Theorem 2. Let \( B \) be a countable set of limit points. Then \( B \) is metrizable.

Proof. Suppose \( B \) is uncountably many members because there are uncountably many disjoint open sets \( (x, y) \) such a space is not metrizable. To prove this, let \( \ell \) be a positive rational number such that \( 2 \ell \) suffices to show that for any \( A \) balls centered at points in \( B \) exists a finite subcovering. Hence it follows that there are uncountably many basis elements, contradicting that \( B \) is second-countable.

Ex. 30.3 (Morten Poulsen). Let \( X \) be second-countable and let \( A \) be an uncountable subset of \( X \). Suppose only countably many points of \( A \) are limit points of \( A \) and let \( A_0 \subset A \) be the countable set of limit points.

For each \( x \in A - A_0 \) there exists a basis element \( U_x \) such that \( x \in U_x \) and \( U_x \cap A = \{x\} \). Hence if \( a \) and \( b \) are distinct points of \( A - A_0 \) then \( U_a \neq U_b \), since \( U_a \cap A = \{a\} \neq \{b\} = U_b \cap A \). It follows that there uncountably many basis elements, contradicting that \( X \) is second-countable.

Note that it also follows that the set of points of \( A \) that are not limit points of \( A \) are countable.

Ex. 30.4 (Morten Poulsen).

Theorem 1. Every compact metrizable space is second-countable.

Proof. Let \( X \) be a compact metrizable space, and let \( d \) be a metric on \( X \) that induces the topology on \( X \).

For each \( n \in \mathbb{Z}_+ \) let \( A^n \) be an open covering of \( X \) with \( 1/n \)-balls. By compactness of \( X \) there exists a finite subcovering \( A^n \).

Now \( B = \bigcup_{n \in \mathbb{Z}_+} A_n \) is countable, being a countable union of finite sets.

\( B \) is a basis: Let \( U \) be an open set in \( X \) and \( x \in U \). By definition of the metric topology there exists \( \epsilon > 0 \) such that \( B_d(x, \epsilon) \subset U \). Choose \( N \in \mathbb{Z}_+ \) such that \( 2/N < \epsilon \). Since \( A_N \) covers \( X \) there exists \( B_d(y, 1/N) \) containing \( x \). If \( z \in B_d(y, 1/N) \) then

\[
d(x, z) \leq d(x, y) + d(y, z) \leq 1/N + 1/N = 2/N < \epsilon,
\]

i.e. \( z \in B_d(x, \epsilon) \), hence \( B_d(y, 1/N) \subset B_d(x, \epsilon) \subset U \). It follows that \( B \) is a basis.

\( \square \)

Ex. 30.5. Let \( X \) be a metrizable topological space.

Suppose that \( X \) has a countable dense subset \( A \). The collection \( \{B(a, r) \mid a \in A, r \in \mathbb{Q}_+\} \) of balls centered at points in \( A \) and with a rational radius is a countable basis for the topology: It suffices to show that for any \( y \in B(x, \epsilon) \) there are \( a \in A \) and \( r \in \mathbb{Q}_+ \) such that \( y \in B(a, r) \subset B(x, \epsilon) \). Let \( r \) be a positive rational number such that \( 2r < \epsilon - d(x, y) \) and let \( a \in A \cap B(a, r) \). Then \( y \in B(a, r) \), of course, and \( B(a, r) \subset B(x, \epsilon) \) for if \( d(a, z) < r \) then \( d(x, z) \leq d(x, y) + d(y, z) \leq d(x, y) + d(y, a) + d(a, z) < d(x, y) + 2r < \epsilon \).

Suppose that \( X \) is Lindelöf. For each positive rational number \( r \), let \( A_r \) be a countable subset of \( X \) such that \( X = \bigcup_{a \in A_r} B(a, r) \). Then \( A = \bigcup_{r \in \mathbb{Q}_+} A_r \) is a dense countable subset: Any open ball \( B(x, \epsilon) \) contains a point of \( A_r \) when \( 0 < r < \epsilon \), \( r \in \mathbb{Q}_+ \).

We now have an extended version of Thm 30.3:

Theorem 2. Let \( X \) be a topological space. Then

\[
X \text{ has a countable dense subset} \iff X \text{ is 2nd countable} \iff X \text{ is Lindelöf}
\]

\[
X \text{ is 1st countable}
\]

If \( X \) is metrizable, the three conditions of the top line equivalent.

Ex. 30.6. \( R_\ell \) has a countable dense subset and is not 2nd countable. According to [Ex 30.5] such a space is not metrizable.

The ordered square \( I^2_\ell \) is compact and not second countable. Any basis for the topology has uncountably many members because there are uncountably many disjoint open sets \( (x \times 0, x \times 1) \), \( x \in I \), and each of them contains a basis open set. (Alternatively, note that \( I^2_\ell \) contains the uncountable discrete subspace \( \{x \times \frac{1}{n} \mid x \in I \} \) so it can not be second countable by [Example 2 p 190].) According to [Ex 30.4] or [30.5(b)] a compact space with no countable basis is not metrizable.
Ex. 30.7. (Open ordinal space and closed ordinal space) Sets of the form \((\alpha, \beta), -\infty \leq \alpha < \beta \leq +\infty\), form bases for the topologies on the open ordinal space \(S_\Omega = [0, \Omega]\) and the closed ordinal space \(\overline{S}_\Omega = [0, \Omega]\) [§14, Thm 16.4]. The sets \((\alpha, \beta) = (\alpha, \beta + 1) = [\alpha + 1, \beta]\) are closed and open. Let \(n\) denote the \(n\)th immediate successor of the first element, 0.

[0, \Omega] is first countable: \([0] = [0, 1)\) is open so clearly \([0, \Omega]\) is first countable at the point 0. For any other element, \(\alpha > 0\), we can use the collection of neighborhoods of the form 
\((\beta , \alpha]\) for \(\beta < \alpha\).

[0, \Omega] does not have a countable dense subset: The complement of any countable subset contains a countable subcollection with empty intersection. Since closed subsets of \([0, \Omega]\) are closed, it follows immediately that closed subspaces of Lindelöf spaces are Lindelöf [Example 3, p. 181].

[0, \Omega] is not second countable: If it were, there would be a countable dense subset [Thm 30.3].

[0, \Omega] is not Lindelöf: The open covering consisting of the sets \([0, \alpha), \alpha < \Omega\), does not contain a countable subcovering.

[0, \Omega] is not first countable at \(\Omega\): This is a consequence of [Lemma 21.2] in that \(\Omega\) is a limit point of \([0, \Omega]\) but not the limit point of any sequence in \([0, \Omega]\) for all such sequences are bounded [Example 3, p. 181].

[0, \Omega] does not have a dense countable subset: for the same reason as for \([0, \Omega]\).

[0, \Omega] is not second countable: It is not even first countable.

[0, \Omega] is Lindelöf: It is even compact [Thm 27.1].

\(S_\Omega = [0, \Omega]\) is limit point compact but not compact [Example 2, p. 179] so it can not be metrizable [Thm 28.2]. \(S_\Omega\) is first countable and limit point compact so it is also sequentially compact [Thm 28.2].

\(\overline{S}_\Omega = [0, \Omega]\) is not metrizable since it is not first countable.

Ex. 30.9. A space \(X\) is Lindelöf if and only if any collection of closed subsets of \(X\) with empty intersection contains a countable subcollection with empty intersection. Since closed subspaces of closed subsets are closed, it follows immediately that closed subspaces of Lindelöf spaces are Lindelöf.

The anti-diagonal \(L \subset R_\ell \times R_\ell\) is a closed discrete uncountable subspace [Example 4 p 193]. Thus the closed subset \(L\) does not have a countable dense subset even though \(R_\ell \times R_\ell\) has a countable dense subset.

Ex. 30.12. Let \(f : X \to Y\) be an open continuous map.

Let \(B\) be a neighborhood basis at the point \(x \in X\). Let \(f(B)\) be the collection of images \(f(B) \subset f(X)\) of members \(B\) of the collection \(B\). The sets in \(f(B)\) are open in \(Y\), and hence also in \(f(X)\), since \(f\) is an open map. Let \(f(x)\) be a point in \(f(X)\). Any neighborhood of \(f(x)\) has the form \(V \cap f(X)\) for some neighborhood \(V \subset Y\) of \(f(x)\). Since \(p^{-1}(V)\) is a neighborhood of \(x\) there is a set \(B\) in the collection \(B\) such that \(x \in B \subset f^{-1}(V)\). Then \(x \in f(B) \subset V \cap f(X)\). This shows that \(f(B)\) is a neighborhood basis at \(f(x) \in f(X)\).

Let \(B\) be a basis for the topology on \(X\). Let \(f(B)\) be the collection of images \(f(B) \subset f(X)\) of members \(B\) of the collection \(B\). The sets in \(f(B)\) are open in \(Y\), and hence also in \(f(X)\), since \(f\) is an open map. Since \(B\) is a covering of \(X\), \(f(B)\) is a covering of \(f(X)\). Suppose that \(f(x) \in f(B_1) \cap f(B_2)\) where \(x \in X\) and \(B_1, B_2\) are basis sets. Choose a basis set \(B_3\) such that \(x \in B_3 \subset f^{-1}(f(B_1) \cap f(B_2))\). Then \(f(x) \in f(B_3) \subset f(B_1) \cap f(B_2)\). This shows that \(f(B)\) is a basis for a topology \(T_{f(B)}\) on \(f(X)\). This topology is coarser than the topology on \(f(X)\) since the basis elements are open in \(f(X)\). Conversely, let \(f(x) \in V \cap f(X)\) where \(V\) is open in \(Y\). Choose a basis element \(B\) such that \(x \in B \subset f^{-1}(V)\). Then \(f(x) \in f(B) \subset V \cap f(X)\). This shows that all open subsets of \(f(X)\) are in \(T_{f(B)}\). We conclude that \(f(B)\) is a basis for the topology on \(f(X)\).

We conclude that continuous open maps preserve 1st and 2nd countability.

Ex. 30.13. Let \(D\) be a countable dense subset and \(U\) a collection of open disjoint subsets. Pick a member of \(D\) inside each of the open open sets in \(U\). This gives an injective map \(U \to D\). Since \(D\) is countable also \(U\) is countable.
Ex. 30.16. For each natural number \( k \in \mathbb{Z}_+ \), let \( D_k \) be the set of all finite sequences

\[(I_1, \ldots, I_k, x_1, \ldots, x_k)\]

where \( I_1, \ldots, I_k \subset I \) are disjoint closed subintervals of \( I \) with rational endpoints and \( x_1, \ldots, x_k \in \mathbb{Q} \) are rational numbers. Since \( D_k \) is a subset of a countable set,

\[D_k \hookrightarrow (\mathbb{Q} \times \mathbb{Q}) \times \cdots \times (\mathbb{Q} \times \mathbb{Q}) \times \mathbb{Q} \times \cdots \times \mathbb{Q} = \mathbb{Q}^{3k},\]

\( D_k \) itself is countable [Cor 7.3]. Put \( D = \bigcup_{k \in \mathbb{Z}_+} D_k \). As a countable union of countable sets, \( D \) is countable [Thm 7.5].

For each element \((I_1, \ldots, I_k, x_1, \ldots, x_k) \in D_k\), let \( x(I_1, \ldots, I_k, x_1, \ldots, x_k) \in \mathbb{R}^I \) be the element given by

\[\pi_t x(I_1, \ldots, I_k, x_1, \ldots, x_k) = \begin{cases} x_j & t \in I_j \text{ for some } j \in \{1, \ldots, k\} \\ 0 & t \notin I_1 \cup \cdots \cup I_k \end{cases}\]

where \( \pi_t : \mathbb{R}^I \to \mathbb{R}, t \in I, \) is the projection map. This defines a map \( x : D \to \mathbb{R}^I \).

(a). The basis open sets in \( \mathbb{R}^I \) are finite intersections \( \bigcap_{i=1}^k \pi^{-1}_{i_j}(U_{i_j}) \) where \( i_1, \ldots, i_k \) are \( k \) distinct points in \( I \) and \( U_{i_1}, \ldots, U_{i_k} \) are \( k \) open subsets of \( \mathbb{R} \). Choose disjoint closed subintervals \( I_j \) such that \( i_j \in I_j \) and choose \( x_j \in U_{i_j} \cap \mathbb{Q}, j = 1, \ldots, k \). Then \( x(I_1, \ldots, I_k, x_1, \ldots, x_k) \in \bigcap_{i=1}^k \pi^{-1}_{i_j}(U_{i_j}) \) for \( \pi_t x(I_1, \ldots, I_k, x_1, \ldots, x_k) = x_j \in U_{i_j} \) for all \( j = 1, \ldots, k \). This shows that any (basis) open set contains an element of \( x(D) \), i.e., that the countable set \( x(D) \) is dense in \( \mathbb{R}^I \).

(b). Let \( D \) be a dense subset of \( \mathbb{R}^J \) for some set \( J \). Let \( f : J \to \mathcal{P}(D) \) be the map from the index set \( J \) to the power set \( \mathcal{P}(D) \) of \( D \) given by \( f(j) = D \cap \pi_{i_j}^{-1}(2003, 2004) \). Let \( j \) and \( k \) be two distinct points of \( J \). Then \( f(j) \neq f(k) \) for

\[f(j) - f(k) = (\pi_{i_j}^{-1}(2003, 2004) - \pi_{i_k}^{-1}(2003, 2004)) \cap D \]

since \( D \) is dense. This shows that \( f \) is injective. Thus \( \text{card } J \leq \text{card } \mathcal{P}(D) \).