( y \) and human capital \( g^j (t) \). This result encourages the life cycle investment advise about shifting the portfolio holdings from stocks to bonds as one grows old.
It is more or less clear how this optimal investment should be implemented institutionally if saving takes place in the bank (in that case just replace $y$ by $x$ in the proportion above.) It is less clear how this portfolio decision should be implemented if saving takes place in a pension institution. In a lot of saving constructions like with-profit life insurance and pension funding, the asset allocation is made at the discretion of the life insurance company or the fund manager, respectively. However, unit-link insurance is an exception where the asset allocation is decided by the policy holder and is therefore the type of product one should think of in this connection. Alternatively, the institution could sell so-called life cycle products where the allocation is, indeed, managed automatically by the institution but in agreement with the policy holder and in accordance with some given profile adapted to his life course.

6.2 Constraints

Depending on the legislative and institutional environment it may be natural to work with constraint on controls and/or controlled processes, i.e. financial wealth. If the pension institution works with a minimum death sum $S_{\text{min}}$, say, the optimization problems should be formulated such that $S$ should be chosen among the admissible controls obeying such a constraint. If the pension institution works with a minimum reserve, say, this has to be incorporated in the set of admissible controls. It is very easy to come up with an example of such a constraint since it is typically assumed that reserves are non-negative. This separates the pension saving business from the loaning business of the individual with the convenient consequence for the pension institution that it never has to worry about the creditworthiness of the policy holder (the policy holder, in turn, has to worry about the creditworthiness of the pension institution, but this is a completely different story).

While it may be more or less foreseeable what happens to the static optimization problem in presence of such constraints, they typically cause greater difficulties in case of dynamic optimization. Constraints on the controls may be tractable, though hard, but constraints on controlled processes may cause insuperable difficulties. Typically, one resorts to different methods, e.g. like the so-called martingale method. See Nielsen and Steffensen (2008) for the martingale method applied to an optimization problem with constraints on the life insurance sum.

We believe that generalizations of the results in this article and Kraft and Steffensen (2008) in the direction of constraints are important. The results will for sure lose on beauty but probably gain applicability.

6.3 Design

If the optimal dynamic control is really dynamic and the optimal utility reserve is larger in dynamic optimization problems than in static ones, then our model advises the policy holders to change the insurance payment coefficients continuously, which in practice means ’frequently’. This may be prevented by institutional constraints or simply be seen as too time consuming for both the policy holder and the institution to be practically implementable. But this creates a demand for designing the payment processes of the insurance contracts such that they contain inherent optimal profiles. It must be beneficial to all involved parties to design the payment profiles optimally such that the number of intermediary transactions is kept to a minimum.

Kraft and Steffensen (2008) derive deterministic differential equations for the optimal controls given that the policy is in a given state. These dynamics are of course heavily dependent on the parameters of the model and the utility function. But their pattern shows some very nice features. For appropriate choices of the weight functions, the optimal controls turn out to be exponentially
increasing or decreasing. The rate of change is determined by the parameters in general and the risk aversion parameter in particular. For the portfolio decision there is a trend from unit-link insurance products where the policy holder himself has to adjust his portfolio, towards life cycle products where the portfolio selection is individual but a part of the contract. There is no reason why the institution should not offer individual payment profiles as well as a part of the contract. E.g. for a given risk aversion, the pension institution implements a given life cycle investment profile and a given profile of an exponentially decreasing death sum or exponentially increasing life annuity benefits or whatever comes out of the formulas.

We believe that individual advice on payment profiles can be substantially improved by the use of the patterns of thinking presented in this article. Such advice is often given on the basis of the policy holder’s life course but without taking the policy holder personal attitudes towards risk into account. With the formulas presented, improvements in this direction are implementable.

References


7 Appendix

Throughout the article we use the notation that we present here in the appendix. This is partly the same as or inspired from traditional actuarial notation. For abbreviation of classical reserves in Section 2, one benefits from formulas with transition probabilities calculated with $\mu^*$ and discounting.

In a general multistate model we use the following notation,

\[ a_{x+t:m-t}^{*jk} = \int_{t}^{m} e^{-\int_{t}^{s} \mu^* (t,s) ds} r p^*_{jk} (t,s) ds, \]

\[ m-t a_{x+t:n}^{*jk} = \int_{m}^{m+n} e^{-\int_{m}^{s} \mu^* (t,s) dt} r p^*_{jk} (t,s) dt, \]

\[ A_{x+t:m-t}^{1*jk} = \int_{t}^{m} e^{-\int_{t}^{s} \mu^* (t,s) \mu^*_{kd} (s) ds} r p^*_{jk} (t,s) ds. \]

In the disability model we abbreviate this general notation to \((j \in \{a,i\})\),

\[ a_{x+t:m-t}^{*j} = a_{x+t;m-t}^{*ja} + a_{x+t:m-t}^{*ji}, \]

\[ m-t a_{x+t:n}^{*j} = m-t a_{x+t;n}^{*ja} + m-t a_{x+t;n}^{*ji}, \]

\[ A_{x+t:m-t}^{1*j} = A_{x+t;m-t}^{1*ja} + A_{x+t;m-t}^{1*ji}. \]

In the survival model we abbreviate even further into

\[ a_{x+t:m-t}^{*} = a_{x+t;m-t}^{*aa}, \]

\[ m-t a_{x+t:n}^{*} = m-t a_{x+t;n}^{*aa}, \]

\[ A_{x+t:m-t}^{1*} = A_{x+t;m-t}^{1*aa}. \]

For abbreviation in the static optimization problem in Section 4, one benefits from formulas with transition probabilities calculated with $\mu$, without discounting but with the utility weights (taken to the power of $\gamma - 1$ and therefore decorated with a prime),

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\[ a_{x:t+m-t}^{w'j} = \int_t^m p_{jk} (t, s) w_j^{1-\gamma} (s) \, ds, \]
\[ m-t a_{x:t+n}^{w'j} = \int_m^{m+n} p_{jk} (t, s) w_j^{1-\gamma} (s) \, dt, \]
\[ A_{x:t+m-t}^{w'j} = \int_0^m p_{jk} (t, s) \mu_{kd} (s) w_k^{1-\gamma} (s) \, ds, \]
\[ a_{x:t+m-t}^{w'j} = a_{x:t+m-t}^{w'ja} + a_{x:t+m-t}^{w'ji}, \]
\[ m-t a_{x:t+n}^{w'j} = m-t a_{x:t+n}^{w'ja} + m-t a_{x:t+n}^{w'ji}, \]
\[ A_{x:t+m-t}^{w'j} = A_{x:t+m-t}^{w'ja} + A_{x:t+m-t}^{w'ji}, \]
\[ a_{x:t+m-t}^{w'} = a_{x:t+m-t}^{w'aa}, \]
\[ m-t a_{x:t+n}^{w'} = m-t a_{x:t+n}^{w'aa}, \]
\[ A_{x:t+m-t}^{w'} = A_{x:t+m-t}^{w'aa}. \]

For abbreviation in the dynamic optimization problem in Section 5, one benefits from formulas with transition probabilities calculated with \( \bar{\mu} \), discounting with \( \bar{r} \), and with the pure utility weights,

\[ \bar{a}_{x+t:m-t}^{wjk} = \int_t^m e^{-\int_t^s \bar{r} \, ds} \bar{p}_{jk} (t, s) w_k (s) \, ds, \]
\[ m-t \bar{a}_{x+t:n}^{wjk} = \int_m^{m+n} e^{-\int_t^s \bar{r} \, ds} \bar{p}_{jk} (t, s) w_k (s) \, ds, \]
\[ \bar{A}_{x+t:m-t}^{wjk} = \int_0^m e^{-\int_t^s \bar{r} \, ds} \bar{p}_{jk} (t, s) \bar{\mu}_{kd} (s) w_k (s) \, ds, \]
\[ \bar{a}_{x+t:m-t}^{wj} = \bar{a}_{x+t:m-t}^{wja} + \bar{a}_{x+t:m-t}^{wji}, \]
\[ m-t \bar{a}_{x+t:n}^{wj} = m-t \bar{a}_{x+t:n}^{wja} + m-t \bar{a}_{x+t:n}^{wji}, \]
\[ \bar{A}_{x+t:m-t}^{wj} = \bar{A}_{x+t:m-t}^{wja} + \bar{A}_{x+t:m-t}^{wji}, \]
\[ \bar{a}_{x+t:m-t}^{w} = \bar{a}_{x+t:m-t}^{waa}, \]
\[ m-t \bar{a}_{x+t:n}^{w} = m-t \bar{a}_{x+t:n}^{waa}, \]
\[ \bar{A}_{x+t:m-t}^{w} = \bar{A}_{x+t:m-t}^{waa}. \]