Munkres §10

Ex. 10.1. If a subset of a well-ordered set has an upper bound, the smallest upper bound is a least upper bound (supremum) for the set. (This proof is a tautology!)

Ex. 10.2.

(a). The smallest successor $x_+$ of any element $x$ is the immediate successor. (The iterated successors of $x$ has the order type of a section of $\mathbb{Z}_+$.)

(b). $\mathbb{Z}$.

Ex. 10.4.

(a). Let $A$ be a simply ordered set containing a subset with the order type of $\mathbb{Z}_-$. Then this subset does not have a smallest element so $A$ is not well-ordered. Conversely, let $A$ be simply ordered set containing a nonempty subset $B$ with no smallest element. Let $b_1$ be any element of $B$. Since $b_1$ is not a smallest element of $B$ there is some element $b_2$ of $B$ such that $b_2 < b_1$. Continuing inductively we obtain an infinite descending chain $\cdots < b_{n+1} < b_n < \cdots < b_2 < b_1$ forming a subset of the same order type as $\mathbb{Z}_-$.

(b). $A$ does not contain a subset with the order type of $\mathbb{Z}_-$.

Ex. 10.6.

(a). For any element $\alpha$ of $S_\Omega$, the set $\{x \in S_\Omega \mid x \leq \alpha\} = S_\alpha \cup \{\alpha\}$ is countable but $S_\Omega$ itself is uncountable [Lemma 10.2].

(b). For any element $\alpha \in S_\Omega$, the set $S_\alpha \cup \{\alpha\}$ is countable so its complement, $\{x \in S_\Omega \mid x > \alpha\} = (\alpha, +\infty)$, in the uncountable set $S_\Omega$, is uncountable [Lemma 10.2, Thm 7.5].

(c). We show the stronger statement [Thm 10.3] that $X_0$ is not bounded from above. We do this by assuming that $X_0$ has an upper bound $\alpha$ and find a contradiction. The (non-empty) simply ordered set $(\alpha, +\infty)$ is well-ordered [p. 63], it has no largest element by (a), and each element of $(\alpha, +\infty)$, except the smallest element, has an immediate predecessor. Thus $(\alpha, +\infty)$ has the order type of $\mathbb{Z}_+$, in particular $(\alpha, +\infty)$ is countable, contradicting (b). (Let $x$ be any element of $(\alpha, +\infty)$. Since $(\alpha, +\infty)$ does not contain an infinite descending chain [Ex 10.4], $\alpha$ is an iterated immediate predecessor of $x$ and $x$ is an iterated immediate successor of $\alpha$.)

Ex. 10.7. We show the contrapositive. Let $J_0$ be any subset of $J$ that is not everything. Let $\alpha$ be the smallest element of the complement $J - J_0$, the smallest element outside $J_0$. This means that $\alpha \notin J_0$ and that any element smaller than $\alpha$ is in $J_0$, i.e. $S_\alpha \subset J_0$. Thus $J_0$ is not inductive.

References