1 Introduction

The mathematics of finance and the mathematics of life insurance were always intersecting.

Life insurance contracts specify an exchange of streams of payments between the insurance company and the contract holder. These payment streams may cover the life time of the contract holder. Therefore, time valuation of money is crucial for any measurement of payments due in the past as well as in the future. Life insurance companies never put their money under the pillow, and accumulation and distribution of capital gains were always part of the insurance business. With respect to the future, appropriate discounting of contractual obligations qualifies the estimates of liabilities.

Financial contracts specify an exchange of streams of payments as well. However, while the life insurance payment stream is partly linked to the state of health of the insured, the financial payment stream is linked to the 'state of health' of an enterprise. That could be the stream of dividends distributed to the owners of the enterprise or the stream of claims contingent on the price of the enterprise paid to the holder of a so-called derivative. The discipline of personal finance is particularly closely linked to life insurance. Decisions on e.g. consumption, investment, retirement, and insurance coverage belong to some of the most substantial life time financial decisions of an individual.

Valuation of payment streams is probably the most important discipline in the intersection between finance and life insurance. Various valuation dogmas are in play here. The principle of no arbitrage and the market efficiency assumption are taken as given in the majority of modern academic approaches to valuation of financial contracts. Life insurance contract valuation typically relies on independence, or at least asymptotic independence, between insured lives. Then the law of large numbers ensures that reasonable estimates can be found if the portfolio of insurance contracts is sufficiently large. Both dogmas reduce the valuation problem to being primarily a matter of calculation of conditional expected values.

Conditional expected values can be approached by several different techniques. E.g. Monte Carlo simulation exploits that conditional expected values can be approximated by empirical means. Sometimes, however, one can go at least part of the way by explicit calculations. E.g. if a series of auxiliary models with explicit expected values converges towards the real model in such a way that the series of explicit expected values converges to the desired quantity. A different route can be taken when the underlying stochastic system is Markovian, i.e. if given the present state, the future is independent of the past. Then solutions to certain systems of deterministic differential equations can often be proved to characterize the conditional expected values. This is the route taken to various valuation problems and optimization problems in finance and life insurance in this exposition. Here, we just state the differential equations and do not discuss possible numerical solutions to these, though.
Valuation is performed by calculation of conditional expected values. However, the claim to be evaluated may contain decision processes in case of which the valuation problem is extended to a matter of calculating extrema of conditional expected values. The extrema are taken over the set of admissible decision processes. However, also extrema of conditional expected values can be characterized by differential equations, albeit more involved. Also decision problems that are not part of a valuation problem are relevant and are studied here. We solve both a problem of minimizing expected quadratic disutility and a problem of maximizing expected power utility. In both cases we state differential equations characterizing the solutions. Actually, from a technical point of view, valuation under decision taking and utility optimization basically only differ by the first measuring streams of payments and the second measuring streams of utility of payments. Even from a qualitative point of view the disciplines are closely related, e.g. in the valuation approach called utility indifference pricing that we shall not deal with here, though.

The models used in this article combine the geometric Brownian motion modelling of financial assets with the finite state Markov chain modelling of the state of a life insurance policy. However, the finite state Markov chain model appears in finance in other connections than life insurance. Therefore the stated differential equations apply to other fields of finance. One example is reduced form modelling of credit risk where the 'state of health', or in this connection creditworthiness, of an enterprise can be modelled by a Markov chain. Another example is valuation of innovative enterprise pipelines. Many types of innovative projects may be modelled by a finite state Markov chain. In e.g. drug development, the drug candidate can be in different states (phases) and certain milestone payments are connected to certain states of the drug candidate.

The list of discoverers in the field of Markov processes and systems of partial differential equations is awe-inspiring: Feller, Kolmogorov and Dynkin are the fathers of the connection between Markov processes and mathematical analysis. After them contributions by Feynman, Kac, Davis, Bensoussan and Lions among others are relevant in the context of this article. However, we concentrate on a few references on more recent applications related to the material of this article and enclose a sectionwise outline.

Section 2: Thiele wrote down in 1875 an ordinary differential equation for the reserve of a life insurance contract. His work was generalized by Hoem (1969) and further by Norberg (1991). The Nobel prize awarded work by Black and Scholes (1973) and Merton (1973) initiated pricing of claims contingent on underlying financial processes. The theory of option pricing has since then turned into one of the larger industries of applied mathematics worldwide. Shortly after, applications to insurance products with contingent claims were suggested by Brennan and Schwartz (1976). The first hybrid between Thiele's and Black and Scholes' differential equations appeared in Aase and Persson (1994). Differential equations for the reserve that connects Hoem (1969) with Aase and Persson (1994) appeared in Steensen (2000). We state and derive the differential equations of Thiele, Black and Scholes and a particular hybrid equation.

Section 3: Applications to more general life insurance products are based on the notions of surplus and dividend distribution. These were studied by Norberg (1999,2001) who also evaluated future dividends by systems of ordinary differential equations. Steffensen (2006b) approached the dividend valuation problem by solving systems of partial differential equations, conforming with a particular specification of the underlying financial market. We state the partial differential equation studied in Steffensen (2006b), including a particular case with a semi-explicit solution.

Section 4: Contingent claims with early exercise options are connected to the theory of optimal stopping and variational inequalities. Grosen and Jørgensen (2000) realized the connection to surrender options in life insurance. In Steffensen (2002), the connection was generalized to general intervention options and the Markov chain model for the insurance policy. We state and prove the
variational inequality for the price of a contingent claim and state the corresponding system for an insurance contract with a surrender option.

Section 5: Optimal arrangement of payment streams in life insurance was first based on the linear regulator. We refer the reader to Fleming and Rishel (1975) for the linear regulator and Cairns (2000) for an overview over its applications to life insurance. The linear regulator was combined with the Markov chain model of an insurance contract in Steensen (2006a). We state and prove the Bellman equation for the linear regulator, and state the Bellman equation derived in Steensen (2006a), including an indication of the solution.

Section 6: The more conventional approach to decision making in finance is based on utility optimization, see Korn (1997) and Merton (1990). Merton (1990) approached decision problems in personal finance and introduced uncertainty of life times. A connection to the Markov chain model of an insurance contract was suggested in Steensen (2004). In Nielsen (2004) a related problem is solved. We state the Bellman equations for the decision problems solved by Merton (1990) and Steensen (2004), including an indication of the solution.

2 The Differential Systems of Thiele and Black-Scholes

2.1 Thiele’s Differential Equation

In this section we state and derive the differential equation for the so-called reserves connected to a life insurance contract with deterministic payments. We give a proof for the differential equation that corresponds to the proofs that will appear in the rest of the article. We end the section by considering the stochastic differential equation for the reserve with application to unit-link life insurance. See Hoem (1969) and Norberg (1991) for differential equations for the reserve.

Let \( B(t) \) denote the total amount of contractual benefits less premiums payable during the time interval \([0, t]\). We assume that it develops in accordance with the dynamics

\[
\frac{d B(t)}{dt} = \sum_{k} b^{j} Z(t) k \frac{d N^k(t)}{dt},
\]

(1)

Here, \( B^j \) is a deterministic and sufficiently regular function specifying payments due during sojourns in state \( j \), and \( b^{j} k \) is a deterministic and sufficiently regular function specifying payments due upon transition from state \( j \) to state \( k \). We assume that each \( B^j \) decomposes into an absolutely continuous part and a discrete part, i.e.

\[
\frac{d B^j(t)}{dt} = b^j(t) dt + \Delta B^j(t).
\]

(2)

Here, \( \Delta B^j(t) = B^j(t) - B^j(t-) \), when different from 0, is a jump representing a lump sum payable at time \( t \) if the policy is then in state \( j \). The set of time points with jumps in \( B^j \) is \( D = \{t_0, t_1, \ldots, t_q\} \) where \( 0 = t_0 < t_1 < \ldots < t_q = n \).
We assume that $Z$ is a time-continuous Markov process on the state space $\mathcal{J}$. Furthermore, we assume that there exist deterministic and sufficiently regular functions $\mu^k(t)$ such that $N^k$ admits the stochastic intensity process $\{\mu^{Z(t)k}(t)\}_{t \in [0,\infty]}$, i.e.

$$M^k(t) = N^k(t) - \int_0^t \mu^{Z(s)k}(s) \, ds$$

constitutes an $\mathcal{F}^Z$-martingale.

![Disability model with mortality, disability, and possibly recovery.](image)

Figure 1: Disability model with mortality, disability, and possibly recovery.

Figure 1 illustrates the disability model used to describe a policy on a single life, with payments depending on the state of health of the insured.

We assume that the investment portfolio earns return on investment by a constant interest rate $r$. We use the notation $\int_s^t = \int_{(s,t]}$ throughout and introduce the short-hand notation $\int_s^t r = \int_s^t r(\tau) \, d\tau = r(s-t)$. Throughout we use subscript for partial differentiation, e.g. $V^j_t = \frac{\partial}{\partial t} V^j_t$.

The insurer needs an estimate of the future obligations stipulated in the contract. The usual approach to such a quantity is to think of the insurer having issued a large number of similar contracts with payment streams linked to independent lives. The law of large numbers then leaves the insurer with a liability per insured that tends to the expected present value of future payments, given the past history of the policy, as the number of policy holders tends to infinity. We say that the valuation technique is based on diversification of risk. The conditional expected present value is called the reserve and appears on the liability side of the insurer’s balance scheme. By the Markov assumption the reserve is given by

$$V^{Z(t)}(t) = E \left[ \int_t^n e^{-\int_t^\tau r} dB(s) \bigg| Z(t) \right].$$

(3)

We can now present the first differential equation, in general spoken of as Thiele’s differential equation.

**Proposition 1** The statewise reserve defined in (3) is characterized by the following deterministic
system of backward ordinary differential equations,
\begin{align*}
0 &= V^j_t(t) + AV^j(t) + \beta^j(t) - rV^j(t), \ t \notin \mathcal{D}, \quad (4a) \\
0 &= R^j_t(t), \ t \in \mathcal{D}, \quad (4b) \\
0 &= V^j(n). \quad (4c)
\end{align*}

In most expositions on the subject, (4a) is written as
\[ V^j_t(t) = rV^j(t) - b^j(t) - \sum_{k: k \neq j} \mu^j_k(t) R^k_t(t), \]
with the so-called sum at risk \( R^k_t(t) \) defined by
\[ R^k_t(t) = b^k(t) + V^k(t) - V^j(t). \]

In the succeeding sections, however, it turns out to be convenient to work with the differential operator abbreviation. We choose to do this already at this stage in order to communicate the cross-sectional similarities.

There are several roads leading to (4). We present a proof that shows that any function solving the differential equation (4) actually equals the reserve defined in (3). Such a result shows that (4) as a sufficient condition on \( V \) in the sense that the differential equation characterizes the reserve uniquely.

Take an arbitrary function \( H^j_t(t) \) solving (4) and consider the process \( H^{Z(t)}_t(t) \). For this process the following line of equalities holds,
\begin{align*}
H^{Z(t)}_t(t) &= - \int_t^n d \left( e^{- \int_s^t r H^{Z(s)}(s)} \right) \\
&= - \int_t^n e^{- \int_s^t r \left( -r H^{Z(s)}(s) ds + dH^{Z(s)}(s) \right)} \\
&= \int_t^n e^{- \int_s^t r \left( dB(s) - \sum_{k: k \neq j} R^j_{H}^{Z(s-k)}(s) dM^k(s) \right)} \\
&\quad - \int_t^n e^{- \int_s^t r \left( H^{Z(s)}(s) + A H^{Z(s)}(s) + \beta^{Z(s)}(s) - r H^{Z(s)}(s) \right) ds} \\
&\quad - \sum_{s \in \{t, n\} \cap \mathcal{D}} e^{- \int_s^t r R^j_{H}^{Z(s)}(s)} \\
&= \int_t^n e^{- \int_s^t r \left( dB(s) - \sum_{k: k \neq j} R^j_{H}^{Z(s-k)}(s) dM^k(s) \right)}. \\
\end{align*}

Here \( R^j_H \) and \( R^j_{H}^{Z(k)} \) are defined as \( R^j \) and \( R^j_{H}^{Z(k)} \) with \( V \) replaced by \( H \). Now, taking conditional expectation on both sides and assuming sufficient integrability, the integral with respect to the martingale vanishes. This leaves us with the conclusion that any solution to (4) equals the reserve,
\[ H^{Z(t)}_t(t) = V^{Z(t)}_t(t). \]

We end this section by reviewing the dynamics of the reserve. Plugging (4) into (5) leads to
\begin{align*}
\frac{dV^{Z(t)}_t(t)}{dt} &= r V^{Z(t)}_t(t) dt - dB^{Z(t)}_t(t) - \sum_{k: k \neq j} \mu^{Z(t)k}_j(t) R^{Z(t)k}_t(t) dt \\
&\quad + \sum_{k: k \neq j} \left( V^k(t) - V^{Z(t-k)}(t) \right) dN^k(t), \\
\end{align*}
that is a backward stochastic differential equation. The term backward refers to the fact that the solution is fixed by the terminal condition (4c), i.e. $V^{Z(n)}(n) = 0$. Usually this terminal condition is rewritten by (4b) into $V^j(n) := \Delta B^j(n)$ where $\Delta B^j(n)$ is a fixed terminal payment. However, one can turn things upside down by taking this terminal condition to be the defining relation of $\Delta B^{Z(n)}(n)$ in terms of $V^{Z(n)}(n)$, i.e. $\Delta B^{Z(n)}(n) := V^{Z(n)}(n)$ with $V^{Z(n)}(n)$ given by (6). Then the terminal condition $V^{Z(n)}(n) = 0$ is fulfilled by construction. We then just need an initial condition on $V$ to consider it as a forward stochastic differential equation. Here, one should take the so-called equivalence relation $V^0(0-) = \Delta B^{Z(n)}(n)$ as initial condition. Hereafter, $V^k(t)$ can be taken to be anything and plays the role as initial condition at time $t$ on $V$, given that the policy jumps into state $k$.

The type of life insurance where terminal payments are linked to the development of the policy is, generally speaking, known as unit-link life insurance. The construction described above is indeed a kind of unit-link life insurance with no guarantee in the sense that there are no predefined bounds on $\Delta B^{Z(n)}(n)$. The simplest implementation turns out by putting $V^k(t) = V^{Z(t-)}(t)$ so that

$$dV^{Z(t)}(t) = rV^{Z(t)}(t)\, dt - dB^{Z(t)}(t) - \sum_{k:k\neq Z(t)} \mu^{Z(t)k}(t) b^{Z(t)k}(t)\, dt.$$  

This means that the reserve is maintained upon transition and the risk sum $R^{jk}(t)$ reduces to the transition payment $b^{jk}(t)$. Then the reserve is really nothing but an account from that the infinitesimal benefits less premiums $dB^{Z(t)}(t)$ are paid and from that the so-called natural risk premium $\sum_{k:k\neq Z(t)} \mu^{Z(t)k}(t) b^{Z(t)k}(t)$ is withdrawn to cover the benefits $b^{Z(t)k}(t), k \neq Z(t)$.

### 2.2 Black-Scholes Differential Equation

In this section we state and prove the differential equation for the value of a financial contract with payments linked to a stock index. See Black and Scholes (1973) and Merton (1973) for the original contributions.

We consider a financial contract issued at time 0 and terminating at a fixed finite time $n$. The payoff from the financial contract is linked to the value of a stock index. Let $X(t)$ denote the stock index at time $t \in [0,n]$. The history of the stock index up to and including time $t$ is represented by the sigma-algebra $\mathcal{F}^X(t) = \sigma\{ X(s), s \in [0,t] \}$. The development of the stock index is formalized by the filtration $\mathcal{F}^X = \{ \mathcal{F}^X(t) \}_{t \in [0,n]}$.

Let $B(t)$ denote the total amount of contractual payments during the time interval $[0,t]$. We assume that it develops in accordance with the dynamics

$$dB(t) = b(t, X(t))\, dt + \Delta B(t, X(t)),$$

where $b(t, x)$ and $\Delta B(t, x)$ are deterministic and sufficiently regular functions specifying payments if the stock value is $x$ at time $t$. The decomposition of $B$ into an absolutely continuous part and a discrete part conforms with (2). Again, we denote the set of time points with jumps in $B$ by $\mathcal{D} = \{ t_0, t_1, \ldots, t_q \}$ where $0 = t_0 < t_1 < \ldots < t_q = n$. The most classical example of a contractual payment function is the European call option given by the following specification of payment coefficients,

$$b(t, x) = 0, \quad \Delta B(t, x) = 0, \quad t < n, \quad \Delta B(n, x) = \max(x - K, 0),$$

for some constant $K$. 

6
We assume that $X$ is a time-continuous Markov process on $\mathbb{R}_+$ with continuous paths. Furthermore, we assume that the dynamics of $X$ are given by the stochastic differential equation,

\[
dX(t) = \alpha X(t) \, dt + \sigma X(t) \, dW(t),
\]

\[
X(0) = x_0,
\]

where $W$ is a Wiener-process, and $\alpha$ and $\sigma$ are constants.

We assume that one may invest in $X$ but, at the same time, a riskfree investment opportunity is available. The riskfree investment opportunity earns return on investment by a constant interest rate $r$, corresponding to the investment portfolio underlying the insurance portfolio in the previous section.

The issuer of the financial contract wishes to calculate the value of the future payments in the contract. The idea of so-called derivative pricing is that the contract value should prevent the contract from imposing arbitrage possibilities, i.e. riskfree capital gains beyond the return rate $r$. The entrepreneurs of modern financial mathematics realized that, in certain financial markets like the one given here, this idea is sufficient to produce the unique value of the financial contract. This contract value equals the conditional expected value,

\[
V(t, X(t)) = \mathbb{E}^Q \left[ \int_t^T e^{-\int_t^s r \, dB(s)} \mid X(t) \right],
\]

where

\[
dX(t) = rX(t) \, dt + \sigma X(t) \, dW^Q(t),
\]

with $W^Q$ being a Wiener-process under the measure $Q$. The measure $Q$ is called a martingale measure because the discounted stock index $e^{-rt}X(t)$ is a martingale under this measure. This construction ensures that the price preventing arbitrage possibilities can be represented in the form (10). Thus, it is actually just a probability theoretical tool for representation.

We introduce the differential operator $\mathcal{A}$, the rate of payments $\beta$, and the updating sum $R$,

\[
\mathcal{A}V(t, x) = V_x(t, x) \, rx + \frac{1}{2} V_{xx}(t, x) \sigma^2 x^2,
\]

\[
\beta(t, x) = b(t, x),
\]

\[
R(t, x) = \Delta B(t, x) + V(t, x) - V(t-, x).
\]

We can now present the second differential equation.

**Proposition 2** The contract value given by (10) is characterized by the following deterministic backward partial differential equation,

\[
0 = V_t(t, x) + \mathcal{A}V(t, x) + \beta(t, x) - rV(t, x), \quad t \notin \mathcal{D}, \tag{11a}
\]

\[
0 = R(t, x), \quad t \in \mathcal{D}, \tag{11b}
\]

\[
0 = V(n, x). \tag{11c}
\]

The usual situation in financial expositions is that there are no payments until termination, in the case of which (12) reduces to

\[
0 = V_t(t, x) + \mathcal{A}V(t, x) - rV(t, x),
\]

\[
V(n-, x) = \Delta B(n, x).
\]
in general spoken of as the *Black-Scholes equation*. For the European call option given by (9), the terminal condition is given by \( V(n-,x) = \max(x-K,0) \). In this case, the system has an explicit solution that is known as the Black-Scholes formula. This can be found in almost any textbook on derivative pricing. As in the previous section we prove that the differential equation is a sufficient condition on the contract value in the sense that any function solving (12) indeed equals the contract value given by (10).

Take an arbitrary function \( H \) solving (12) and consider the process \( H(t;X(t)) \). For this process the following line of equalities holds,

\[
H(t;X(t)) = \int_t^n d\left(e^{-\int_s^t r} H(s,X(s))\right)
\]

\[
= -\int_t^n e^{-\int_s^t r} (-rH(s,X(s)) + dH(s,X(s)))
\]

\[
= \int_t^n e^{-\int_s^t r} \left( dB(s) - H_x(s,X(s)) \sigma X(s) dW^Q(s) \right)
\]

\[
- \int_t^n e^{-\int_s^t r} \left( H_x(s,X(s)) + \alpha H(s,X(s)) + \beta(s,X(s)) - rH(s,X(s)) \right) ds
\]

\[
- \sum_{s \in \{t,n\} \cap \mathbb{D}} e^{-\int_s^t r} R_H(s,X(s))
\]

\[
= \int_t^n e^{-\int_s^t r} \left( dB(s) - H_x(s,X(s)) \sigma X(s) dW^Q(s) \right).
\]

Now, taking conditional expectation on both sides and assuming sufficient integrability, the integral with respect to the martingale vanishes. This leaves us with

\[
H(t;X(t)) = V(t;X(t)).
\]

Thus, any function solving (12) equals the contract value, and the differential equation is then a sufficient condition to characterize the contract value.

### 2.3 A Hybrid Equation

In this section we state the differential equation for the reserves connected to a life insurance contract with payments linked to a stock index. We end the section by considering a stochastic differential equation for the reserve with applications to unit-link life insurance. See Brennan and Schwartz (1976), Aase and Persson (1994), and Steensen (2000) for the original ideas and the general hybrid equations, respectively.

As in Section 2.1, we consider an insurance policy issued at time 0 and terminating at a fixed finite time \( n \) with a payment stream given by (1). However, instead of letting each \( B^j \) and each \( b^{jk} \) be deterministic functions of time, we introduce dependence on the stock index as formalized in Section 2.2. We assume that the accumulated payment process develops in accordance with the dynamics

\[
\frac{dB(t)}{dt} = dB^Z(t;X(t)) + \sum_{k,k \neq Z(t-)} b^{Z(t-)} b^{Z(t-)} X(t;X(t)) dN^k(t),
\]  

where

\[
\frac{dB^j(t,x)}{dt} = b^j(t,x) dt + \Delta B^j(t,x),
\]

with sufficiently regular functions \( b^{jk}(t,x), b^j(t,x), \) and \( \Delta B^j(t,x) \).

As in the previous sections, we are interested in valuation of the future payments in the payment process. The question is now how we should integrate the two approaches to risk pricing presented
there. In Section 2.1, we assumed insured risk to obey the law of large numbers and based the risk valuation on diversification. This left us with a conditional expected present value under the objective probability measure. In Section 2.2, we based the risk valuation on the no arbitrage paradigm of derivative pricing. This left us with a conditional expected present value under an artificial measure \( Q \) called the martingale measure. Which measure should we now use for valuation of integrated insurance and financial risk in the payment process (13)?

The prevention of arbitrage possibilities is not sufficient to get a unique martingale measure. Instead, this idea leaves us with an infinite set of martingale measures. From these measures, some can be said to play more important roles than others. Probably the most important role is played by the product measure that combines the objective measure of insurance risk with the martingale measure of financial risk. We denote, with a slight misuse of notation, also this product measure by \( Q \). This particular martingale measure appears both in several so-called quadratic hedging approaches and in the theory of asymptotic arbitrage. Typically, this measure is applied for valuation of integrated financial and insurance risk. Here, we simply take this measure for given and proceed.

It should be mentioned that the differential equation below holds for a much larger class of martingale measures in the following sense: Instead of valuating insurance risk under the objective measure one could change this measure and still have a martingale measure. However changing the measure of insurance risk is just a matter of changing the transition intensities for \( Z \). So changing the intensities in the formulas below corresponds to picking out an alternative martingale measure to the product measure described in the previous paragraph.

We can now define the reserve by

\[
V_{Z(t)}(t, X(t)) = E^Q\left[ \int_t^n e^{-\int_t^s \gamma dB(s)} \big| Z(t), X(t) \right].
\]  

(14)

Note here that we choose the term reserve for the hybrid (14) of the reserve given in (3) and the contract value given in (??). This reflects that the reserve (14) typically appears on the liability side of an insurance company’s balance scheme.

We introduce the differential operator \( A \), the payment rate \( \beta \) and the updating sum \( R \),

\[
A V^j(t, x) = \sum_{k, k \neq j} \mu^{jk}(t) (V^k(t, x) - V^j(t, x)) + V^j_x(t, x) r x + \frac{1}{2} V^j_{xx}(t, x) \sigma^2 x^2,
\]

\[
\beta^j(t, x) = b^j(t, x) + \sum_{k, k \neq j} \mu^{jk}(t) b^k(t, x),
\]

\[
R^j(t, x) = \Delta B^j(t, x) + V^j(t, x) - V^j(t, -),
\]

(15a)  

(15b)  

(15c)

We can now present the third differential equation.

**Proposition 3** The reserve given by (14) is characterized by the following deterministic system of backward partial differential equations,

\[
0 = V^j_t(t, x) + A V^j(t, x) + \beta^j(t, x) - r V^j(t, x), \quad t \notin D,
\]

\[
0 = R^j(t, x), \quad t \in D,
\]

\[
0 = V^j(n, x).
\]

(16a)  

(16b)  

(16c)
We shall not go through the derivation of the differential equation sufficient condition for characterizing the reserve. The recipe and the calculations can be copied from the previous section but they become more messy as the valuation problem expands. But it is worthwhile to realize that the differential equation (16) is a true generalization of both (4) and (12). The specialization of (16) into (4) comes from erasing all stock index dependence. The specialization into (12) comes from erasing all state dependences and all payments triggered by transitions of \( Z \).

We end this section by studying the special insurance contract introduced at the end of Section 2.1 in the presence of stock index dependence. The backward stochastic differential equation corresponding to (6) describing the dynamics of the reserve turns into

\[
dV^Z(t, X(t)) = \left( rV^Z(t, X(t)) + (\alpha - r)V^Z_x(t, X(t)) X(t) \right) dt + V^Z(t, X(t)) \sigma X(t) dW(t) - dB^Z(t, X(t)) - \sum_{k:k \neq Z(t-)} \mu^{Z(t)k}(t) R^Z(t)k(t, X(t)) dt + \sum_{k:k \neq Z(t-)} \left( V^k(t, X(t)) - V^{Z(t-)}(t, X(t)) \right) dN^k(t)
\]

with

\[
R^Z(t, x) = h^Z(t, x) + V^k(t, x) - V^j(t, x).
\]

As in Section 2.1 we let \( \Delta B^Z(n)(n) := V^Z(n)(n-) \) be the defining relation implying that the terminal condition \( V^Z(n)(n) = 0 \) is fulfilled by construction.

Furthermore, we assume that from the reserve a proportion \( \pi(t) \) is invested in the stock index at time \( t \). Then, letting \( h \) denote the number of stock indices held at time \( t \) and noting that \( \pi(t) V^Z(t)(t, X(t)) = h(t) X(t) \), we then have that

\[
V^Z_x(t, X(t)) = \frac{h(t)}{\pi(t)} = \frac{V^Z(t)(t, X(t))}{X(t)}.
\]

Plugging this relation into (17) gives us a general version of (6). We write down here the special case coming from \( V^k(t) = V^{Z(t-)}(t) \), corresponding to (7),

\[
dV^Z(t, X(t)) = \left( r + \pi(t)(\alpha - r) \right) V^Z(t, X(t)) dt + \sigma \pi(t) V^Z(t, X(t)) dW(t) - dB^Z(t, X(t)) - \sum_{k:k \neq Z(t)} \mu^{Z(t)k}(t) b^Z(t)k(t, X(t)) dt.
\]

Now, this is an investment account with the proportion \( \pi \) invested in the stock index and with a flow of payments corresponding to (7), except for the possibility of stock index dependence in all payments.

3 Surplus and Dividends

3.1 The Dynamics of the Surplus

In this section we introduce the notion of surplus that measures the excess of assets over liabilities. Also the notion of dividends that allows the insured to participate in the performance of the insurance contract, is introduced. For the succeeding sections, only the process of dividends and
the derived dynamics of the surplus are important. See Norberg (1999,2001) and Steffensen (2004) for detailed studies of the notions of surplus and dividends.

Life insurance contracts are typically long-term contracts with time horizons up to half a century or more. Calculation of reserves is based on assumptions on interest rates and transition intensities until termination. Two difficulties arise in this connection. Firstly, these are quantities that are difficult to predict even on a shorter-term basis. Secondly, the policy holder may be interested in participating in returns on risky assets rather than riskfree assets. At the end of Section 2.3 we gave one approach to the second difficulty: Let the terminal lump sum payment be defined by the terminal value of the reserve. Then the prospective expected value given by (14) can be calculated retrospectively. The unit-linked insurance without a guarantee is hereby constructed. For various reasons, however, only few life insurance contracts were constructed like that in the past.

Instead the insurer makes a first prudent guess on the future interest rates and transition intensities in order to be able to put up a reserve, knowing quite well that realized returns and transitions differ. This first guess on interest rates and transition intensities, here denoted by $(r^*, \mu^*)$, is called the first order basis, and gives rise to the first order reserve, $V^*$. The set of payments $B$ settled under the first order basis is called the first order payments or the guaranteed payments.

However, the insurer and the policy holder agree that the realized returns and transitions should be reflected in the realized payment stream. For this reason the insurer adds to the first order payments a dividend payment stream. We denote this payment stream by $D$ and assume that its structure corresponds to the structure of $B$, i.e.

$$
\begin{align*}
    dD (t) &= dD Z^k (t) + \sum_{k, k \neq Z(t-)} \delta^k (t) dN^k (t), \\
    dD^j (t) &= \delta^j (t) dt + \Delta D^j (t).
\end{align*}
$$

Here, however, the coefficients of $D$, $\delta^j (t)$, $\delta^j (t)$, and $\Delta D^j (t)$, are not assumed to be deterministic. In contrast, the dividends should reflect realized returns and transitions relative to the first order basis assumptions.

One can now categorize basically all types of life and pension insurance by their specification of $D$. Such a specification includes possible constraints on $D$, the way $D$ is settled, and the way in that $D$ materializes into payments for the policy holder or others. We shall not give a thorough exposition of the various types of life insurance existing but just give a few hints to what we mean by categorization. When dividends are constrained to be to the benefit of the policy holder, i.e. $D$ is positive and increasing, one speaks of participating or with-profit life insurance. In so-called pension funding there is no such constraint. There, however, often the insured himself is not affected by dividends. In return, an employer pays or receives dividends. No matter whether dividends affect the insured or his employer, the dividends do not necessarily materialize into cash payments. The insurer may convert them into adjustments to first order payments. Such a conversion is then agreed upon in the contract. In participating life insurance this adjustment of first order payments is called bonus. We could continue the categorization of life insurance contracts but we stop here. For all types of contracts, however, remains the question: How should dividends reflect the realized returns and transitions?

A natural measure of realized performance is the surplus given by excess of assets over liabilities. Assuming that payments are invested in a portfolio with value process $Y$ and that liabilities are measured by the first order reserve, we get the surplus

$$
X (t) = \int_0^t Y (s) d (-(B + D) (s)) - V^{Z(t)*} (t),
$$

11
where the first part is the total payments in the past accumulated with capital gains from investing in $Y$. Note that $X$ in this section is defined as the surplus, in contrast to the previous section where $X$ was the stock index.

We now assume that a proportion of $Y$ given by

$$
\alpha(t, X(t)) = \frac{\pi(t, X(t)) X(t)}{X(t) + V^{Z(t)}(t)}
$$

is invested in a risky asset modelled as in Section 2.2. Then the dynamics of $Y$ are given by

$$
dY(t) = rY(t) \, dt + \sigma \frac{\pi(t, X(t)) X(t)}{X(t) + V^{Z(t)}(t)} Y(t) \, dW^Q(t).
$$

Note that we choose to specify the dynamics of $Y$ directly in terms of $W^Q$, the Wiener process under the valuation measure. Deriving the dynamics of $X$, using these dynamics for $Y$, one arrives, after a number of rearrangements and abbreviations, at

$$
dX(t) = rX(t) \, dt + \alpha(t, X(t)) X(t) \, dW(t) + d(C - D)(t),
$$

where $C$ is a surplus contribution process with a structure corresponding to the structure of $B$ and $D$, i.e.

$$
dC(t) = dC^{Z(t)}(t) + \sum_{k \neq Z(t)} c^{Z(t)-j,k} \, dN^{k}(t),
$$
$$
dC^j(t) = c^j(t) \, dt + \Delta C^j(t).
$$

The dynamics of $X$ show that $\alpha$ is actually the proportion of the surplus invested in the risky asset. This is the reason for starting out with the proportion $\pi(t, X(t)) X(t) / (X(t) + V^{Z(t)}(t))$. The elements $c^{j,k}$, $c^j$, and $\Delta C^j$ of $C$ are deterministic functions. They are, of course, important for a closer study on the elements of the surplus. However, they are not crucial for derivation and comprehension of the formulas in what follows.

Having introduced the surplus above as a performance measure, a natural next step is to link the dividend payments directly to the surplus, i.e.

$$
\delta^j(t) = \alpha(t, X(t)),
$$

$$
\delta^{j,k}(t) = \alpha^{j,k}(t, X(t)),
$$

$$
\Delta D^j(t) = \Delta D^j(t, X(t)),
$$

where we, with a slight misuse of notation, use the same notation for the dividend payments and their functional dependence on $(t, X(t))$. This formalization of dividends would certainly be a way of getting realized returns (in $Y$) and transitions (in $N$) reflected in the dividend payments.

We could have introduced other performance measures than the surplus defined above. However, other well-founded performance measures would typically also follow the dynamics given by (19) with appropriate definition of the coefficients in $C$. The formulas derived below would hold true. Thus, in this respect, the story about first order quantities and surplus can be seen as just one example of the state process $X$ underlying the dividend payments.

### 3.2 The Differential Equation for the Market Reserve

In this section we state the differential equation for the reserves connected to a life insurance contract with dividend payments linked to the surplus. This formalizes most practical life insurance contracts where dividends are linked to the performance of the insurance contract. Furthermore, for the special case of dividends that are linear in the surplus, we separate variables of the reserves. Thereby one system of partial differential equations is reduced to two systems of ordinary
differential equations. See Steffensen (2006b) for further studies on partial differential equations for evaluation of surplus-linked dividends.

The insurer is interested in valuation of the total future liabilities. We introduce as reserve the expected present value of future total payments given the past history of the policy. The expectation is taken under the product measure $Q$ introduced in Section 2.2. Since future payments depend on $(Z(t), X(t))$ only and $(Z(t), X(t))$ is a Markov process, the reserve is given by

$$V^{Z(t)}(t, X(t)) = E^Q \left[ \int_t^\infty e^{-r(s-t)} d(B + D)(s) \right| (Z(t), X(t))].$$

(21)

We introduce the differential operator $A$, the payment rate $\beta$ and the updating sum $R$,

$$A^j(t, x) = \sum_{k:k\neq j} \mu_{jk}(t) \left( V^k(t, X + c^k(t) - \delta^k(t, x)) - V^j(t, x) \right)$$

$$+ V^j(t, x) (rx + c^j(t) - \delta^j(t, x)) + \frac{1}{2} V^j_{xx}(t, x) \pi^2(t, x) \sigma^2 x^2,$$

(22a)

$$\beta^j(t, x) = b^j(t) + \delta^j(t, x) + \sum_{k:k\neq j} \mu_{jk}(t) \left( b^k(t) + \delta^k(t, x) \right),$$

(22b)

$$R^j(t, x) = \Delta B^j(t) + \Delta D^j(t, x)$$

$$+ V^j(t, x + \Delta C^j(t) - \Delta D^j(t, x)) - V^j(t-, x).$$

(22c)

We are now ready to present the fourth differential equation.

**Proposition 4** The reserve given by (14) is characterized by the following deterministic system of backward partial differential equations,

$$0 = V^j(t, x) + A^j(t, x) + \beta^j(t, x) - r V^j(t, x), \quad t \notin D,$$

(23a)

$$0 = R^j(t, x), \quad t \in D,$$

(23b)

$$0 = V^j(n, x).$$

(23c)

As in Section 2.3 we shall not go through the derivation of the differential equation. The calculations are even more messy than those leading to the system (16), but the basic ingredients remain the same. However, we explain how (23) generalizes (16) in several respects.

Firstly, compare the differential operators (15a) and (22a). In (15a), the change in the reserve corresponding to a transition from $j$ to $k$ is reflected in the difference $V^k(t, x) - V^j(t, x)$. In this section a state transition also affects the variable $X$ such that after a jump from $j$ to $k$ at time $t$, $X(t) = X(-) + c^k(t) - \delta^k(t, X(-))$. In (22a), this is seen in the change in the reserve by an updating of the variable $x$ accordingly. A similar difference appears between (15c) and (22c). In (15c) the state process $X$ is not affected by a lump sum payment at a deterministic point in time. This leads to a change in the reserve of $V^j(t, x) - V^j(t-, x)$. In this section, a lump sum payment at time $t$ yields $X(t) = X(-) + \Delta C^j(t) - \Delta D^j(t, X(-))$. This is then seen in (22c) by an updating of the variable $x$ accordingly.

Secondly, in (15a) the coefficient on $V^j(t, x)$, $rx$, stems from the systematic return rate on investment $r X(t)$. In this section, the systematic rate of increments of $X$, given sojourn in state $j$, equals $r X(t) + c^j(t) - \delta^j(t, X(t))$. This is then reflected in the coefficient on $V^j(t, x)$, $rx +
Finally, we have in this section allowed for a certain proportional investment of the surplus in the risky asset. The volatility \( \pi(t, X(t)) \sigma X(t) \, dW(t) \) leads to a different coefficient on \( V_{ix}(t, x) \) in (22a) than in (15a).

Apart from the difference between the differential operators, the systems (23) and (16) are almost identical. In this section we have added the two payment streams \( B \) and \( D \), of which only \( D \) is linked to \( X \). In Section 2.3 the payment stream \( B \) was linked to what \( X \) presented there. This is reflected in the according replacement of payments in (15b) and (15c), such that (22b) and (22c) appear.

So far we have just presented the differential equation characterizing the reserve. We have not discussed which functional dependence of dividends on \( X \) that might be relevant. For such a discussion we need to know the insurer’s and the policy holder’s agreement on reflection of performance in dividends. In practice, dividends are always increasing in \( X \). Then a good performance is shared between the two parties by the insurer paying back part of the surplus as positive dividends. A bad performance is shared between the two parties by the insurer collecting part of the deficit as negative dividends. Since there may be constraints on \( D \), e.g. \( D \) increasing, these qualitative estimates are not necessarily strict, though. There are only few examples of a functional dependence that allow for more explicit calculations. However, luckily the most important one allows us to take an important step further. We end this section by specifying a particular functional dependence of dividends on \( X \) that allows for more explicit calculations of the reserve.

We introduce dividends that are linear in the surplus in the sense that

\[
\delta^j(t) = p^j(t) + q^j(t) X(t), \\
\delta^{jk}(t) = p^{jk}(t) + q^{jk}(t) X(t), \\
\Delta D^j(t) = \Delta P^j(t) + \Delta Q^j(t) X(t),
\]

where \( p^j, p^{jk}, \Delta P^j, q^j, q^{jk}, \) and \( \Delta Q^j \) are positive deterministic functions. It is an easy exercise to plug these dividends into the system (23). The next step is then to suggest a useful separation of variables in \( V \). Linearity of dividends inspires a guess on the form

\[
V^j(t, x) = f^j(t) + g^j(t) x.
\]

Plugging this guess and its derivatives into (23) and collecting all terms including and excluding \( x \), respectively, gives us systems of ordinary differential equations for \( f \) and \( g \). We leave it to the reader to verify that the differential equations covering \( f \) and \( g \) are similar in structure to (4). This makes further studies, interpretations, and representations possible. In this exposition we just notify the separation of variables of the reserve function for linear dividends. This separation reduces the system (23) of partial differential equations to two systems of ordinary differential equations characterizing \( f \) and \( g \).

4 Intervention

4.1 Optimal Stopping and Early Exercise Options

In this section we state and prove the differential equation for the value of a financial contract with payments linked to a stock index and with an early exercise option. The proof shows that the differential equation is sufficient for a characterization of the contract value.

In Section 2.2 we studied the price of a financial contract where the payment rates and lump sum payments at deterministic points in time were linked to a stock index. Typically, there is the
additional feature to such a contract that the contract holder can, at any point in time \( t \) until termination, close the contract. He then receives a payoff that depends on the stock value upon closure. This feature is known as the premature or early exercise option, since it gives the contract holder the opportunity to convert future payments into an immediate premature payment.

Recall the payment stream (8) in Section 2.2. Now assume that, given exercise at time \( t \), all future payments are converted into one exercise payment, due at time \( t \), and denoted by

\[ \Phi(t) = \Phi(t, X(t)) \]

where we, with a slight misuse of notation, use \( \Phi \) for both the process and its sufficiently regular functional dependence on \((t, X(t))\). We are now interested in calculating the value of the contract.

It is possible to give an arbitrage argument for the unique contract value,

\[ V(t, X(t)) = \sup_{\tau \in [t, n]} E^{Q} \left[ \int_{t}^{\tau} e^{-\int_{s}^{\tau} \kappa(t) \, ds} dW(t) + e^{-\int_{t}^{\tau} \kappa(t) \, ds} X(t) \right] . \]  

(24)

The decision not to exercise prematurely is included in the supremum in (24) by specifying

\[ \Phi(n) = 0 \]  

(25)

and letting the decision not to exercise prematurely be presented by \( \tau = n \).

Assume that \( X \) is modelled as in Section 2.2 and that the market available is as in Section 2.2. One cannot from the results in the previous sections immediately see how the differential equation from there can be generalized to the situation in this section. For a fixed \( \tau \) the valuation problem is the same as in Section 2.2 with \( n \) replaced by \( \tau \) but how does the supremum affect the results? Does there still exist a deterministic differential equation characterizing the contract value?

We define the differential operator \( \mathcal{A} \), the rate of payments \( \beta \), and the sum \( R \) as in Section 2.2, and introduce furthermore the sum \( \varphi \) by

\[ \varphi(t, x) = \Delta B(t, x) + \Phi(t, x) - V(t-, x) . \]

We can now present the fifth differential equation.

**Proposition 5** The contract value given by (24) is characterized by the following deterministic backward partial variational inequality,

\[
\begin{align*}
0 & \geq V_t(t, x) + \mathcal{A} V(t, x) + \beta(t, x) - r V(t, x), \ t \notin \mathcal{D}, \quad (26a) \\
0 & \geq \Phi(t, x) - V(t, x), \ t \notin \mathcal{D}, \quad (26b) \\
0 & = (V_t(t, x) + \mathcal{A} V(t, x) + \beta(t, x) - r V(t, x)) (V(t, x) - \Phi(t, x)), \ t \notin \mathcal{D}, \quad (26c) \\
0 & \geq R(t, x), \ t \in \mathcal{D}, \quad (26d) \\
0 & \geq \varphi(t, x), \ t \in \mathcal{D}, \quad (26e) \\
0 & = R(t, x) \varphi(t, x), \ t \in \mathcal{D}, \quad (26f) \\
0 & = V(n, x). \quad (26g)
\end{align*}
\]

This system should be compared with (12). Firstly, (11a) is replaced by (26a)-(26c). The equation in (11a) turns into an inequality in (26a). An additional inequality (26b) states that the contract value always exceeds the exercise payoff. This is reasonable, since one of the possible
exercise strategies is to exercise immediately and this would give an immediate exercise payoff. The equality (26c) is the mathematical version of the following statement: At any point in the state space \((t, x)\) at least one of the inequalities in (26a) and (26b) must be an equality.

Secondly, (11b) is replaced by (26d)-(26f). The equation in (11b) turns into an inequality in (26d). An additional inequality (26e) states that the contract value on the time set \(D\) exceeds the lump sum plus the exercise payoff falling due. The equality (26f) states that at least one of the inequalities in (26d) and (26e) must be an equality. Note that (26d)-(26f) easily can be written as

\[
V(t-, x) = \Delta B(t, x) + \max \{V(t, x), \Phi(t, x)\}, \quad t \in D,
\]

while there is no such abbreviation available for (26a)-(26c). However, we choose the version (26d)-(26f) to illustrate the symmetry with (26a)-(26c).

The usual situation in financial expositions is that there are no payments until exercise or termination whatever comes first. In that case \((t, x)\) disappears from (26a)-(26c) and (26d)-(26g) reduce to

\[
V(n, x) = B(n, x) + \max \{V(n, x), \Phi(n, x)\}.
\]

while there is no such abbreviation available for (26a)-(26c). However, we choose the version (26d)-(26f) to illustrate the symmetry with (26a)-(26c).

By the variational inequality (26) one can divide the state space into two regions, possibly intersecting. In the first region, (26a) and (26d) are equalities. This region consists of the states where the optimal stopping strategy for the contract holder is not to stop. In this region the contract value follows a differential equation as if there were no exercise option. In the second region (26b) and (26e) are equalities. This region consists of the states where the optimal stopping strategy for the contract holder is to stop. Thus, in this region the value of the contract equals the exercise payoff.

It is possible to show that (26) is a necessary condition on the contract value. However, instead we go directly to verifying that (26) is also a sufficient condition. The proof starts out in the same way as the verification argument in Section 2.2. Take an arbitrary function \(H\) solving (26) and consider the process \(H(t, X(t))\). Then we can write, by replacing \(n\) by \(\tau\) in (5),

\[
H(t, X(t)) = e^{-\int_t^\tau r \, d\tau} H(\tau, X(\tau)) + \int_t^\tau e^{-\int_t^s r \, ds} \left( dB(s) - H_x(s, X(s)) \sigma X(s) \, dW^Q(s) \right)
- \int_t^\tau e^{-\int_t^s r \, ds} \left( H_s(s, X(s)) + A H(s, X(s)) + \beta(s, X(s)) - r H(s, X(s)) \right) \, ds
- \sum_{s \in \{t, \tau\} \cap D} e^{-\int_t^s r \, ds} R_H(s, X(s)).
\]

Now consider an arbitrary stopping time \(\tau\). For this stopping time we know from (26a), (26b) and (26d) that

\[
H(t, X(\tau)) \geq \int_t^\tau e^{-\int_t^s r \, ds} dB(s) + e^{-\int_t^\tau r \, \Phi(\tau)}
- \int_t^\tau e^{-\int_t^s r \, dh_z(s, X(s)) \sigma X(s) \, dW^Q(s)}.
\]

Taking, firstly, conditional expectation given \(X(t)\) on both sides and then taking supremum over \(\tau\) gives that

\[
H(t, X(\tau)) \geq \sup_{\tau \in [t,n]} E^Q \left[ \int_t^\tau e^{-\int_t^s r \, ds} dB(s) + e^{-\int_t^\tau r \, \Phi(\tau)} \right| X(t) \right]. (28)
\]

Now consider instead the stopping time defined by

\[
\tau^* = \inf_{s \in [t,n]} \{ H(s, X(s)) = \Phi(s, X(s)) \}.
\]
This stopping time is indeed well-defined since, from (25) and (26g), \( H(n, X(n)) = \Phi(n, X(n)) = 0 \), so that \( \tau^* \) occurs no later than \( n \). We now know from (26c) and (26f) that
\[
0 = H_s(s, X(s)) + AH_s(s, X(s)) + \beta(s, X(s)) - rH_s(s, X(s)), \quad s \in [t, \tau^*],
\]
such that
\[
H(t, X(t)) = \int_t^{\tau^*} e^{-\int_t^s r \, dB(s)} + e^{-\int_t^s r \Phi(\tau^*)} - \int_t^{\tau^*} e^{-\int_t^s r H_s(s, X(s))} X(s) \, dW^Q(s).
\]
Taking, firstly, conditional expectation given \( X(t) \) on both sides and then estimating over all possible stopping times yields the inequality
\[
H(t, X(t)) \leq \sup_{\tau \in [t, n]} E^Q \left[ \int_t^{\tau^*} e^{-\int_t^s r \, dB(s)} + e^{-\int_t^s r \Phi(\tau)} \right] X(t).
\]
By (28) and (29), we conclude that \( H(t, X(t)) = V(t, X(t)) \). Thus, any function solving (26) characterizes the contract value. Note that the proof also produces the optimal exercise strategy. The contract holder should exercise according to the stopping time \( \tau^* \). However, in order to know when to exercise, one must be able to calculate the value. Only rarely, the variational inequality (26) has an explicit solution. However, there are several numerical procedures developed for this purpose. One may e.g. use Monte Carlo techniques, general partial differential equation approximations, or certain specific approximations developed for specific functions \( \Phi \).

4.2 Intervention Options in Life and Pension Insurance

In this section we state the differential equation for the reserve of a life insurance contract with dividends linked to the surplus and with a surrender option. Furthermore we comment on the generalization to general intervention options. See Grosen and Jørgensen (2000) and Steffensen (2002) for results on the surrender options and general intervention options.

In correspondence with the previous section, also the holder of a life insurance contract can, typically, terminate his policy prematurely. The act of terminating a life insurance policy is called surrender, and the exercise option is in this context called a surrender option. We consider the insurance contract described in Section 3, i.e. a contract with the total accumulated payments given by \( B + D \). Assume now that the contract holder can terminate his policy at any point in time. Given that he does so at time \( t \), he receives the surrender value
\[
\Phi(t) = \Phi(Z(t), X(t)),
\]
for a sufficiently regular function \( \Phi(t, x) \). Here, we take \( X \) to be the surplus process introduced in Section 3.

We are now interested in calculating the value of future payments specified in the policy. We consider the reserve,
\[
V_{Z(t)}(t, X(t)) = \sup_{\tau \in [t, n]} E^Q \left[ \int_t^{\tau^*} e^{-\int_t^s r \, dB(s)} + e^{-\int_t^s r \Phi(\tau)} \right] Z(t), X(t),
\]
where \( Q \) is the product measure described in Section 2.3. As in the previous section, one cannot immediately see how the differential equation (23) generalizes to this situation. The results in the
previous section indicate, however, that the differential equation can be replaced by a variational inequality.

We define the differential operator $A$, the payment rate $\beta$, and the updating sum $R$ as in Section 3, and introduce furthermore the sum $q^j$ by

$$q^j(t, x) = \Delta B^j(t, x) + \Phi^j(t, x) - V^j(t, x).$$

We can now present the sixth differential equation.

**Proposition 6** The reserve given by (30) is characterized by the following deterministic system of backward partial variational inequalities,

$$0 \geq V^j(t, x) + AV^j(t, x) + \beta^j(t, x) - rV^j(t, x), \quad t \notin D,$$

$$0 \geq \Phi^j(t, x) - V^j(t, x), \quad t \notin D,$$

$$0 = \left( V^j(t, x) + AV^j(t, x) - \beta^j(t, x) - rV^j(t, x) \right) \left( V^j(t, x) - \Phi^j(t, x) \right), \quad t \notin D,$$

$$0 \geq R^j(t, x), \quad t \in D,$$

$$0 \geq q^j(t, x), \quad t \in D,$$

$$0 = R^j(t, x) q^j(t, x), \quad t \in D,$$

$$0 = V^j(n, x).$$

This differential equation can be compared with (23) in the same way as (26) was compared with (12). Its verification goes in the same way as the verification of (??) although it becomes somewhat more involved. We shall not go through this here. As in the previous section, one can now divide the state space into two regions, possibly intersecting. In the first region, the reserve follows a differential equation as if surrender were not possible. This region consists of states from where immediate surrender is suboptimal. In the second region, the reserve equals the surrender value, This region consists of the states where immediate surrender is optimal.

The surrender value is often in practice given by the first order reserve defined in Section 3, in the sense that

$$\Phi^j(t, x) = V^j(t),$$

and is, thus, not surplus dependent.

The title of this section is Intervention. So far we have only dealt with stopping, first of a financial contract in the previous subsection, and second of an insurance contract in this subsection. In practice the insurance policy holder typically holds other options that in some respects are similar in nature to the surrender option but in other respects not. The most important one is the free policy option that allows the policy holder to stop all premium payments but continue the contract in a so-called free policy state. Exercising a free policy option leads to a reduction of the first order benefits that were settled under the assumption of full premium payment. Thus, exercising a free policy option does not stop the insurance policy that continues under free policy conditions, but stops only the premium payments. Therefore, one should rather speak of intervention in than stopping of the insurance policy. Of course, stopping is a special example of intervention.

For a stopping or surrender option, there is always only one control act, namely the act of stopping since hereafter the contract has expired. Given that the policy has been converted into a free policy, the policy holder may still hold a surrender option. Thus, introducing interventions, the policy holder may choose between different series of interventions. This feature produces technical challenges in the verification of a variational inequality characterizing the reserve. However, the basic structure of the resulting variational inequality remains the same.
5 Quadratic Optimization

5.1 Portfolio Quadratic Optimization of Dividends

In this section we state and prove the differential equation for a value function of an optimization problem where preferences over surplus and dividends are specified by a quadratic disutility function. We speak of the value process as a disutility reserve. The surplus introduced in Section 3 is here approximated by a considerably simpler process. We also indicate the solution to the differential equation and the optimal dividend strategy. The control problem studied in this section is known as the linear regulator. See Fleming and Rishel (1975) for the linear regulator in general and Cairns (2000) for its applications to life insurance.

In Section 3, we introduced the notion of surplus. The surplus accumulates a stochastic process of surplus contributions $C$ and capital gains from investment in a Black-Scholes market. From the surplus is withdrawn redistributions to the policy holders in terms of dividends. We modelled the process of dividends similarly to the underlying payment process $B$ (and the process of surplus contributions $C$). In (22) a deterministic differential equation for the reserve was presented where the coefficients in the dividend process are linked to the surplus. We concluded Section 3 by proposing dividends to be affine in the surplus. This led to a reserve that is affine in the surplus. Thus, Section 3 dealt with valuation of certain dividend plans. The question that we did not address was whether, or rather when, surplus linked dividends, or dividends affine in the surplus for that matter, are particularly attractive. Questions of that kind appear in the discipline of optimization rather than valuation.

We approximate the surplus by a diffusion process on the basis of the following list of adaptations:

- We assume that the surplus is invested in the riskfree asset exclusively.
- We approximate the process of surplus contributions by a Brownian motion with volatility $\rho$ and drift $c$.
- We assume that accumulated dividends are absolutely continuous and paid out by the rate $\delta$.

These adaptations give us the following surplus dynamics,

$$
\text{d}X (t) = rX (t) \text{d}t + d(C - D) (t),
X (0) = x_0,
$$

where

$$
\text{d}C (t) = c(t) \text{d}t + \rho (t) dW (t),
\text{d}D (t) = \delta (t) \text{d}t.
$$

We are now interested in deciding on a dividend rate $\delta$ that we prefer over other dividend rates according to some preference criterion. For this purpose we introduce a process of accumulated disutilities $U$, that is absolutely continuous with disutility rate $u(t, \delta(t), X(t))$, i.e.

$$
\text{d}U (t) = u (t, \delta (t), X (t)) \text{d}t.
$$

We now introduce a certain quadratic disutility criterion,

$$
u (t, \delta, x) = p (t) (\delta - a (t))^2 + q (t) x^2.
(31)$$
This criterion punishes quadratic deviations of the present dividend rate from a dividend target rate \( a \) and deviations of the surplus from 0. Such a disutility criterion reflects a trade-off between policy holders preferring stability of dividends, relative to \( a \), over non-stability, and the insurance company preferring stability of the surplus relative to 0. The preference over the surplus could be driven by regulatory rules stating that earned surplus contributions should be redistributed upon earning in some sense. The deterministic functions \( p \) and \( q \) give weights to these preference formalizations.

At time \( t \) the future disutilities are measured by their conditional expectation. We define the *disutility reserve* as the infimum of all such conditional expectations over all admissible dividend payment streams, i.e.

\[
V(t, X(t)) = \inf_D E \left[ \int_t^\infty dU(s) \mid X(t) \right].
\] (32)

Except for the infimum over \( D \), note the similarity with e.g. (14). The primary difference is that, instead of measuring an expected (present) value of payment rates, we now measure an expected disutility function of payment rates, \( p(t) (\delta(t) - a(t))^2 \). Hereto we add an expected disutility function of the position of the surplus, \( q(t) X(t)^2 \).

Now, we introduce the differential operator \( \mathcal{A} \) and the rate of disutilities \( \beta \),

\[
\mathcal{A}V(t, x) = V_x(t, x)(rx + c(t) - \delta) + \frac{1}{2}V_{xx}(t, x) \rho^2, \\
\beta(t, x) = u(t, \delta, x).
\]

We are now ready to present the *seventh differential equation* that is a so-called *Bellman equation*.

**Proposition 7** The disutility reserve given by (32) is characterized by the following Bellman equation,

\[
0 = V_t(t, x) + \inf_\delta [AV(t, x) + \beta(t, x)],
\] (33a)

\[
0 = V(n, x).
\] (33b)

An appendix to this differential equation is the specification of the optimal dividend stream, i.e. the dividend stream that actually minimizes the disutility reserve (32). This optimal dividend stream, specified by the optimal rate \( \delta^* \), is simply the argument of the supremum in (33a), i.e.

\[
\delta^* = \arg\inf_\delta [AV(t, x) + \beta(t, x)].
\] (34)

It is worthwhile to comment on the connection between (33) and e.g. the variational inequality (26). In (26a)-(26b) and in (26d)-(26e), we had two inequalities, corresponding to two different actions, stopping and not stopping. From (26c) and (26f) one of the inequalities must be an equality. The structure of (33a) is the same in the sense that (33a) represents an infinite set of inequalities, corresponding to each possible dividend rate. However, one of the inequalities must hold with equality. Since for each dividend rate, the disutility reserve is described by the same partial differential equation, we can write this in the very compact way (33a). This compact way actually corresponds to the compact writing of (26d)-(26f) in (27).
We now go to the verification of (33a) being a sufficient condition for characterization of the disutility reserve. We start out in the same way as in Section 2. Given a function \( H(t,x) \) solving (33a) and an arbitrary dividend strategy \( \delta \), we can write

\[
H(t,X(t)) = -\int_t^\infty dH(s,X(s)) = \int_t^\infty (dU(s) - H_x(s,X(s)) \rho dW(s)) - \int_t^\infty (H_x(s,X(s)) + AH(s,X(s)) + \beta(s,X(s))) ds.
\]

Note that, given sufficient integrability, we could now, by taking conditional expectation on both sides of (35), conclude the following: If the disutility reserve were defined for an exogenously given dividend payment stream, then (33a) would characterize the disutility reserve with this stream plugged in and without the supremum over \( \delta \). This result is obtained by the methodology used in Section 2. We now argue how the extremum in (32) imposes the extremum in (33a).

Firstly, consider an arbitrary strategy \( \delta \). For this strategy we know, by (33a), that

\[
0 \leq H_t(t,X(t)) + AH(t,X(t)) + \beta(t,X(t))
\]

such that, by (35),

\[
H(t,X(t)) \leq \int_t^\infty (dU(s) - H_x(s,X(s)) \rho(s)dW(s)).
\]

Now, assuming sufficient integrability, taking conditional expectation on both sides and then taking infimum over \( D \), gives us the inequality

\[
H(t,X(t)) \leq \inf_D \mathbb{E} \left[ \int_t^\infty dU(s) \bigg| X(t) \right].
\]

Secondly, for the specific strategy, \( \delta^* = \arg\inf_{\delta} [-A H(t,X(t)) - \beta(t,X(t))] \), we know from (33a) that \( H_t(t,X(t)) + AH(t,X(t)) + \beta(t,X(t)) = 0 \). Inserting this in (35) yields

\[
H(t,X(t)) = \int_t^\infty (dU(s) - H_x(s,X(s)) \rho(s)dW(s)).
\]

Now, taking conditional expectation on both sides and then estimating over all possible dividend strategies yields the inequality

\[
H(t,X(t)) \geq \inf_D \mathbb{E} \left[ \int_t^\infty dU(s) \bigg| X(t) \right].
\]

That \( H(t,X(t)) = V(t,X(t)) \) now follows from (36) and (37).

We shall not go into the methodology of solving (33a), but just state that it actually has a solution in explicit form. The solution is given by

\[
V(t,X(t)) = f(t) (X(t) - g(t))^2 + h(t),
\]

that is just a certain parametrization of a second order polynomial function in \( X(t) \). The functions \( f \), \( g \), and \( h \) are deterministic functions solving certain differential equations. We choose this parametrization in order to write the optimal dividend rate as

\[
\delta^*(t,x) = a(t) + \frac{f(t)}{p(t)} (x - g(t))
\]
that leads to the following interpretation. Firstly, the dividends contains the target rate \( a \) taking into consideration the preferences over present dividends. Secondly, the preferences over the present and future surplus are hidden in an adjustment to this control. This adjustment controls \( X \) towards \( g \) that can be considered as the optimal position for \( X \) at time \( t \). This adjustment happens with the force \( f/p \) that somehow weighs the future preferences over \( X \) through \( f \) against the present preferences over \( \delta \) through \( p \). The functions \( a, p, \) and \( q \) appear in the differential equations for \( f, g, \) and \( h \).

The optimal dividend rate is affine in \( X \). So we can conclude that if redistribute according to the specifications in this section, then it makes sense to work with affine dividend strategies. In general, disutility rates that are functions of the dividend rates and the surplus always lead to optimal dividend rates that are linked to the surplus. This is a consequence of the Markov property. Thus, it does make sense in general to work with the system (22).

5.2 Statewise Quadratic Optimization of Dividends

In this section we state the differential equation for the disutility reserve of an optimization problem where preferences over surplus and dividends are specified by a quadratic disutility function. The surplus is modelled as in Section 3. We also indicate the solution to the differential equation and the optimal dividend strategy. See Steensen (2006a) for the generalization of the linear regulator to Markov chain driven payments.

In the previous section, we approximated the surplus by a diffusion process and controlled it by an absolutely continuous dividend process \( D \). We now take the step back to the original surplus process with dynamics given by (19). Again, however, we skip investment in the risky asset such that (19) reduces to

\[
\begin{align*}
\frac{dX(t)}{dt} &= rX(t)dt + d(C - D)(t), \\
X(0) &= x_0,
\end{align*}
\]

where \( C \) and \( D \) are the contribution and dividend processes given in (20) and (18), respectively. As in the previous section, we now introduce a process \( U \) of accumulated disutilities. However, due to the structure of \( C \) and \( D \), we allow for lump sum disutilities at the discontinuities of \( C \) and \( D \). Thus, inheriting the structure of the payment processes, \( U \) is taken to have the dynamics

\[
\begin{align*}
\frac{dU(t)}{dt} &= \frac{dU^Z(t)}{dt} + \sum_{k,k \neq Z(t-)} \frac{dU^{Z(t)-k}(t)}{dt} + \frac{dU^j(t)}{dt} dN^k(t), \\
\frac{dU^j(t)}{dt} &= w^j(t, \delta, x) dt + \Delta U^j(t, \Delta D, x).
\end{align*}
\]

Inspired by the quadratic disutility functions introduced in the previous section, we form the coefficients in the process of accumulated disutilities \( U \) accordingly, i.e.

\[
\begin{align*}
w^j(t, \delta, x) &= p^j(t) (\delta - a^j(t))^2 + q^j(t) x^2, \\
w^{jk}(t, \delta, x) &= p^{jk}(t) (\delta - a^{jk}(t))^2 + q^{jk}(t) x^2, \\
\Delta U^j(t, \Delta D, x) &= \Delta P^j(t) (\Delta D - \Delta A^j(t))^2 + \Delta Q^j(t) x^2.
\end{align*}
\]

These coefficients should be compared with (31). Firstly, there are now three coefficients corresponding to disutility rates, lump sum disutilities upon transitions of \( Z \) and lump sum disutilities
at deterministic points in time. Secondly, for each type of dividend payment, we allow the target to be state dependent. Thirdly, the weights on disutility of dividend deviations against disutility of surplus deviations are also allowed to be state dependent.

The idea is now, with the generalized process of accumulated disutilities, to solve the corresponding optimization problem associated with the disutility reserve

$$V^Z(t, X(t)) = \inf_{D} E \left[ \int_{t}^{\infty} dU(s) \right] Z^t(t, X(t)).$$

We introduce the differential operator $A$, the utility rate $\beta$ and the updating sum $R$,

$$AV^j(t, x) = \sum_{k:k \neq j} \mu^j_k(t) \left( V^k(t, x + c^j_k(t) - \delta^k) - V^j(t, x) \right) + V^j_x(t, x) \left( rx + c^j(t) - \delta \right) ,$$

$$\beta^j(t, x) = p^j(t) (\delta - a^i(t))^2 + q^j(t) x^2 + \sum_{k:k \neq j} \mu^j_k(t) \left( p^k(t) \left( \delta^k - a^j_k(t) \right)^2 + q^k(t) \left( x + c^j_k(t) - \delta^k \right)^2 \right) ,$$

$$R^j(t, x) = \Delta P^j(t) (\Delta D - \Delta A^j(t))^2 + \Delta Q^j(t) x^2 + V^j(t, x + \Delta C^j(t) - \Delta D) - V^j(t-, x) .$$

We are now ready to present the eighth differential equation that is a generalized version of the Bellman equation (33a).

**Proposition 8** The disutility reserve given by (38) is characterized by the following Bellman equation,

$$0 = V^j(t, x) + \inf_{\delta, \delta^k} AV^j(t, x) + \beta^j(t, x) , \quad t \notin D ,$$

(39a)

$$0 = \inf_{\Delta D} R^j(t, x) , \quad t \in D ,$$

(39b)

$$0 = V^j(n, x) .$$

(39c)

The methodology needed for verification of (39) as a sufficient condition for characterization of the disutility reserve is the same as in the previous section. However, the state dependence makes the calculations somewhat more involved.

In the previous section, we proposed an appropriately parametrized second order polynomial function as solution to the Bellman equation. It is very convenient that this simple structure is inherited by the solution to (39). The only generalization of the proposed solution is that the coefficient functions $f$, $g$, and $h$ should be state dependent, i.e.

$$V^Z(t, X(t)) = f^Z(t) \left( X(t) - g^Z(t) \right)^2 + h^Z(t) .$$

Now, it is possible to derive systems of ordinary differential equations for $f$, $g$, and $h$, that can be solved numerically. The optimal dividend payments, given that the policy is in state $j$ at time $t$,
The optimal lump sum dividend payment on \( D, \Delta D^{*j}(t) \), follows a formula similar in structure to the formula for \( \delta^{*j}(t) \). Due to the parametrization of the second order polynomial solution the following interpretations of \( \delta^{*j}(t) \) and \( \delta^{*jk}(t) \), respectively, apply:

The optimal dividend rate should be interpreted in the same way as in the previous section. The rate is given by the target rate and an adjustment that takes care of future preferences over \( X \). The adjustment moves \( X \) towards its optimal position at time \( t \), \( g^j(t) \), with the force \( f^j(t) = p^j(t) \). Now consider the optimal lump sum payment upon transition. This is actually a weighted average of three quantities corresponding to three considerations. Firstly, a dividend payment equal to its target is preferred with the first weight \( p^{jk}(t) + q^{jk}(t) + f^k(t) \). Secondly, a payment pushing \( X(t) \) towards its target 0 is preferred with the second weight \( q^{jk}(t) \). Thirdly, the consideration of the position of \( X \) in the future leads to an adjustment that brings \( X \) close to its optimal position after the transition, \( g^k(t) \), by a force equal to the third weight, \( f^k(t) \). A similar interpretation applies for the lump sum payment at deterministic points in time.

6 Utility Optimization

6.1 Merton’s Optimization Problem

In this section we state the differential equation for a value function, here called the utility reserve, of an optimization problem where preferences over surplus and dividends is specified by a power utility function. The surplus introduced in Section 3 is here approximated by a considerably simpler process. We also indicate the solution to the differential equation and the optimal dividend strategy. See Korn (1997) and Merton (1990) for original contributions.

In Section 5, we approximated the surplus introduced in 3. This led to a portfolio version of the quadratic optimization problem of a life insurance company. Here again, we formulate the redistribution problem as a control problem. However, we now add a decision variable. We do not assume that surplus is invested in the riskfree asset only. Instead, we consider the proportion invested in risky assets as a decision variable.

Here, we start out by approximating the surplus introduced in Section 3 on the basis of the following list of adaptations:

- We assume that the process of contributions to the surplus is absolutely continuous and accumulates by the rate \( c \).
- We assume that accumulated dividends are absolutely continuous and paid out by the rate \( \delta \).
This gives us the following surplus dynamics,

\[
\begin{align*}
    dX(t) &= (r + \pi(t) (\alpha - r)) \cdot X(t) \, dt + \pi(t) \cdot \sigma X(t) \, dW(t) + d(C-D)(t), \\
    X(0) &= x_0,
\end{align*}
\]

where

\[
\begin{align*}
    dC(t) &= c(t) \, dt, \\
    dD(t) &= \delta(t) \, dt.
\end{align*}
\]

As in Section 5, we introduce a preference criterion to decide on a dividend rate \( \delta \) and an investment proportion \( \pi \). We introduce a process of accumulated utilities \( U \) and a power utility rate \( u(t, \delta(t)) \), i.e. for \( \gamma < 1 \),

\[
\begin{align*}
    dU(t) &= u(t, \delta(t)) \, dt, \\
    u(t, \delta) &= \frac{1}{\gamma} a(t)^{1-\gamma} \delta^\gamma. \quad (40)
\end{align*}
\]

The criterion (40) rewards high dividend rates without consideration to the surplus. The deterministic function \( a \) weighs the utility of dividends over time. Without further specifications such a problem has no solutions since it would be optimal to pay out infinite dividend rates. However, adding the constraint that the terminal surplus must be non-negative, the problem makes sense.

We now measure the future utilities by the utility reserve,

\[
V(t, X(t)) = \sup_{\pi, D} \mathbb{E} \left[ \int_t^T dU(s) \cdot X(s) \right]. \quad (41)
\]

Note the similarity with e.g. (14) where we now, instead of measuring an expected (present) value of the payment rates \( \delta \), measure an expected utility function of the payment rates. Hereto, we have added the supremum that leaves us with an optimization problem.

Now, we introduce the differential operator \( \mathcal{A} \), and the rate of disutilities \( \beta \),

\[
\begin{align*}
    \mathcal{A} V(t, x) &= V_x(t, x) \left( (r + \pi (\alpha - r)) x + c(t) - \delta \right) \\
    &\quad + \frac{1}{2} V_{xx}(t, x) \pi^2 \sigma^2 x^2, \\
    \beta(t) &= \frac{1}{\gamma} a(t)^{1-\gamma} \delta^\gamma.
\end{align*}
\]

We are now ready to present the ninth differential equation that is a Bellman equation.

**Proposition 9** The utility reserve given by (41) is characterized by the following Bellman equation,

\[
\begin{align*}
    0 &= V_t(t, x) + \sup_{\delta, \pi} \left[ \mathcal{A} V(t, x) + \beta(t) \right], \quad (42a) \\
    0 &= V(n, x). \quad (42b)
\end{align*}
\]

The optimal dividend stream and the optimal investment strategy, specified by the optimal rate \( \delta^* \) and the optimal proportion \( \pi^* \), are simply the arguments of the infimum in (42), i.e.

\[
\begin{align*}
    \delta^* &= \arg \sup_{\delta} \left[ \mathcal{A} V(t, x) + \beta(t) \right], \\
    \pi^* &= \arg \sup_{\delta} \left[ \mathcal{A} V(t, x) + \beta(t) \right].
\end{align*}
\]
The verification of (42) as a sufficient condition characterizing the utility reserve goes in exactly the same way as in Section 5. The only difference is that all inequalities are turned around since we are now solving a maximization problem instead of a minimization problem.

As we did in Section 5, we can separate the variables of the solution. In this case, the solution is given by

\[ V(t, X(t)) = \frac{1}{\gamma} f(t)^{1-\gamma} (X(t) + g(t))^\gamma. \]

With this parametrization of the solution, both \( f \) and \( g \) have solutions that can be interpreted as present values. There exists an artificial rate \( r^* \), that depends on all parameters in the model, such that

\[ f(t) = \int_t^\infty e^{-\int_t^s r^* (\sigma) \, ds} \, ds, \]
\[ g(t) = \int_t^\infty e^{-\int_t^s r^* (\sigma) \, ds} \, ds. \]

The function \( f \) is a present value that says something about the value of investing and smoothing out the surplus over the residual time to maturity. The time weights in the function \( a \) appear in \( f \). The function \( g \) is the present value of future contributions to the surplus. From its appearance in the utility reserve, the insurance company could activate all future surplus contributions and account for them in the surplus.

The optimal controls become

\[ \delta^*(t, x) = \frac{a(t)}{f(t)} (x + g(t)), \]
\[ \pi^*(t, x) x = \frac{1}{1 - \gamma} \frac{\alpha - r}{\sigma^2} (x + g(t)). \]

These strategies are easy to interpret. One should pay out, at any point in time, a fraction \( a(t) / f(t) \) of \( X \) that weights the future preferences over dividends through \( f \) against the present preferences over dividends through \( a \). The amount optimally invested in stocks \( \pi^*(t, X(t)) X(t) \) is a proportion of the surplus plus activated future surplus contributions.

Note that for \( t \to n \), \( f \to 0 \), such that the proportion of \( X \) paid out as dividends tends to infinity. The consequence is that the optimally controlled surplus ends at 0.

6.2 Statewise Power Utility Optimization of Dividends

In this section we state the differential equation for the utility reserve of an optimization problem where preferences over surplus and dividends is specified by a power utility function. The surplus is modelled as in Section 3. We also indicate the solution to the differential equation and the optimal dividend strategy. See Steensen (2004) for the generalization of Merton’s optimization problem to Markov chain driven payments.

In the previous section, we approximated the surplus by modelling both contributions and dividends as absolutely continuous processes. We now take a step back to the original surplus process with dynamics given by (19). We now model the surplus as in (19) with the exception that we still take \( C \) to be approximated by a deterministic function, i.e.

\[ dC(t) = c(t) \, dt + \Delta C(t), \]

allowing for no state dependence in the surplus contribution. The full return to (19) with \( C \) given by (20) is not immediately possible. The process of accumulated utilities is, on the other hand, given by
\[ dU(t) = dU^Z(t, \delta(t), \Delta D(t)) + \sum_{k: k \neq Z(t^-)} u(Z(t^-)^k) \left( t, \delta^k(t) \right) dN^k(t), \]
\[ dU^J(t, \delta, \Delta D) = u^J(t, \delta) dt + \Delta U^J(t, \Delta D). \]

We generalize the power utility function to state-dependent utility functions in the sense that
\[ u^J(t, \delta) = \frac{1}{\gamma} a^J(t)^{1-\gamma} \delta^\gamma, \]
\[ u^{jk}(t, \delta^k) = \frac{1}{\gamma} a^{jk}(t)^{1-\gamma} (\delta^k)^\gamma, \]
\[ \Delta U^J(t, \Delta D) = \frac{1}{\gamma} \Delta A^J(t)^{1-\gamma} (\Delta D)^\gamma. \]

These coefficients should be compared with (40). Due to the structure of the dividend payments, there is now one coefficient for each type of dividend payment. Furthermore, we allow the coefficient functions to depend on the state of \( Z \).

We now introduce the utility reserve
\[ V^Z(t, X(t)) = \sup_{D, \pi} \mathbb{E} \left[ \int_t^T dU(s) \left| Z(t), X(t) \right. \right]. \tag{43} \]

We introduce the differential operator \( A \), the utility rate \( \beta \) and the updating sum \( R \),
\[ A^J(t, x) = \sum_{k: k \neq j} \mu^{jk}(t) \left( V^k(t, x - \delta^k) - V^J(t, x) \right) \]
\[ + V^J_j(t, x) \left( (r + \pi (\alpha - r)) x + c(t) - \delta \right) \]
\[ + \frac{1}{2} V^J_{xx}(t, x) \pi^2 \sigma^2 x^2, \]
\[ \beta^J(t) = \frac{1}{\gamma} a^J(t)^{1-\gamma} \delta^\gamma + \sum_{k: k \neq j} \mu^{jk}(t) \frac{1}{\gamma} a^{jk}(t)^{1-\gamma} (\delta^k)^\gamma, \]
\[ R^J(t, x) = \frac{1}{\gamma} \Delta A^J(t)^{1-\gamma} (\Delta D)^\gamma + x^2 + V^J(t, x + \Delta C(t) - \Delta D) - V^J(t, x). \]

We are now ready to present the tenth - and final - differential equation that is a generalized version of the Bellman equation (42).

**Proposition 10** The utility reserve given by (43) is characterized by the following Bellman equation,
\[ 0 = V_j^J(t, x) + \sup_{\pi, \delta, \delta^k} \left[ A^J(t, x) + \beta^J(t) \right], \quad t \notin D, \tag{44a} \]
\[ 0 = \sup_{\Delta D} R^J(t, x), \quad t \in D, \tag{44b} \]
\[ 0 = V^J(n, x). \tag{44c} \]

The methodology needed for verifying (44) as a sufficient condition for the characterization of the utility reserve is the same as in Section 5.
In Section 5, we separated variables of the utility reserve. In Section 5, introducing state dependence led to a separation of variables such that parts depending on time became state dependent as well. The question is whether this trick works here again. Indeed,

\[ V(Z(t), X(t)) = \frac{1}{\gamma} f(Z(t))^{1-\gamma} (X(t) + g(t))^\gamma. \]

In the previous section, the function \( f \) could be interpreted as an artificial present value of the stream of coefficients \( a \). Here again, the resulting differential equation for \( f \) leads to similar possibilities for interpretations. However, the conclusion becomes somewhat involved and is not pursued further here. On the other hand, we still have that

\[ g(t) = \int_t^T e^{-\int_s^t r} dC(s) ds, \]

and the insurance company can again activate all future deterministic surplus contributions and account for them in the surplus.

The optimal amount invested in stocks is still

\[ \pi^*(t, X(t)) X(t) = \frac{1}{1-\gamma} \frac{\alpha - r}{\sigma^2} (X(t) + g(t)), \]

whereas the optimal dividend payments are formalized by

\[ \delta^{ij}(t, x) = \frac{a^{ij}(t)}{f^j(t)} x, \]
\[ \delta^{ijk}(t, x) = \frac{a^{ijk}(t)}{a^{ijk}(t) + f^k(t)} x, \]
\[ \Delta D^{ij}(t, x) = \frac{\Delta A^j(t)}{\Delta A^j(t) + f^j(t)} x. \]

Again, we can interpret the optimal fraction of surplus in the optimal dividend rate as a trade-off between present considerations in \( a \) and future considerations in \( f \). The same interpretation applies for the optimal lump sum dividends. The numerator concerns the present preferences while the denominator concerns the future preferences, including the present. For all considerations the state dependence of \( f \) is reflected in the state dependent optimal dividend payments.

In Section 5, we ended up with \( X(n) = 0 \) due to infinite dividend proportions of the surplus as we get closer to maturity. In this case, the same conclusion is a consequence of the terminal condition \( f^j(n) = 0 \).

References


