Munkres §4

Ex. 4.2. We assume that there exists a set $\mathbb{R}$ equipped with two binary operations, + and ·, and a linear order $<$ such that

(1) $(\mathbb{R}, +, \cdot)$ is a field.
(2) $x < y \Rightarrow x + z < y + z$ and $0 < x, 0 < y \Rightarrow 0 < xy$
(3) $(\mathbb{R}, <)$ is a linear continuum

Using these axioms we can establish all the usual rules of arithmetic.

(c): $\Rightarrow$: Assume that $x > 0$. Adding $-x$ to this gives $0 > -x$. 
$\Leftarrow$: Assume that $-x < 0$. Adding $x$ to this gives $0 < x$.

(g): Since $0 \neq 1$ in a field, we have either $0 < 1$ or $1 < 0$ by Comparability. We rule out the latter possibility. If $1 < 0$, then $-1 > 0$ so also $1 = (-1) \cdot (-1) > 0$, a contradiction. Thus we have $0 < 1$ and then also $-1 < 0$ by point (c).

Ex. 4.3 (Morten Poulsen).

(a). Let $A$ be a collection of inductive sets. Since $1 \in A$ for all $A \in A$, it follows that $1 \in \cap_{A \in A} A$. Let $a \in \cap_{A \in A} A$. Since $A$ is inductive for all $A \in A$, it follows that $a + 1 \in A$ for all $A \in A$, hence $a + 1 \in \cap_{A \in A} A$. So $\cap_{A \in A} A$ is inductive.

(b). By definition $\mathbb{Z}^+ = \cap_{A \in A} A$, where $A$ is the collection of all inductive subsets of the real numbers.

Proof of (1): The set $\mathbb{Z}^+$ is inductive by (a).

Proof of (2): Suppose $A \subset \mathbb{Z}^+$ is inductive. Since $A$ inductive, it follows by the definition of $\mathbb{Z}^+$ that $\mathbb{Z}^+ \subset A$, i.e. $A = \mathbb{Z}^+$.

Ex. 4.4 (Morten Poulsen).

(a). Let $A$ be the set of $n \in \mathbb{Z}_{+}$ for which the statement holds.

The set $A$ is inductive: It is clear that $1 \in A$, since the only nonempty subset of $\{1\}$ is $\{1\}$. Suppose $n \in A$. Let $B$ be a nonempty subset of $\{1, \ldots, n + 1\}$. If $n + 1 \in B$ then $n + 1$ is the largest element in $B$. If $n + 1 \notin B$ then the set $B \cap \{1, \ldots, n\}$ contains a largest element, since $n \in A$.

So $A \subset \mathbb{Z}_{+}$ is inductive, by the principle of induction, it follows that $A = \mathbb{Z}_{+}$, as desired.

(b). Consider!

Ex. 4.5 (Morten Poulsen).

(a). Let $a \in \mathbb{Z}_{+}$. Let

$$X = \{ x \in \mathbb{R} | a + x \in \mathbb{Z}_{+} \} .$$

The set $X$ is inductive: $1 \in X$, since $a \in \mathbb{Z}_{+}$ and $\mathbb{Z}_{+}$ inductive. Suppose $x \in X$. Since $a + (x + 1) = (a + x) + 1$, $a + x \in \mathbb{Z}_{+}$ and $\mathbb{Z}_{+}$ inductive, it follows that $x + 1 \in X$.

By ex. 4.3(a) it follows that $X \cap \mathbb{Z}_{+} \subset \mathbb{Z}_{+}$ is inductive. By the principle of induction, it follows that $X \cap \mathbb{Z}_{+} = \mathbb{Z}_{+}$, which proves (a).

(b). Let $a \in \mathbb{Z}_{+}$. Let

$$X = \{ x \in \mathbb{R} | ax \in \mathbb{Z}_{+} \} .$$

The set $X$ is inductive: $1 \in X$, since $a1 = a \in \mathbb{Z}_{+}$. Suppose $x \in X$. Since $a(x + 1) = ax + a$ and $ax, a \in \mathbb{Z}_{+}$, it follows by (a) that $x + 1 \in X$.

As above, it follows that $X \cap \mathbb{Z}_{+} = \mathbb{Z}_{+}$, which proves (b).
(c). Let 

\[ X = \{ x \in \mathbb{R} \mid x - 1 \in \mathbb{Z}_+ \cup \{0\} \}. \]

The set \( X \) is inductive: \( 1 \in X \), since \( 1 - 1 = 0 \in \mathbb{Z}_+ \cup \{0\} \). Suppose \( x \in X \). Note that \( (x+1) - 1 = (x-1) + 1 \). If \( x - 1 = 0 \) then \( (x-1) + 1 = 1 \in \mathbb{Z}_+ \cup \{0\} \). If \( x - 1 \in \mathbb{Z}_+ \) then, since \( \mathbb{Z}_+ \) is inductive, \( (x-1) + 1 \in \mathbb{Z}_+ \subseteq \mathbb{Z}_+ \cup \{0\} \). So \( x + 1 \in X \).

As above, it follows that \( X \cap \mathbb{Z}_+ = \mathbb{Z}_+ \), which proves (c).

(d). Let \( c \in \mathbb{Z} = \mathbb{Z}_- \cup \{0\} \cup \mathbb{Z}_+ \), where \( \mathbb{Z}_- \) is negatives of the elements of \( \mathbb{Z}_+ \). First we prove the result for \( d = 1 \):

(i) \( c + 1 \in \mathbb{Z} \): If \( c \in \mathbb{Z}_+ \) the result follows from (a). It is clear if \( c = 0 \). If \( c \in \mathbb{Z}_- \) then \( c + 1 = -(-c + 1) \), since \( -c \in \mathbb{Z}_+ \), it follows from (c) that \( -c - 1 \in \mathbb{Z}_+ \cup \{0\} \), hence \( c + 1 \in \mathbb{Z} \).

(ii) \( c - 1 \in \mathbb{Z} \): If \( c \in \mathbb{Z}_+ \) the result follows from (c). It is clear if \( c = 0 \). If \( c \in \mathbb{Z}_- \) then \( c - 1 = -(-c + 1) \), since \( -c \in \mathbb{Z}_+ \), it follows from (a) or by the inductivity of \( \mathbb{Z}_+ \) that \( -c + 1 \in \mathbb{Z}_+ \), hence \( c - 1 \in \mathbb{Z} \).

Next we prove the result for \( d \in \mathbb{Z}_+ \): Let

\[ X = \{ x \in \mathbb{R} \mid c + x \in \mathbb{Z} \} \]

and

\[ Y = \{ y \in \mathbb{R} \mid c - y \in \mathbb{Z} \}. \]

The set \( X \) is inductive: \( 1 \in X \), c.f. (i). Suppose \( x \in X \). Since \( c + (x+1) = (c+x) + 1 \), \( x \in \mathbb{Z} \) and (i), it follows that \( x+1 \in X \).

The set \( Y \) is inductive: \( 1 \in X \), c.f. (ii). Suppose \( y \in Y \). Since \( c - (y+1) = (c-y) - 1 \), \( c \in \mathbb{Z} \) and (ii), it follows that \( y+1 \in X \).

As above, it follows that \( X \cap \mathbb{Z}_+ = \mathbb{Z}_+ \) and \( Y \cap \mathbb{Z}_+ = \mathbb{Z}_+ \). This proves the result for \( d \in \mathbb{Z}_+ \).

The result is clear if \( d = 0 \). The case \( d \in \mathbb{Z}_- \) is now easy: Since \( c + d = c - (-d) \) and \( -d \in \mathbb{Z}_+ \), it follows that \( c + d \in \mathbb{Z} \). Since \( c - d = c + (-d) \) and \( -d \in \mathbb{Z}_+ \), it follows that \( c - d \in \mathbb{Z} \).

(e). Let \( c \in \mathbb{Z} \). Let

\[ X = \{ x \in \mathbb{R} \mid cx \in \mathbb{Z} \}. \]

The set \( X \) is inductive: \( 1 \in X \), since \( c1 = c \in \mathbb{Z} \). Suppose \( x \in X \). Since \( c(x+1) = cx + c \) and \( cx \in \mathbb{Z} \), it follows, by (d), that \( x + 1 \in X \).

As above, it follows that \( X \cap \mathbb{Z}_+ = \mathbb{Z}_+ \). Thus the result is proved for \( d \in \mathbb{Z}_+ \) and is clear if \( d = 0 \). Since \( cd = (-c)(-d) \), \( -c \in \mathbb{Z} \), hence if \( d \in \mathbb{Z}_- \) then \( -d \in \mathbb{Z}_+ \), this proves the case \( d \in \mathbb{Z}_- \).

References