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Munkres §3

Ex. 3.12 (Morten Poulsen). It might help to think of (ii) and (iii) as rotated dictionary orders and drawing a diagram might help as well.

(i). Let $(x_0, y_0) \in \mathbf{Z}_+ \times \mathbf{Z}_+$. Immediate predecessors:

- If $y_0 > 1$ then the immediate predecessor is $(x_0, y_0 - 1)$.
- If $y_0 = 1$ then $(x_0, 1)$ has no immediate predecessor.

The smallest element is $(1, 1)$.

(ii). Let $(x_0, y_0) \in \mathbf{Z}_+ \times \mathbf{Z}_+$. Immediate predecessors:

- If $x_0 = 1$ then $(1, y_0)$ has no immediate predecessor.
- If $x_0 > 1$ and $y_0 = 1$ then the immediate predecessor is $(x_0 - 1, 1)$.
- If $x_0 > 1$ and $y_0 > 1$ then the immediate predecessor is $(x_0 - 1, y_0 - 1)$.

There are no smallest element.

(iii). Let $(x_0, y_0) \in \mathbf{Z}_+ \times \mathbf{Z}_+$. Immediate predecessors:

- If $y_0 > 1$ then the immediate predecessor is $(x_0 + 1, y_0)$.
- If $x_0 > 1$ and $y_0 = 1$ then the immediate predecessor is $(x_0 - 1, y_0)$.
- The element $(1, 1)$ has no immediate predecessor.

The smallest element is $(1, 1)$.

Since (i) has a smallest element, but (ii) hasn't a smallest element, it follows that the order types of (i) and (ii) are different. Similarly are the order types of (ii) and (iii) different. Since (i) has more than one element (actually countably infinite many elements) without an immediate predecessor and (iii) has only one element without an immediate predecessor, it follows that they have different order types.

Ex. 3.13 (Morten Poulsen).

Theorem 1. *If an ordered set A has the least upper bound property, then it has the greatest lower bound property.*

Proof. Assume $A_0 \subset A$ is nonempty and has a lower bound $b \in A$. Let

$$B_0 = \{\, a \in A \mid \forall\, a_0 \in A_0 : a \leq a_0 \,\},$$

i.e. B_0 is the set of all lower bounds for A_0. [Want to show that B_0 has a largest element].

Now $B_0 \subset A$ is nonempty, since $b \in B_0$, and has an upper bound, e.g. every element in A_0. Since A has the least upper bound property the set C_0 of all upper bounds for B_0, i.e.

$$C_0 = \{\, a \in A \mid \forall\, b_0 \in B_0 : b_0 \leq a \,\},$$

has a smallest element $c \in C_0$. Since $A_0 \subset C_0$ it follows that c is a lower bound for A_0, hence $c \in B_0$. It follows that c is the largest element in B_0, this means by definition that A_0 has a greatest lower bound, hence A has the greatest lower bound property. □

Ex. 3.15 (Morten Poulsen). Assume that $\mathbf{R}$ has the least upper bound property.

(a). By a argument similar to the one in example 13, it follows that the sets $[0, 1]$ and $[0, 1)$ have the least upper bound property.

(b). The set $X = [0,1] \times [0,1]$ in the dictionary order has the least upper bound property: Suppose $A \subset X$ is nonempty and has an upper bound. Since $[0,1]$ has the least upper bound property the nonempty set

$$X_0 = \{\, x \in [0,1] \mid \exists\, y \in [0,1] : (x,y) \in A \,\} \subset [0,1]$$

has a least upper bound x_0. Let

$$Y_0 = \{\, y \in [0,1] \mid (x_0, y) \in A \,\} \subset [0,1].$$

If Y_0 is empty then $(x_0, 0)$ is clearly the least upper bound of A. If Y_0 is nonempty then Y_0 is bounded above by 1, hence Y_0 has a least upper bound y_0. It follows, by construction, that (x_0, y_0) is the least upper bound of A. Thus X has the least upper bound property.

The set $Y = [0,1] \times [0,1)$ in the dictionary order has not the least upper bound property: Let B be the set $[0, \frac{1}{2}] \times [0,1)$, B is clearly bounded above. But the set of upper bounds for B has no smallest element, since no element of the form $(\frac{1}{2}, y)$, $y \in [0,1)$, is an upper bound for B and given $\varepsilon > 0$ then $(\frac{1+\varepsilon}{2}, 0) < (\frac{1}{2} + \varepsilon, 0)$. Thus Y hasn't the least upper bound property.

The set $Z = [0,1) \times [0,1]$ has the least upper bound property by a argument similar to the one for X.

References