Ex. 25.1. \( \mathbb{R}^\ell \) is totally disconnected [Ex 23.7]; its components and path components [Thm 25.5] are points. The only continuous maps \( f : \mathbb{R} \to \mathbb{R}^\ell \) are the constant maps as continuous maps on connected spaces have connected images.

Ex. 25.2.

\( \mathbb{R}^\ell \) in product topology: Let \( X \) be \( \mathbb{R}^\ell \) in the product topology. Then \( X \) is is path connected (any product of path connected spaces is path connected [Ex 24.8]) and hence also connected.

\( \mathbb{R}^\ell \) in uniform topology: Let \( X \) be \( \mathbb{R}^\ell \) in the uniform topology. Then \( X \) is not connected for \( X = B \cup U \) where both \( B \), the set of bounded sequences, and \( U \), the complementary set of unbounded sequences, are open as any sequence within distance \( \frac{1}{2} \) of a bounded (unbounded) sequence is bounded (unbounded).

We shall now determine the path components of \( X \). Note first that for any sequence \( (y_n) \) we have

\[(0) \text{ and } (y_n) \text{ are in the same path component } \iff (y_n) \text{ is a bounded sequence}\]

\[\Rightarrow: \text{ Let } u : [0,1] \to X \text{ be a path from } (0) \text{ to } (y_n). \text{ Since } u(0) = (0) \text{ is bounded, also } u(1) = (y_n) \text{ is bounded for the connected set } u([0,1]) \text{ can not intersect both subsets in a separation of } X.\]

\[\Leftarrow: \text{ The formula } u(t) = (ty_n) \text{ is a path from } (0) \text{ to } (y_n). \text{ To see that } u \text{ is continuous note that } d(u(t_1), u(t_0)) = \sup \{ |(t_1 - t_0)y_n|, 1 \} = |t_1 - t_0| M \text{ when } |t_1 - t_0| < M^{-1} \text{ where } M = \sup \{ |y_n| | n \in \mathbb{Z}_+ \} \text{ and } d \text{ is the uniform metric.} \]

Next observe that \( (y_n) \to (x_n) + (y_n) \) is an isometry or \( X \) to itself [Ex 20.7]. It follows that in fact

\[(x_n) \text{ and } (y_n) \text{ are in the same path component } \iff (y_n - x_n) \text{ is a bounded sequence}\]

for any two sequences \( (x_n), (y_n) \in \mathbb{R}^\ell \).

This describes the path components of \( X \). It also shows that balls of radius \( < 1 \) are path connected. Therefore \( X \) is locally path connected so that the path components are the components [Thm 25.5].

\( \mathbb{R}^\ell \) in box topology: Let \( X \) be \( \mathbb{R}^\ell \) in the box topology. Then \( X \) is not connected for the box topology is finer than the uniform topology [1, Thm 20.4, Ex 23.1]; in fact, \( X = B \cup U \) where both \( B \), the set of bounded sequences, and \( U \), the complementary set of unbounded sequences, are open as they are open in the uniform topology or as any sequence in the neighborhood \( \prod (x_n - 1, x_n + 1) \) is bounded (unbounded) if \( (x_n) \) is bounded (unbounded), see [1, Example 6, p 151].

The (path) components of \( X \) can be described as follows:

\[(x_n) \text{ and } (y_n) \text{ in the same (path) component } \iff x_n = y_n \text{ for all but finitely many } n\]

\[\Rightarrow: \text{ Suppose that } x_n \text{ and } y_n \text{ are different for infinitely many } n \in \mathbb{Z}_+. \text{ For each } n, \text{ choose a homeomorphism } h_n : \mathbb{R} \to \mathbb{R} \text{ such that } h_n(x_n) = 0 \text{ and } h_n(y_n) = n \text{ in case } x_n \neq y_n. \text{ Then } h = \prod h_n : X \to X \text{ is a homeomorphism with } h(x_n) = (0) \text{ and } h(y_n) = n \text{ for infinitely many } n. \text{ Since a homeomorphism takes (path) components to (path) components and } h(x_n) = (0) \in B \text{ and } h(y_n) \in U \text{ are not in the same (path) component, } (x_n) \text{ and } (y_n) \text{ are not in the same (path) component either.}\]

\[\Leftarrow: \text{ The map } u(t) = (1 - t)x_n + ty_n, \ t \in [0,1], \text{ is constant in all but finitely many coordinates. From this we see that } u : [0,1] \to X \text{ is a continuous path from } (x_n) \text{ to } (y_n). \text{ Therefore, } (x_n) \text{ and } (y_n) \text{ are in the same (path) component.} \]

\( X \) is not locally connected since the components are not open [1, Thm 25.3]. The component of the constant sequence \( (0) \) is \( \mathbb{R}^\ell \).

\( \mathbb{R}^\ell \) in the box topology is an example of a space where the components and the path components are the same even though the space is not locally path connected, cf [1, Thm 25.5].
Ex. 25.3. A connected and not path connected space cannot be locally path connected [Thm 25.5]. Any linear continuum is locally connected (the topology basis consists of intervals which are connected in a linear continuum [Thm 24.1]). The subsets \( \{x\} \times [0, 1] = \{x \times 0, x \times 1\}, x \in [0, 1] \), are path connected for they are homeomorphic to \([0, 1]\) in the usual order topology [Thm 16.4]. There is no continuous path starting in \([x \times 0, x \times 1]\) and ending in \([y \times 0, y \times 1]\) when \(x \neq y\) for the same reason as there is no path from \(0 \times 0\) to \(1 \times 1\) [Example 6, p 156]. Therefore these sets are the path components of \(I_o^2\). Since the path components are not open we see once again that \(I_o^2\) is not locally path connected [Thm 25.4]. \((I_o^2\) is an example of a space with one component and uncountable many path components.)

Ex. 25.4. Any open subset of a locally path connected space is locally path connected. In a locally path connected space, the components and the path components are the same [Thm 25.5].

Ex. 25.8. Let \(p: X \to Y\) be a quotient map where \(X\) is locally (path-)connected. The claim is that \(Y\) is locally (path-)connected.

Let \(U\) be an open subspace of \(Y\) and \(C\) a (path-)component of \(U\). We must show that \(C\) is open in \(Y\), i.e., that \(p^{-1}(C)\) is open in \(X\). But \(p^{-1}(C)\) is a union of (path-)components of the open set \(p^{-1}(U)\) and in the locally (path-)connected space \(X\) open sets have open (path-)components.

References