Munkres §32

Ex. 32.1. Let \( Y \) be a closed subspace of the normal space \( X \). Then \( Y \) is Hausdorff [Thm 17.11]. Let \( A \) and \( B \) be disjoint closed subspaces of \( Y \). Since \( A \) and \( B \) are closed also in \( X \), they can be separated in \( X \) by disjoint open sets \( U \) and \( V \). Then \( Y \cap U \) and \( V \cap Y \) are open sets in \( Y \) separating \( A \) and \( B \).

Ex. 32.3. Look at [Thm 29.2] and [Lemma 31.1]. By [Ex 33.7], locally compact Hausdorff spaces are even completely regular.

Ex. 32.4. Let \( A \) and \( B \) be disjoint closed subsets of a regular Lindelöf space. We proceed as in the proof of [Thm 32.1]. Each point \( a \in A \) has an open neighborhood \( U_a \) with closure \( \overline{U_a} \) disjoint from \( B \). Applying the Lindelöf property to the open covering \( \{U_a\}_{a \in A} \cup \{X - A\} \) we get a countable open covering \( \{U_i\}_{i \in \mathbb{Z}_+} \) of \( A \) such that the closure of each \( U_i \) is disjoint from \( B \). Similarly, there is a countable open covering \( \{V_i\}_{i \in \mathbb{Z}_+} \) of \( B \) such that the closure of each \( V_i \) is disjoint from \( A \). Now the open set \( \bigcup U_i \) contains \( A \) and \( \bigcup V_i \) contains \( B \) but these two sets are not necessarily disjoint. If we put \( U'_i = U_1 - V_1, U'_2 = U_2 - V_1 - V_2, \ldots, U'_i = U_i - V_1 - \cdots - V_i, \ldots \) we subtract no points from \( A \) so that the open sets \( \{U'_i\} \) still form an open covering of \( A \). Similarly, the open sets \( \{V'_i\} \), where \( V'_i = V_i - U_1 - \cdots - U_i \), cover \( B \). Moreover, the open sets \( \bigcup U'_i \) and \( \bigcup V'_i \) are disjoint for \( U'_i \) is disjoint from \( V_1 \cup \cdots \cup V_i \) and \( V'_i \) is disjoint from \( U_1 \cup \cdots \cup U_i \).

Ex. 32.5. \( R^\omega \) (in product topology ) is metrizable [Thm 20.5], in particular normal [Thm 32.2]. \( R^\omega \) in the uniform topology is, by its very definition [Definition p. 124], metrizable, hence normal.

Ex. 32.6. Let \( X \) be completely normal and let \( A \) and \( B \) be separated subspaces of \( X \); this means that \( A \cap B = \emptyset = \overline{A} \cap B \). Note that \( A \) and \( B \) are contained in the open subspace \( X - (\overline{A} \cap \overline{B}) = (X - A) \cup (X - B) \) where their closures are disjoint. (The closure of \( A \) in \( X - (\overline{A} \cap \overline{B}) \) is \( \overline{A} - \overline{B} \) [Thm 17.4].) The subspace \( X - (\overline{A} \cap \overline{B}) \) is normal so it contains disjoint open subsets \( U \supset A \) and \( V \supset B \). Since \( U \) and \( V \) are open in an open subspace, they are open [Lemma 16.2].

Conversely, suppose that \( X \) satisfies the condition (and is a \( T_1 \)-space). Let \( Y \) be any subspace of \( X \) and \( A \) and \( B \) two disjoint closed subspaces of \( Y \). Since \( \overline{A} \cap Y \) and \( \overline{B} \cap Y \) are disjoint [Thm 17.4], \( \overline{A} \cap B = \overline{A} \cap (Y \cap B) = (\overline{A} \cap Y) \cap (B \cap Y) = \emptyset \), and, similarly, \( A \cap \overline{B} = \emptyset \). By assumption, \( A \) and \( B \) can then be separated by disjoint open sets. If we also assume that \( X \) is \( T_1 \) then it follows that \( Y \) is normal.

References